

RELATIONS BETWEEN UNION AND INTERSECTION OF IDEALS AND THEIR CORRESPONDING IDEAL TOPOLOGIES¹

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Abstract. The concept of ideals on topological spaces is studied by Vaidyanathaswamy, Kuratowski, Noiri and many others. In this paper we discuss the relationship between the topologies generated by intersection and union of ideals on a topological space. We also prove some results on connected topologies in the context of ideal topological space.

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1. Introduction

Ideals in topological spaces were studied by Vaidyanathaswamy [11], Kuratowski [8], Noiri [2, 3, 4] and many others [10, 5]. In 1990, Janković and Hamlett [6] investigated further properties of ideal topological spaces. In a topological space X , an ideal is defined as a nonempty collection \mathcal{I} of subsets of X which is closed under finite union and such that if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$. These two properties are satisfied by the collection of nowhere dense subsets of any topological space.

With the help of an ideal in a topological space X , we can construct an operator on the collection $\mathcal{P}(X)$, the set of all subsets of X , which will become a closure operator. The closure operator gives a topology \mathcal{T}^* on X (see [7]). This new topology is finer than the starting topology.

In almost all of the earlier works on ideal topological spaces some subsets of the topological space are declared as \mathcal{I} -open, α - \mathcal{I} -open, Semi- \mathcal{I} -open, or pre- \mathcal{I} -open and so on. Properties of such sets were studied. Noiri [2, 3, 4] and others [6, 10] call the open subsets of the topological space (X, \mathcal{T}^*) as $*$ -open sets and study their properties. In these works only one ideal is considered on a topological space. No significant work is available in the literature which consider more than one ideal and the topologies generated by them on a single topological space (X, \mathcal{T}) . In this paper we consider several ideals on the same topological space (X, \mathcal{T}) and construct new ideals using the set operations \cup

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and \cap . Also we study the relationship among the topologies generated by these ideals.

In Section 2 we recall definitions and results from the literature; in Section 3 we discuss the relationship between the union and intersection of ideals and topologies and in Section 4 we prove some results on ideal topological spaces in the context of connectedness.

2. Preliminary Definitions and Results

We start with the definition of an ideal in a topological space.

Definition 2.1. Let X be any set. An ideal in X is a nonempty collection \mathcal{I} of subsets of X satisfying the following.

- i. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
- ii. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

If (X, \mathcal{T}) is a topological space and \mathcal{I} is an ideal on X , then the triplet $(X, \mathcal{T}, \mathcal{I})$ is called an ideal topological space.

Unless otherwise stated X will denote a topological space, \mathcal{T} and \mathcal{I} denote the topology and the ideal on X under consideration. If (X, \mathcal{T}) is a topological space and $x \in X$, $\mathcal{T}(x)$ denote the set $\{U \in \mathcal{T} | x \in U\}$. We denote the complement $X - A$ of A in X by A^c .

A closure operator on a set X is a function on the collection of all subsets of X taking A to \bar{A} satisfying the following conditions:

- i. $\bar{\emptyset} = \emptyset$
- ii. For each A , $A \subseteq \bar{A}$
- iii. $\overline{\bar{A}} = \bar{A}$
- iv. For any A and B , $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

These four conditions are called Kuratowski closure axioms [7]. If “ $-$ ” is a closure operator on a set X , \mathcal{F} is the family of all subsets A of X for which $\bar{A} = A$, and if \mathcal{T} is the family of complements of members of \mathcal{F} , then \mathcal{T} is a topology on X and \bar{A} is the \mathcal{T} -closure of A for each subset A of X . This topology is called the topology generated by the closure operator “ $-$ ”.

Definition 2.2. ([8]) For any subset A of X , define

$$A^*(\mathcal{I}, \mathcal{T}) = \{x \in X | U \cap A \notin \mathcal{I} \text{ for every } U \in \mathcal{T}(x)\}.$$

The operator “ $-$ ” defined by $\bar{A} = A \cup A^*$ is a Kuratowski closure operator which gives a topology on X called the topology generated by \mathcal{I} .

This topology is usually denoted as \mathcal{T}^* , but as we are going to have many topologies simultaneously on one space (X, \mathcal{T}) , we prefer to denote it by $\mathcal{T}_{\mathcal{I}}$.

Let (X, \mathcal{T}) be a topological space. Then the following results hold trivially.

- i. If $\mathcal{I}_1 \subseteq \mathcal{I}_2$ are two ideals on X , then $\mathcal{T}_{\mathcal{I}_1} \subseteq \mathcal{T}_{\mathcal{I}_2}$.
- ii. If $\mathcal{I} = \{\emptyset\}$, then $\mathcal{T}_{\mathcal{I}} = \mathcal{T}$.
- iii. If A is closed in $\mathcal{T}_{\mathcal{I}}$, then $A^* \subseteq A$.
- iv. If $A \in \mathcal{I}$, then $A^* = \emptyset$ and A is closed in $(X, \mathcal{T}_{\mathcal{I}})$.

Definition 2.3. ([1]) A topological space (X, \mathcal{T}) is called Alexandroff space if the intersection of arbitrary family of open sets is open, or equivalently, the union of arbitrary family of closed sets is closed.

3. Union and Intersections of Ideal Topological Spaces

If $\{\mathcal{I}_\alpha\}$ is a collection of ideals on X , then $\cap \mathcal{I}_\alpha$ is an ideal and the union $\cup \mathcal{I}_\alpha$ need not be an ideal. But if $\{\mathcal{I}_\alpha\}$ is a chain of ideals, then $\cup \mathcal{I}_\alpha$ is an ideal.

If $\{\mathcal{I}_\alpha\}$ is a collection of ideals, then for simplicity we denote the topology generated by the ideal \mathcal{I}_α by \mathcal{T}_α .

Theorem 3.1. Let $\{\mathcal{I}_\alpha\}$ be a collection of ideals on (X, \mathcal{T}) . If $\mathcal{I} = \cap \mathcal{I}_\alpha$ then

$$(3.1) \quad \mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}_0$$

where $\mathcal{T}_0 = \cap \mathcal{T}_\alpha$.

Proof. That \mathcal{T}_0 is a topology follows obviously. As $\mathcal{I} = \cap \mathcal{I}_\alpha$, $\mathcal{I} \subseteq \mathcal{I}_\alpha$ and hence $\mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}_\alpha$ for every α . This implies $\mathcal{T}_{\mathcal{I}} \subseteq \cap \mathcal{T}_\alpha = \mathcal{T}_0$. \square

Strict inequality may hold in (3.1) and equality holds if and only if (X, \mathcal{T}) is an Alexandroff space as seen below.

Example 3.2. Let \mathcal{T} be the usual topology on \mathbb{R} . For every positive integer n , let \mathcal{I}_n be the collection of all subsets of the open interval $(0, \frac{1}{n})$. Let $\mathcal{I} = \cap \mathcal{I}_n$ and let $\mathcal{T}_0 = \cap \mathcal{T}_n$. Then $\mathcal{T}_{\mathcal{I}} \subsetneq \mathcal{T}_0$.

Let us justify our claim in the example. Clearly \mathcal{I}_n is an ideal, $\mathcal{I} = \{\emptyset\}$ and hence $\mathcal{T}_{\mathcal{I}} = \mathcal{T}$. We claim that $(-\infty, 0] \in \mathcal{T}_n$ for all n . As $(0, \frac{1}{n}) \in \mathcal{I}_n$, $(0, \frac{1}{n})$ is closed in \mathcal{T}_n ; as $\mathcal{T} \subseteq \mathcal{T}_n$, $[\frac{1}{n}, \infty)$ is closed in \mathcal{T}_n . Hence $(0, \frac{1}{n}) \cup [\frac{1}{n}, \infty)$ is closed in \mathcal{T}_n which shows that $(0, \infty)$ is closed in \mathcal{T}_n . Thus $(-\infty, 0] \in \mathcal{T}_n$ for all n and hence $(-\infty, 0] \in \cap \mathcal{T}_n = \mathcal{T}_0$. But $(-\infty, 0] \notin \mathcal{T}_{\mathcal{I}}$. Therefore $\mathcal{T}_{\mathcal{I}} \subsetneq \mathcal{T}_0$.

Theorem 3.3. A topological space (X, \mathcal{T}) is Alexandroff if and only if for any collection $\{\mathcal{I}_\alpha\}$ of ideals on X , $\mathcal{T}_{\mathcal{I}} = \mathcal{T}_0$ where $\mathcal{I} = \cap \mathcal{I}_\alpha$ and $\mathcal{T}_0 = \cap \mathcal{T}_\alpha$.

Proof. Assume that (X, \mathcal{T}) is Alexandroff. Let $\{\mathcal{I}_\alpha\}$ be a collection of ideals and $\mathcal{I} = \cap \mathcal{I}_\alpha$. As $\mathcal{I} \subseteq \mathcal{I}_\alpha$, $\mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}_\alpha$ for every α . This implies that

$$(3.2) \quad \mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}_0.$$

We claim that $\mathcal{T}_0 \subseteq \mathcal{T}_{\mathcal{I}}$. Let $V \in \mathcal{T}_0$ and $A = V^c$. As $V \in \mathcal{T}_\alpha$, A is closed in \mathcal{T}_α for all α . Thus

$$(3.3) \quad A^*(\mathcal{I}_\alpha, \mathcal{T}) \subseteq A \text{ for every } \alpha.$$

We aim to prove that A is closed in $\mathcal{T}_{\mathcal{I}}$. It is enough to prove that $A^*(\mathcal{I}, \mathcal{T}) \subseteq A$. Assume that $x \notin A$. This implies $x \notin A^*(\mathcal{I}_\alpha, \mathcal{T})$ for every α by (3.3). Then for all α there exists $U_\alpha \in \mathcal{T}(x)$ such that

$$(3.4) \quad U_\alpha \cap A \in \mathcal{I}_\alpha.$$

Let $U = \bigcap U_\alpha$. Then $x \in U$ and U is open in (X, \mathcal{T}) as (X, \mathcal{T}) is Alexandroff. As $U \subseteq U_\alpha$, $U \cap A \subseteq U_\alpha \cap A$ for every α . Therefore by (3.4), $U \cap A \in \mathcal{I}_\alpha$ for every α . That is, $U \cap A \in \bigcap \mathcal{I}_\alpha = \mathcal{I}$. Thus we obtained an open set U in (X, \mathcal{T}) such that $U \cap A \in \mathcal{I}$. Therefore $x \notin A^*(\mathcal{I}, \mathcal{T})$ and hence $A^*(\mathcal{I}, \mathcal{T}) \subseteq A$. Thus A is closed in $\mathcal{T}_{\mathcal{I}}$. That is, $V \in \mathcal{T}_{\mathcal{I}}$. Therefore $\mathcal{T}_0 \subseteq \mathcal{T}_{\mathcal{I}}$. This together with (3.2), implies that $\mathcal{T}_{\mathcal{I}} = \mathcal{T}_0$.

To prove the converse assume that for any collection $\{\mathcal{I}_\alpha\}$ of ideals on (X, \mathcal{T}) , $\mathcal{T}_{\mathcal{I}} = \mathcal{T}_0$ where $\mathcal{I} = \bigcap \mathcal{I}_\alpha$ and $\mathcal{T}_0 = \bigcap \mathcal{T}_\alpha$. To prove (X, \mathcal{T}) is Alexandroff let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in (X, \mathcal{T}) . Let $U = \bigcap U_\alpha$. We claim that U is open in (X, \mathcal{T}) .

For all $\alpha \in \Lambda$, let $A_\alpha = U_\alpha - U$ and let \mathcal{I}_α be the collection of all the subsets of the set A_α . Then \mathcal{I}_α is an ideal on X . If A is any set in \mathcal{I} , then for all α , $A \in \mathcal{I}_\alpha$ and hence $A \subseteq U_\alpha - U$. As A is a subset of U_α for all α , $A \subseteq U$; as $A \subseteq U_\alpha - U$, $A = \emptyset$. Hence $\mathcal{I} = \bigcap \mathcal{I}_\alpha = \{\emptyset\}$. Therefore $\mathcal{T}_{\mathcal{I}} = \mathcal{T}$. By our assumption, $\mathcal{T}_{\mathcal{I}} = \mathcal{T}_0$ and hence $\mathcal{T}_0 = \mathcal{T}$.

As $A_\alpha \in \mathcal{I}_\alpha$, A_α is closed in \mathcal{T}_α . Since U_α is open in \mathcal{T} , it is open in \mathcal{T}_α and as A_α is closed in \mathcal{T}_α , $U_\alpha - A_\alpha$ is open in \mathcal{T}_α . Hence $U = U_\alpha - A_\alpha$ is open in \mathcal{T}_α for all $\alpha \in \Lambda$. Thus $U \in \mathcal{T}_\alpha$ and hence $U \in \mathcal{T}_0$. Thus $U \in \mathcal{T}$. Therefore (X, \mathcal{T}) is Alexandroff. \square

Now we turn our attention towards union of ideals. The union of a chain of ideals is always an ideal, but the union of an arbitrary collection of ideals need not be an ideal.

If $\{\mathcal{I}_\alpha\}$ is a collection of ideals on X , then the collection $\mathcal{P}(X)$ of all subsets of X is an ideal containing all members of all the ideals \mathcal{I}_α ; so, the family \mathcal{C} of all ideals on X containing all members of all \mathcal{I}_α 's is nonempty and hence the intersection of all ideals in \mathcal{C} is the smallest ideal containing all members of all these ideals \mathcal{I}_α . So, for any collection of ideals there exists a smallest ideal containing all these ideals.

Fact: Let $\{\mathcal{I}_\alpha\}$ be a collection of ideals and let \mathcal{I} be the smallest ideal containing all \mathcal{I}_α 's. Then \mathcal{I} is the collection of all sets formed by taking the union of finitely many members from $\bigcup \mathcal{I}_\alpha$.

Proof. Let \mathcal{I}_0 be the collection of all sets formed by taking the union of finitely many members from $\bigcup \mathcal{I}_\alpha$. We claim that $\mathcal{I}_0 = \mathcal{I}$. We first show that \mathcal{I}_0 is an ideal. Let $A, B \in \mathcal{I}_0$. Then A and B are the union of finitely many members from $\bigcup \mathcal{I}_\alpha$ and hence $A \cup B$ is also so.

Let $A \in \mathcal{I}_0$ and $B \subseteq A$. Then $A = A_1 \cup A_2 \cup \dots \cup A_n$ where $A_i \in \mathcal{I}_{\alpha_i}$ and hence

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n).$$

But $(B \cap A_i)$ is a subset of A_i and hence in \mathcal{I}_{α_i} . Thus B is the union of finitely many members from $\cup \mathcal{I}_{\alpha}$ and hence $B \in \mathcal{I}_0$. Thus \mathcal{I}_0 is an ideal containing all members of all \mathcal{I}_{α} 's.

Let \mathcal{I}' be an ideal containing all members of all \mathcal{I}_{α} 's. If A is in \mathcal{I}_0 , then $A = A_1 \cup A_2 \cup \dots \cup A_n$ where $A_i \in \mathcal{I}_{\alpha_i}$. As $A_i \in \mathcal{I}_{\alpha_i}$, $A_i \in \mathcal{I}'$; an ideal is closed under finite union; hence $A_1 \cup A_2 \cup \dots \cup A_n \in \mathcal{I}'$. That is, $A \in \mathcal{I}'$. Thus $\mathcal{I}_0 \subseteq \mathcal{I}'$. This proves that \mathcal{I}_0 is the smallest ideal containing all \mathcal{I}_{α} 's. \square

Theorem 3.4. *Let (X, \mathcal{T}) be a topological space. Let $\{\mathcal{I}_{\alpha}\}$ be a collection of ideals on X . Let \mathcal{I} be the smallest ideal containing all \mathcal{I}_{α} 's. Let \mathcal{T}_0 be the smallest topology containing all \mathcal{T}_{α} 's. Then $\mathcal{T}_{\mathcal{I}} = \mathcal{T}_0$.*

Proof. As $\{\mathcal{I}_{\alpha}\}$ is a collection of ideals and \mathcal{I} is the smallest ideal containing \mathcal{I}_{α} 's, $\mathcal{I}_{\alpha} \subseteq \mathcal{I}$ and hence $\mathcal{T}_{\alpha} \subseteq \mathcal{T}_{\mathcal{I}}$. But \mathcal{T}_0 is the smallest topology containing all \mathcal{T}_{α} 's. Therefore

$$(3.5) \quad \mathcal{T}_0 \subseteq \mathcal{T}_{\mathcal{I}}.$$

Next we claim that $\mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}_0$. Let $V \in \mathcal{T}_{\mathcal{I}}$ and $x \in V$; we shall construct a \mathcal{T}_0 -open set $W \subseteq V$ containing x . Let $A = V^c$; then A is closed in $\mathcal{T}_{\mathcal{I}}$ and $x \notin A$. Then $A^*(\mathcal{I}, \mathcal{T}) \subseteq A$. This implies that $x \notin A^*(\mathcal{I}, \mathcal{T})$. Then there exists an open set $U \in \mathcal{T}(x)$ such that $U \cap A \in \mathcal{I}$. Let $K = U \cap A$. Then $K \in \mathcal{I}$. Now, by Fact stated above,

$$K = K_1 \cup K_2 \cup \dots \cup K_n$$

where $K_i \in \mathcal{I}_{\alpha_i}$ for some finitely many α_i 's. As $K_i \in \mathcal{I}_{\alpha_i}$, K_i is closed in \mathcal{T}_{α_i} and hence K_i is closed in \mathcal{T}_0 . Therefore $\bigcup_{i=1}^n K_i$ is closed in \mathcal{T}_0 . That is, K is closed in \mathcal{T}_0 . Hence $K^c \in \mathcal{T}_0$. Since $U \in \mathcal{T}$, we have $U \in \mathcal{T}_0$. Let $W = U \cap K^c$. Then $W \in \mathcal{T}_0$ and $x \in W$. Also

$$W \cap A = (U \cap K^c) \cap A = (U \cap A) \cap K^c = K \cap K^c = \emptyset.$$

Thus $W \cap A = \emptyset$. That is, $W \subseteq A^c$. Hence W is an open set in \mathcal{T}_0 containing x such that $W \subseteq A^c$. Therefore A^c is open in \mathcal{T}_0 . This implies that V is open in \mathcal{T}_0 . Therefore $\mathcal{T}_{\mathcal{I}} \subseteq \mathcal{T}_0$. This, together with (3.5), implies that $\mathcal{T}_{\mathcal{I}} = \mathcal{T}_0$. \square

We know that the union of a collection of topologies need not be a topology. Even if the collection of topologies is a chain, the union need not be a topology. For example, on \mathbb{R} , if for $n \in \mathbb{N}$,

$$\mathcal{T}_n = \{\emptyset, \mathbb{R}, \{1\}, \{1, 2\}, \{1, 2, 3\}, \dots, \{1, 2, 3, \dots, n\}\}$$

then $\{\mathcal{T}_n\}$ is a chain of topologies on \mathbb{R} ; if $U_n = \{1, 2, 3, \dots, n\}$, then $U_n \in \mathcal{T}_n$ and hence $U_n \in \mathcal{T} = \cup \mathcal{T}_k$; but $\bigcup_{n=1}^{\infty} U_n \notin \mathcal{T}$; hence \mathcal{T} is not a topology.

But if $\{\mathcal{T}_{\alpha}\}$ is a chain of topologies generated by a chain of ideals, then the union is a topology as seen in the following theorem.

Theorem 3.5. Let $\{\mathcal{I}_\alpha\}$ be a chain of ideals. Let $\mathcal{T}_0 = \cup \mathcal{T}_\alpha$ and $\mathcal{I} = \cup \mathcal{I}_\alpha$. Then $\mathcal{T}_\mathcal{I} = \mathcal{T}_0$ and hence \mathcal{T}_0 is a topology.

Proof. As $\mathcal{I} = \cup \mathcal{I}_\alpha$, $\mathcal{I}_\alpha \subseteq \mathcal{I}$ and hence $\mathcal{T}_\alpha \subseteq \mathcal{T}_\mathcal{I}$. This implies that

$$(3.6) \quad \mathcal{T}_0 \subseteq \mathcal{T}_\mathcal{I}$$

Next we claim that $\mathcal{T}_\mathcal{I} \subseteq \mathcal{T}_0$. Let $V \in \mathcal{T}_\mathcal{I}$ and $x \in V$. Let $A = V^c$; then A is closed in $\mathcal{T}_\mathcal{I}$, and $x \notin A$. Then $A^*(\mathcal{I}, \mathcal{T}) \subseteq A$. This implies that $x \notin A^*(\mathcal{I}, \mathcal{T})$. Thus there exists an open set $U \in \mathcal{T}(x)$ such that $U \cap A \in \mathcal{I}$. Let $K = U \cap A$. Then $K \in \mathcal{I}$. This implies that $K \in \cup \mathcal{I}_\alpha$. That is, $K \in \mathcal{I}_\alpha$ for some α . Therefore K is closed in \mathcal{T}_α and hence $K^c \in \mathcal{T}_\alpha$. Since $U \in \mathcal{T}$, we have $U \in \mathcal{T}_\alpha$. Let $W = U \cap K^c$. Then $W \in \mathcal{T}_\alpha$ and $x \in W$. As in the proof of Theorem 3.4, $W \cap A = \emptyset$. That is $W \subseteq A^c$. Therefore W is an open set in \mathcal{T}_α containing x such that $W \subseteq A^c$. Therefore A^c is open in \mathcal{T}_α . This implies that $V \in \mathcal{T}_\alpha$ and hence $V \in \cup \mathcal{T}_\alpha$. Therefore $\mathcal{T}_\mathcal{I} \subseteq \mathcal{T}_0$. This together with (3.6), implies that $\mathcal{T}_\mathcal{I} = \mathcal{T}_0$. \square

If $\{\mathcal{T}_\alpha\}$ is an increasing chain of topologies and if $\cup \mathcal{T}_\alpha$ is a topology on X , then we may not find a chain of ideals $\{\mathcal{I}_\alpha\}$ such that \mathcal{T}_α is the topology generated by \mathcal{I}_α 's. The following example proves this.

Example 3.6. On \mathbb{R} , let $\mathcal{T}_0 = \{\emptyset, \mathbb{R}\}$ and for positive integers n , let

$$\mathcal{T}_n = \{\emptyset, \mathbb{R}, (-1, 1), (-2, 2), \dots, (-n, n)\}.$$

Then \mathcal{T}_n are topologies whose union $\mathcal{T} = \{\emptyset, \mathbb{R}, (-1, 1), (-2, 2), \dots\}$ is also a topology on \mathbb{R} . But there is no sequence $\{\mathcal{I}_n\}$ of ideals on \mathbb{R} so that \mathcal{T}_n are ideal topologies generated by \mathcal{I}_n on any topology on \mathbb{R} .

Indeed if at all there is such a sequence and a topology \mathcal{T} on \mathbb{R} , then \mathcal{T} must be $\{\emptyset, \mathbb{R}\}$ as \mathcal{T}_0 should be finer than \mathcal{T} ; \mathcal{I}_1 cannot be $\{\emptyset\}$; if A is a nonempty set in \mathcal{I}_1 , then all of its subsets are in \mathcal{I}_1 and hence their complements must be in \mathcal{I}_1 which is not so.

We have a question: If \mathcal{T} and \mathcal{T}' are topologies on a set X so that \mathcal{T}' is finer than \mathcal{T} , is there any ideal \mathcal{I} so that $\mathcal{T}' = \mathcal{T}_\mathcal{I}$? The answer to this question is "No" (See Example 3.7). Mahathalenal [9] defined the concept of semi-ideals and proved that whenever \mathcal{T}' is finer than \mathcal{T} there is a semi-ideal \mathcal{J} so that the topology generated by \mathcal{J} is \mathcal{T}' . But a nice characterization of topologies having this property is given in Theorem 3.8.

Example 3.7. Let

$$X = \{1, 2, 3, 4\} \quad \mathcal{T} = \{\emptyset, X, \{1, 2\}\} \quad \text{and} \quad \mathcal{T}' = \{\emptyset, X, \{1\}, \{1, 2\}\}.$$

Then there is no ideal \mathcal{I} such that $\mathcal{T}_\mathcal{I} = \mathcal{T}'$.

Proof. Suppose that there is an ideal \mathcal{I} such that $\mathcal{T}_\mathcal{I} = \mathcal{T}'$. Clearly $\mathcal{I} \neq \{\emptyset\}$. So there is a nonempty set A in \mathcal{I} .

Since $A^c \in \mathcal{T}_\mathcal{I}$, A must be either X , $\{2, 3, 4\}$ or $\{3, 4\}$. In all the cases $\{4\} \subseteq A$. Thus $\{4\} \in \mathcal{I}$ which implies that $\{4\}^c \in \mathcal{T}_\mathcal{I}$ which is not so. \square

Theorem 3.8. *Let (X, \mathcal{T}) be a topological space. Let \mathcal{T}' be a topology finer than \mathcal{T} . Then $\mathcal{T}' = \mathcal{T}_{\mathcal{I}}$ for some ideal \mathcal{I} if and only if there exists a chain of ideals $\{\mathcal{I}_\alpha\}$ such that $\cup \mathcal{T}_\alpha = \mathcal{T}'$.*

Proof. Let $\mathcal{T}' = \mathcal{T}_{\mathcal{I}}$ for some ideal \mathcal{I} . Then $\{\mathcal{I}\}$ itself is a chain of ideals having the desired properties. Conversely, let there be a chain of ideals $\{\mathcal{I}_\alpha\}$ such that $\cup \mathcal{T}_\alpha = \mathcal{T}'$. Let $\mathcal{I} = \cup \mathcal{I}_\alpha$. By Theorem 3.5, $\mathcal{T}_{\mathcal{I}} = \cup \mathcal{T}_\alpha$. Then $\mathcal{T}_{\mathcal{I}} = \mathcal{T}'$. \square

4. Connectedness and Ideal Topological Spaces

If $\{\mathcal{T}_\alpha\}$ is a collection of connected topologies on a set X , then the smallest topology containing all \mathcal{T}_α 's need not be connected, even if $\{\mathcal{T}_\alpha\}$ is a chain of connected topologies as seen in the following example.

Example 4.1. Let

$$\mathcal{T}_n = \mathcal{P}(\{1, 2, 3, \dots, n\}) \cup \{\mathbb{N}\}$$

where $\mathcal{P}(A)$ denotes the collection of all subsets on A . Each \mathcal{T}_n is a connected topology on \mathbb{N} . But the smallest topology containing all the \mathcal{T}_n 's is the discrete topology on \mathbb{N} which is not connected.

But if $\{\mathcal{T}_\alpha\}$ is a chain of connected topologies generated by a chain of ideals, then the union is a connected topology as seen in the following theorem.

Theorem 4.2. *Let $\{\mathcal{I}_\alpha\}$ be a chain of ideals so that each (X, \mathcal{T}_α) is connected. Then $(X, \cup \mathcal{T}_\alpha)$ is connected.*

Proof. Let $\mathcal{I} = \cup \mathcal{I}_\alpha$. By Theorem 3.5, $\mathcal{T}_{\mathcal{I}} = \cup \mathcal{T}_\alpha$ and is a topology on X . Suppose that $(X, \mathcal{T}_{\mathcal{I}})$ is not connected. Then $X = A \cup B$ where A and B are nonempty open sets in $\mathcal{T}_{\mathcal{I}}$ and $A \cap B = \emptyset$.

Now $A, B \in \mathcal{T}_{\mathcal{I}}$. This implies that $A, B \in \cup \mathcal{T}_\alpha$. Then $A, B \in \mathcal{T}_\alpha$ for some α as $\{\mathcal{T}_\alpha\}$ is a chain. Thus we have two nonempty open sets $A, B \in \mathcal{T}_\alpha$ such that $X = A \cup B$ and $A \cap B = \emptyset$. This implies that (X, \mathcal{T}_α) is not connected which is not so. Therefore $(X, \cup \mathcal{T}_\alpha)$ is connected. \square

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