

GENERALIZED SOLUTIONS TO STOCHASTIC SYSTEMS WITH MULTIPLICATIVE NOISE IN GELFAND–SHILOV SPACES

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Abstract. The Cauchy problem for systems of differential equations with multiplicative random perturbations in the form of infinite-dimensional Ito integrals is studied. For the systems correct by Petrovskii, conditionally correct and incorrect we point out Gelfand–Shilov spaces of generalized functions where a generalized solution coincides with a mild solution.

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1. Introduction

One of the modern trends of recent research is the study of problems with random perturbations. An important place among them is occupied by differential equations in infinite-dimensional spaces that contain white noise perturbations, additive and multiplicative. For the stochastic problems with multiplicative noise the main object of the research is the first order Cauchy problem in the Ito statement:

$$dX(t) = AX(t)dt + B(X(t))dW(t), \quad t \in [0; T], \quad X(0) = f.$$

The problem was studied in the case when A is the generator of a semigroup of class C_0 in a Hilbert space H , and W is a Wiener process with values in another Hilbert space \mathbb{H} , and $B(X) : \mathbb{H} \rightarrow H$ [3, 5, 8]. Nevertheless, important models in physics, biology and financial mathematics (see, e.g., [1, 4, 13]) reduce to the case of equations with operators that do not generate semigroups of class C_0 and corresponding homogeneous Cauchy problems that are ill-posed.

The present paper is devoted to the stochastic Cauchy problem with differential operators $A = A\left(i\frac{\partial}{\partial x}\right)$:

$$(1.1) \quad \begin{aligned} dX(t, x) &= A\left(i\frac{\partial}{\partial x}\right)X(t, x)dt + B(X(t, x))dW(t, x), \quad t \in [0; T], \\ X(0, x) &= f(x), \quad x \in \mathbb{R}. \end{aligned}$$

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It turns out that such operators A generate R -semigroups and the problem requires studying generalized (with respect to x) solutions. The presence of operator B depending on the generalized solution in this case brings a lot of questions even at the stage of the problem formulation.

2. Statement of the problem and preliminary results

The main object of the paper is the stochastic Cauchy problem (1.1) for system of differential equations of order m with multiplicative random perturbations. In the integral form the problem is written as follows:

$$(2.1) \quad X(t, x) - f(x) = \int_0^t A \left(i \frac{\partial}{\partial x} \right) X(s, x) ds + \int_0^t B(X(s, x)) dW(s, x),$$

$$t \in [0; T], \quad x \in \mathbb{R},$$

and (1.1) is the short form of (2.1). Here

$$A \left(i \frac{\partial}{\partial x} \right) : H = L_m^2(\mathbb{R}) := L^2(\mathbb{R}) \times \dots \times L^2(\mathbb{R}) \rightarrow H$$

is an $m \times m$ operator-matrix whose elements are linear differential operators of orders not exceeding p , initial data $f(\cdot) = f(\cdot, \omega)$, $\omega \in (\Omega, \mathcal{B}(\Omega); P)$, is an H -valued random value, $W(t) = W(t, \cdot) = W(t, \cdot, \omega)$, $t \geq 0$, is a Q -Wiener process with values in $\mathbb{H} = L_n^2(\mathbb{R})$ or a cylindrical (weak) Wiener process.

The \mathbb{H} -valued Q -Wiener process $\{W_Q(t) = W(t, \cdot), t \geq 0\}$ is defined as the Fourier series convergent in \mathbb{H} :

$$W_Q(t) = \sum_{j \in \mathbb{N}} \sigma_j \beta_j(t) e_j,$$

where β_j are independent Brownian motions and $\{e_j\}$ is an orthogonal basis in \mathbb{H} consisting of eigenvectors of a trace class operator Q : $Qe_j = \sigma_j^2 e_j$. The \mathbb{H} -valued cylindrical Wiener process $\{W(t), t \geq 0\}$ is defined as the series: $W(t) = \sum_{j \in \mathbb{N}} \beta_j(t) e_j$, weakly convergent in \mathbb{H} (see, e.g., [3, 11, 2]).

The operator $B = B(X)$, due to which we have not additive but multiplicative disturbances, is supposed to be under the condition ensuring the convergence of the Ito integral $\int_0^t B(X(s, \cdot)) dW(s, \cdot)$ in (2.1):

$$(2.2) \quad \mathbf{E} \left[\int_0^T \|B(X)\|_{\mathbb{H}\mathbb{S}}^2 dt \right] < \infty,$$

where $\|B\|_{\mathbb{H}\mathbb{S}}$ is the norm in the space of Hilbert–Schmidt operators acting from $\mathbb{H}_Q = \{f = Q^{1/2}v \mid v \in \mathbb{H}, \|f\|_{\mathbb{H}_Q} := \|v\|_{\mathbb{H}}\}$ to H for the case of a Q -Wiener process W and acting from \mathbb{H} to H for the case of a weak Wiener process W .

The results of studies of the homogeneous deterministic Cauchy problem

$$(2.3) \quad \frac{\partial}{\partial t} u(t, x) = A \left(i \frac{\partial}{\partial x} \right) u(t, x), \quad t \in [0; T], \quad x \in \mathbb{R}, \quad u(0, x) = f(x),$$

indicate that in the general case the differential operator $A = A(i\frac{\partial}{\partial x})$ does not generate a class C_0 semigroup in H and the problem is not well-posed in H .

The generalized Fourier transform techniques provide that a unique generalized solution for (2.3) exists in an appropriate space of Gelfand–Shilov generalized with respect to x functions Ψ' [7, 9]. This solution can be represented by the family of solution operators $\{U(t), t \in [0; T]\}$:

$$(2.4) \quad u(t, x) = (U(t)f)(x) := (G(t, \cdot) * f(\cdot))(x),$$

where the matrix-function $G(t, x)$ defined as the inverse generalized Fourier transform of $e^{tA(\sigma)}$:

$$G(t, x) := \mathcal{F}^{-1}[e^{tA(\sigma)}](x),$$

is the Green function to (2.3), and $e^{tA(\sigma)}$ is defined as a formal series with respect to powers of $A(\sigma)$, which is the Fourier transform of $A(i\frac{\partial}{\partial x})$. The choice of the space Ψ' is determined by the behavior of matrix-function $e^{tA(\sigma)}$ in the Fourier transformed space $\tilde{\Psi}'$. Namely, we prove that if $e^{tA(\sigma)}$ is a multiplier from $\tilde{\Phi}$ to $\tilde{\Psi}$, then $G(\cdot, t)$ is a convolutor from Φ' to Ψ' (Theorem 3.5). On the basis of this theorem we specify spaces Φ' and Ψ' for every class of the differential system (2.3), where the classification of the systems is given in [7] by the growth of $e^{tA(s)}$, $s \in \mathbb{C}$.

To describe this classification, consider the problem dual to (2.3) with respect to the Fourier transform:

$$(2.5) \quad \frac{\partial}{\partial t} \tilde{u}(t, \sigma) = A(\sigma) \tilde{u}(t, \sigma), \quad t \in [0; T], \quad \sigma \in \mathbb{R}, \quad \tilde{u}(0, \sigma) = \tilde{f}(\sigma).$$

As shown in [7], properties of the solution operators $e^{tA(s)}$ of this problem are determined by the function $\Lambda(s) := \max_j \operatorname{Re} \lambda_j(s)$, $s = \sigma + i\tau$, where $\lambda_j(s)$ are eigenvalues of $A(s)$, and this is a consequence of the estimate:

$$e^{t\Lambda(s)} \leq \left\| e^{tA(s)} \right\|_m \leq C(1 + |s|)^{p(m-1)} e^{t\Lambda(s)}, \quad t \in [0; T].$$

On the basis of $\Lambda(\cdot)$ growth the following classes of the systems (2.3) are allocated:

- *Petrovskii correct systems:* $\Lambda(\sigma) \leq C_1$. This implies the estimate

$$(2.6) \quad \left\| e^{tA(\sigma)} \right\|_m \leq C_2(1 + |\sigma|)^h, \quad t \in [0; T], \quad \sigma \in \mathbb{R},$$

where $C_2 = Ce^{TC_1}$, and $h \leq p(m - 1)$ is the least of $l \in \mathbb{N}$ that provide the estimate $\|e^{tA(\sigma)}\|_m \leq C_2(1 + |\sigma|)^l$.

- *conditionally correct systems:* $\Lambda(\sigma) \leq C|\sigma|^h + C_1$, $h \in (0; 1)$. In this case for $e^{tA(\sigma)}$ the following estimate is true

$$(2.7) \quad \left\| e^{tA(\sigma)} \right\|_m \leq C_2 e^{a|\sigma|^h}, \quad t \in [0; T], \quad \sigma \in \mathbb{R}, \quad \text{where } a > CT.$$

- *incorrect systems:* $\Lambda(\sigma) \leq C|\sigma|^{p_0} + C_1$. In this case

$$(2.8) \quad \left\| e^{tA(\sigma)} \right\|_m \leq C_2 e^{a|\sigma|^{p_0}}, \quad t \in [0; T], \quad \sigma \in \mathbb{R}, \quad \text{where } p_0 \leq p, \quad a > CT.$$

Although $A = A(i \frac{\partial}{\partial x})$ does not generate a semigroup of class C_0 in H in the general case, it generates an R -semigroup $\{S(t), t \in [0; \tau)\}, \tau \leq \infty$ [12]:

$$(2.9) \quad S(t)f = \left(\mathcal{F}^{-1} \left[e^{tA(\sigma)} \tilde{K}(\sigma) \right] \right) * f, \quad t \in [0; \tau), f \in H,$$

where the infinitely differentiable function $\tilde{K}(\cdot)$ is selected according to the growth of $e^{tA(\sigma)}$, and the regularizing operator R is defined as follows

$$(2.10) \quad (Rf)(x) := \left(\mathcal{F}^{-1} \left[\tilde{K}(\sigma) \tilde{f}(\sigma) \right] \right) (x) = \left(\mathcal{F}^{-1} \left[\tilde{K}(\sigma) \right] * f \right) (x), \quad f \in H.$$

In this case solution operators $U(t) = R^{-1}S(t)$ to the problem (2.3) are not bounded in H , but, due to the generalized well-posedness of (2.3), they become bounded in spaces of generalized functions Φ', Ψ' . This leads us to generalized solutions of (2.3), and we consider them in the following sense:

$$\langle \psi, U(t)f \rangle = \langle \psi, R^{-1}S(t)f \rangle = \langle (R^{-1})^* \psi, S(t)f \rangle, \quad t \in [0; T], T < \tau, \\ \psi \in \Psi, f \in H \subset \Phi'.$$

According to the theory of abstract stochastic problems with a multiplicative noise, the solution of (2.1) with A being a generator of a C_0 -semigroup $\{U(t), t \geq 0\}$ in H can be written in the following form

$$(2.11) \quad X(t) = U(t)f + \int_0^t U(t-s)B(X(s))dW(s), \quad t \in [0; T], f \in \text{dom } A.$$

We extend this result to the case of R -semigroups and show (Theorem 3.2) that if A generates an R -semigroup $\{S(t), t \in [0; \tau)\}$, then the family of generalized functions $\{X(t, \cdot), t \in [0; T]\}$ for $T < \tau$ defined by the equation

$$(2.12) \quad \langle \psi, X(t) \rangle = \langle (R^{-1})^* \psi, S(t)f \rangle + \langle (R^{-1})^* \psi, \int_0^t S(t-s)B(X(s))dW(s) \rangle, \\ \psi \in \Psi, f \in H \subset \Phi',$$

is a solution of the generalized stochastic Cauchy problem

$$(2.13) \quad \langle \psi, X(t) \rangle - \langle \psi, f \rangle = \int_0^t \langle A^* \psi, X(s) \rangle ds + \langle \psi, \int_0^t B(X(s))dW(s) \rangle, \\ t \in [0; T], \psi \in \Psi.$$

3. Main results: connection of mild solutions with generalized solutions; specification of spaces where a generalized solution can be found

Definition 3.1. Let A be a linear closed operator in a Hilbert space H and R be a linear bounded operator in H . A strongly continuous with respect to

$t \in [0; \tau)$, $\tau \leq \infty$, family of linear bounded operators $\{S(t), t \in [0; \tau)\}$ in H satisfying the equations

$$(3.1) \quad A \int_0^t S(s)f ds = S(t)f - Rf, \quad t \in [0; \tau), \quad f \in H,$$

$$(3.2) \quad S(t)Af = AS(t)f, \quad t \in [0; \tau), \quad f \in \text{dom } A,$$

is called an R -semigroup generated by A ; the semigroup is *local* if $\tau < \infty$.

Some equivalent definitions and properties of R -semigroups are given in [10]. The property we will use further is that A and R^{-1} commute on $\text{dom } R^{-1}A$.

Theorem 3.2. *Let A generate an R -semigroup $\{S(t), t \in [0; \tau)\}$ in $H = L_m^2(\mathbb{R})$, operators $R^{-1}, R^{-1}A$ act from Ψ' to Ψ' and $T < \tau$. Then in the space Ψ' a solution of (2.12) is a solution to (2.13) with operator $B = B(X)$ under the condition (2.2).*

Proof. Note, that action of operators R^{-1} and $R^{-1}A$ from Ψ' to Ψ' is understood as follows:

$$\langle \psi, R^{-1}Ag \rangle = \langle A^*(R^{-1})^*\psi, g \rangle, \quad \psi \in \Psi, \quad g \in \Psi'.$$

To prove that a solution of (2.12) is a solution to (2.13), first we prove this for the case of $B = 0$. Since a generalized solution of (2.3) has the form (2.4), then $X(t) = R^{-1}S(t)f$ is a generalized solution of (2.3) and hence it is a solution of (2.12) with $B = 0$. Then by the strong continuity of bounded operators $S(t), t \in [0, T]$, we obtain

$$\int_0^t \langle A^*\psi, X(s) \rangle ds = \int_0^t \langle A^*\psi, R^{-1}S(s)f \rangle ds = \langle (R^{-1})^* A^*\psi, \int_0^t S(s)f ds \rangle, \\ \psi \in \Psi, \quad P_{a.s.}$$

Since R^{-1} and A commute on $\text{dom } R^{-1}A$, the operators A^* and $(R^{-1})^*$ commute on Ψ , which we suppose belong to $\text{dom } A \subset H$. This and the equality (3.1) imply

$$\begin{aligned} \langle (R^{-1})^* A^*\psi, \int_0^t S(s)f ds \rangle &= \langle (R^{-1})^* \psi, A \int_0^t S(s)f ds \rangle \\ &= \langle (R^{-1})^* \psi, (S(t)f - Rf) \rangle \\ &= \langle \psi, X(t) \rangle - \langle \psi, f \rangle, \quad t \in [0; T], \quad \psi \in \Psi, \quad P_{a.s.} \end{aligned}$$

for any $f \in H$. Thus we get (2.13) with $B = 0$.

Now we show that the second term in the right-hand side of (2.12) satisfies (2.13) with $f = 0$. Consider

$$\int_0^t \langle A^*\psi, X(s) \rangle ds = \int_0^t \langle (R^{-1})^* A^*\psi, \int_0^s S(s-r)B(X(r))dW(r) \rangle ds, \quad \psi \in \Psi.$$

Again, due to the strong continuity of the R -semigroup of bounded operators $S(t), t \in [0; T]$, and the continuity on Ψ of the functional in the right hand side, we have

$$\int_0^t \langle A^* \psi, X(s) \rangle ds = \langle (R^{-1})^* A^* \psi, \int_0^t \int_0^s S(s-r)B(X(r))dW(r) ds \rangle.$$

Taking into account boundedness of R -semigroup operators and applying the stochastic Fubini theorem [5], we change the order of integration:

$$\begin{aligned} \langle (R^{-1})^* A^* \psi, \int_0^t \int_0^s S(s-r)B(X(r))dW(r) ds \rangle \\ = \langle A^* (R^{-1})^* \psi, \int_0^t \int_r^t S(s-r) ds B(X(r))dW(r) \rangle. \end{aligned}$$

At last, we move the closed operator A under the integral sign and by (3.1) obtain

$$\begin{aligned} \langle A^* (R^{-1})^* \psi, \int_0^t \int_r^t S(s-r) ds B(X(r))dW(r) \rangle \\ = \langle (R^{-1})^* \psi, \int_0^t A \int_0^{t-r} S(h) dh B(X(r))dW(r) \rangle \\ = \langle (R^{-1})^* \psi, \int_0^t S(t-r)B(X(r))dW(r) \rangle - \langle (R^{-1})^* \psi, \int_0^t RB(X(r))dW(r) \rangle \\ = \langle \psi, X(t) \rangle - \langle \psi, \int_0^t B(X(r))dW(r) \rangle, \quad t \in [0; T], \quad \psi \in \Psi, \quad P_{a.s.} . \end{aligned}$$

Thus, generalized Ψ' -valued process $\{X(t), t \in [0; T]\}$ defined by (2.12) is a solution of (2.13).

A solution of (2.12) is called a mild solution, thus we have proved that a mild solution is a generalized one for the problem (2.1). □

Now we extend the notions of multiplier and convolutor from [6, 7] to a couple of spaces.

Definition 3.3. Let multiplication by a real-valued function g be a linear continuous operator from Ψ to Φ , i.e. $g\psi \in \Phi$ for any $\psi \in \Psi$ and $\varphi_n = g\psi_n \rightarrow 0$ in Φ if $\psi_n \rightarrow 0$ in Ψ . Then g is called a *multiplier from Ψ to Φ* . If g is a multiplier from Ψ to Φ , then *product gf* for each $f \in \Phi'$ is defined as follows:

$$\langle \psi, gf \rangle := \langle g\psi, f \rangle, \quad \psi \in \Psi,$$

and g is called a *multiplier from Φ' to Ψ'* .

Definition 3.4. A generalized function $G \in \Psi'$ is called a *convolutor from Ψ to Φ* if

$$(G \bar{*} \psi)(x) := \langle \psi(x + \xi), G(\xi) \rangle = \varphi(x) \in \Phi^3$$

³Note, that when we write an argument to a generalized function (here $G(\xi)$), we mean that the generalized function acts on test functions depending on this argument.

for any $\psi \in \Psi$ and the map $G^* : \Psi \rightarrow \Phi$ is continuous, i.e. $\varphi_n = G^* \psi_n \rightarrow 0$ in Φ if $\psi_n \rightarrow 0$ in Ψ . If G is a convolutor from Ψ to Φ , then *convolution of $G \in \Psi'$ with $f \in \Phi'$* is defined as follows:

$$\langle \psi, G * f \rangle := \langle G^* \psi, f \rangle, \quad \psi \in \Psi,$$

and G is called a *convolutor from Φ' to Ψ'* .

Remark. It follows from the last definition and the definition of convergence in spaces Φ', Ψ' that a convolutor maps Φ' to Ψ' continuously, i.e. $G * f_n \rightarrow 0$ in Ψ' for any sequence $f_n \rightarrow 0$ in Φ' :

$$\langle \psi, G * f_n \rangle := \langle G^* \psi, f_n \rangle = \langle \varphi, f_n \rangle \rightarrow 0, \quad \psi \in \Psi.$$

Remark. In the framework of this paper all mentioned spaces are related as follows:

$$\Psi \subseteq \Phi \subset L_m^2(\mathbb{R}) \subset \Phi' \subseteq \Psi'.$$

In particular, in the stochastic equations we suppose f to be from the Hilbert space $H = L_m^2(\mathbb{R})$, not from a wider space Φ' , because W in (2.13) takes values in another Hilbert space $\mathbb{H} = L_n^2(\mathbb{R})$ and is mapped into $L_m^2(\mathbb{R})$ by $B(\cdot)$; hence the Ito integral in (2.12) takes place in $L_m^2(\mathbb{R})$.

Theorem 3.5. *Let Φ, Ψ be spaces of test functions with continuous shift and $\tilde{\Phi}, \tilde{\Psi}$ be dual spaces of Φ, Ψ with respect to the Fourier transform. If g is a multiplier from $\tilde{\Phi}'$ to $\tilde{\Psi}'$, then $G = \mathcal{F}^{-1}[g]$ is a convolutor from Φ' to Ψ' and*

$$\mathcal{F}[G * f] = \mathcal{F}[G] \cdot \mathcal{F}[f], \quad f \in \Phi'.$$

Proof. Note first that

$$\begin{aligned} \mathcal{F}[\psi(x + \xi)](\sigma) &= \int_{\mathbb{R}} e^{i(\xi, \sigma)} \psi(x + \xi) d\xi \\ &= \int_{\mathbb{R}} e^{i(y, \sigma)} e^{-i(x, \sigma)} \psi(y) dy = e^{-i(x, \sigma)} \tilde{\psi}(\sigma), \quad \sigma \in \mathbb{R}, \end{aligned}$$

for any ψ from Ψ or Φ . It follows that $e^{-i(x, \sigma)}$ is a multiplier in $\tilde{\Psi}$ and in $\tilde{\Phi}$.

Now, by the definition of convolution and that of generalized Fourier transform, we have

$$(G^* \psi)(x) = \langle \psi(x + \xi), G(\xi) \rangle = \frac{1}{2\pi} \langle e^{-i(x, \sigma)} \tilde{\psi}(\sigma), g(\sigma) \rangle.$$

The definition of a multiplier implies

$$\frac{1}{2\pi} \langle e^{-i(x, \sigma)} \tilde{\psi}(\sigma), g(\sigma) \rangle = \frac{1}{2\pi} \langle g(\sigma) e^{-i(x, \sigma)} \tilde{\psi}(\sigma), 1 \rangle = \frac{1}{2\pi} \langle e^{-i(x, \sigma)} \tilde{\varphi}(\sigma), 1 \rangle,$$

where $\tilde{\varphi} := g\tilde{\psi} \in \tilde{\Phi}$. It follows that

$$(G^* \psi)(x) = \frac{1}{2\pi} \langle e^{-i(x, \sigma)} \tilde{\varphi}(\sigma), 1 \rangle = \langle \varphi(x + \xi), \delta(\xi) \rangle = \varphi(x) \in \Phi,$$

thus G^* maps Ψ to Φ . The continuity of the Fourier transform and of the multiplication imply:

$$\begin{aligned} \psi_n \rightarrow 0 \text{ in } \Psi &\Rightarrow \widetilde{\psi}_n \rightarrow 0 \text{ in } \widetilde{\Psi} &\Rightarrow g\widetilde{\psi}_n \rightarrow 0 \text{ in } \widetilde{\Phi} \\ &&\Rightarrow \mathcal{F}^{-1}[g\widetilde{\psi}_n] = G^*\psi_n \rightarrow 0 \text{ in } \Phi, \end{aligned}$$

hence, G^* is continuous from Ψ to Φ , that is G is a convolutor from Φ to Ψ and, consequently, from Φ' to Ψ' .

Finally, consider $\mathcal{F}[G * f]$. For any $\psi \in \Psi$ and $f \in \Phi'$ we have

$$\begin{aligned} \langle \widetilde{\psi}, \mathcal{F}[G * f] \rangle &= 2\pi \langle \psi, G * f \rangle = 2\pi \langle G^*\psi, f \rangle \\ &= \langle \mathcal{F}[G^*\psi], \mathcal{F}[f] \rangle = \langle g\widetilde{\psi}, \mathcal{F}[f] \rangle = \langle \widetilde{\psi}, g\mathcal{F}[f] \rangle. \end{aligned}$$

This implies $\mathcal{F}[G * f] = \mathcal{F}[G] \cdot \mathcal{F}[f]$. □

Now following [6] we define some spaces of test functions, which we need below.

1. Denote by $\mathcal{S}_{\alpha, A}$ ($\alpha > 0, A > 0$) the space of all infinitely differentiable functions satisfying for any $\varepsilon > 0$ the condition

$$|x^k \varphi^{(q)}(x)| \leq C_{q, \varepsilon} (A + \varepsilon)^k k^{k\alpha}, \quad k, q \in \mathbb{N}_0, \quad x \in \mathbb{R},$$

with some constant $C_{q, \varepsilon} = C_{q, \varepsilon}(\varphi)$. This space is a perfect countably normed space with the system of norms

$$(3.3) \quad \|\varphi\|_{q, p} = \sup_{k \in \mathbb{N}_0} \sup_{x \in \mathbb{R}} \frac{|x^k \varphi^{(q)}(x)|}{\left(A + \frac{1}{p}\right)^k k^{k\alpha}}, \quad p \in \mathbb{N}, \quad q \in \mathbb{N}_0.$$

A set \mathcal{G} is bounded in $\mathcal{S}_{\alpha, A}$ with system of norms (3.3) if for any $p \in \mathbb{N}, q \in \mathbb{N}_0$ there exists a constant $C_{q, p}$ independent on elements of \mathcal{G} and such that $\|\varphi\|_{q, p} \leq C_{q, p}$ for all $\varphi \in \mathcal{G}$.

Convergence to zero of a sequence $\varphi_n \in \mathcal{S}_{\alpha, A}$ means that it is bounded in the space and for any $q \in \mathbb{N}_0$ functions $\varphi_n^{(q)}(\cdot)$ converge to zero uniformly on any segment $|x| \leq x_0 < \infty$.

The space $\mathcal{S}_{\alpha, A}$ can be equivalently defined as the set of all infinitely differentiable functions satisfying the condition

$$|\varphi^{(q)}(x)| \leq C'_{q, \rho} e^{-(a-\rho)|x|^{1/\alpha}}, \quad q \in \mathbb{N}_0, \quad x \in \mathbb{R},$$

for any $\rho > 0$, with $C'_{q, \rho} = C'_{q, \rho}(\varphi), a = \frac{\alpha}{e A^{1/\alpha}}$.

It follows from the definition of spaces $\mathcal{S}_{\alpha, A}$ that $\mathcal{S}_{\alpha, A_1} \subset \mathcal{S}_{\alpha, A_2}$ if $A_1 < A_2$. Besides, any sequence convergent in $\mathcal{S}_{\alpha, A_1}$ converges in $\mathcal{S}_{\alpha, A_2}$ as well. Moreover, $\mathcal{S}_{\alpha, A_1}$ is a subspace of $\mathcal{S}_{\alpha, A_2}$.

2. Denote by $\mathcal{S}^{\beta, B}$ ($\beta > 0, B > 0$) the space of all infinitely differentiable functions satisfying for any $\delta > 0$ the condition

$$(3.4) \quad |x^k \varphi^{(q)}(x)| \leq C_{k, \delta} (B + \delta)^q q^{q\beta}, \quad k, q \in \mathbb{N}_0, \quad x \in \mathbb{R},$$

with some constant $C_{k,\delta} = C_{k,\delta}(\varphi)$. This space is a perfect countably normed space with the system of norms

$$(3.5) \quad \|\varphi\|_{k,m} = \sup_{q \in \mathbb{N}_0} \sup_{x \in \mathbb{R}} \frac{|x^k \varphi^{(q)}(x)|}{\left(\mathbb{B} + \frac{1}{m}\right)^q q^{q\beta}}, \quad k \in \mathbb{N}_0, \quad m \in \mathbb{N}.$$

A set \mathcal{G} is bounded in $\mathcal{S}^{\beta,\mathbb{B}}$ if for any $k \in \mathbb{N}_0, m \in \mathbb{N}$ there exists a constant $C_{k,m}$ independent on elements of \mathcal{G} and such that $\|\varphi\|_{k,m} \leq C_{k,m}$ for all $\varphi \in \mathcal{G}$.

Convergence to zero of a sequence $\varphi_n \in \mathcal{S}^{\beta,\mathbb{B}}$ means that it is bounded in the space and for any $q \in \mathbb{N}_0$ the sequences of functions $\varphi_n^{(q)}(\cdot)$ converge to zero uniformly on any segment $|x| \leq x_0 < \infty$.

It follows from the definition of space $\mathcal{S}^{\beta,\mathbb{B}}$ that for $\mathbb{B}_1 < \mathbb{B}_2$ the inclusion $\mathcal{S}^{\beta,\mathbb{B}_1} \subset \mathcal{S}^{\beta,\mathbb{B}_2}$ takes place, in addition, any convergent in $\mathcal{S}^{\beta,\mathbb{B}_1}$ sequence converges in $\mathcal{S}^{\beta,\mathbb{B}_2}$ and $\mathcal{S}^{\beta,\mathbb{B}_1}$ is a subspace of $\mathcal{S}^{\beta,\mathbb{B}_2}$.

3. The space \mathcal{S} consists of all infinitely differentiable functions satisfying the condition

$$(3.6) \quad |x^k \varphi^{(q)}(x)| \leq C_{k,q}, \quad k, q \in \mathbb{N}_0, \quad x \in \mathbb{R},$$

with some constant $C_{k,q} = C_{k,q}(\varphi)$. Thus, the space \mathcal{S} is the space of infinitely differentiable functions decreasing faster than any degree of $\frac{1}{|x|}$ does as $|x| \rightarrow \infty$. This space is a perfect countably normed space with the system of norms

$$\|\varphi\|_p = \sup_{k,q \leq p} \sup_{x \in \mathbb{R}} |x^k \varphi^{(q)}(x)|, \quad p \in \mathbb{N}.$$

A set \mathcal{G} is bounded in \mathcal{S} if for any $p \in \mathbb{N}_0$ there exists a constant C_p independent on the elements of \mathcal{G} and such that $\|\varphi\|_p \leq C_p$ for all $\varphi \in \mathcal{G}$.

Convergence to zero of a sequence $\varphi_n \in \mathcal{S}$ means that it is bounded in \mathcal{S} and for any $q \in \mathbb{N}_0$ the sequence of functions $\varphi_n^{(q)}(\cdot)$ converges to zero uniformly on any segment $|x| \leq x_0 < \infty$.

The space \mathcal{S} is the widest among all spaces considered above: the existence of constants $C_{k,q}(\varphi)$ defining the behavior of functions from \mathcal{S} is sufficient here and it does not matter what type they are.

If we denote by $\widetilde{\Phi}$ the space consisting of classical Fourier transforms of functions from Φ , then for any parameters $\alpha, \mathbb{A}, \beta, \mathbb{B}$

$$\widetilde{\mathcal{S}}_{\alpha,\mathbb{A}} = \mathcal{S}^{\alpha,\mathbb{A}}, \quad \widetilde{\mathcal{S}}^{\beta,\mathbb{B}} = \mathcal{S}_{\beta,\mathbb{B}}, \quad \widetilde{\mathcal{S}} = \mathcal{S}.$$

Using these relations we obtain the following result.

Theorem 3.6. Let $A(i \frac{\partial}{\partial x})$ define a differential system (2.3) of a class satisfying (2.6), (2.7) or (2.8). Then X defined by (2.12) is a solution to (2.13) in the case of

- 1) Petrovskii correct systems, in the space $\Psi' = \mathcal{S}'$;
- 2) conditionally correct systems, in $\Psi' = (\mathcal{S}^{\alpha,\mathbb{A}})'$, where $\alpha = \frac{1}{h}$, $\mathbb{A} = \frac{1}{(h e (a_0 - a + b))^{1/h}}$ for arbitrary $a_0 > a$ and $b > a$;

3) incorrect systems, in $\Psi' = (\mathcal{S}^{\alpha, A})'$, where $\alpha = \frac{1}{p_0}$ and $A = \frac{1}{(p_0 e^{(a_0 - a + b)})^{1/p_0}}$ for arbitrary $a_0 > a$ and $b > a$.

Proof. Let us explain the main idea of the proof. We consider $e^{tA(\sigma)}$, $\sigma \in \mathbb{R}$, and on the base of the estimates (2.6), (2.7) or (2.8) point out spaces $\tilde{\Psi}_1$ and $\tilde{\Phi}_1$, where the exponent defines a multiplier from $\tilde{\Psi}_1$ to $\tilde{\Phi}_1$. Then A generates an R -semigroup $\{S(t), t \in [0; T]\}$ in H defined by (2.9) with $Rf = K * f$, where the choice of $\tilde{K}(\sigma)$, $\sigma \in \mathbb{R}$, provides that the matrix-function $e^{tA(\sigma)} \tilde{K}(\sigma)$ is bounded. We show that for each class of the system (2.3), $\tilde{K}(\sigma)$ can be taken so that it defines a multiplier from $\tilde{\Phi}_1$ to a space $\tilde{\Psi}_2$. Then by Theorem 3.5, K is a convolutor from Ψ'_2 to Φ'_1 , hence the operator R is bounded in these spaces. Consider the inverse operator R^{-1} . Since

$$z(x) = (Rf)(x) = (K * f)(x) = \mathcal{F}^{-1}[\tilde{K}(\sigma)\tilde{f}(\sigma)] = \mathcal{F}^{-1}[\tilde{z}(\sigma)],$$

we get

$$(R^{-1}z)(x) = f(x) = \mathcal{F}^{-1}[\tilde{f}(\sigma)] = \mathcal{F}^{-1} \left[\frac{\tilde{z}(\sigma)}{\tilde{K}(\sigma)} \right]$$

and

$$(AR^{-1}z)(x) = Af(x) = \mathcal{F}^{-1}[A(\sigma)\tilde{f}(\sigma)] = \mathcal{F}^{-1} \left[A(\sigma) \frac{\tilde{z}(\sigma)}{\tilde{K}(\sigma)} \right].$$

We specify the space Φ_2 providing that $\tilde{K}^{-1}(\sigma)$ and $A(\sigma)\tilde{K}^{-1}(\sigma)$ are multipliers from $\tilde{\Psi}_2$ to $\tilde{\Phi}_2$. Then by Theorem 3.5, operators R^{-1} and AR^{-1} become bounded from Φ'_2 to Ψ'_2 . Thus, the space $\Psi' = \Psi'_2$ is the required one and Theorem 3.2 completes the proof in this space.

Now we realize this idea for each class of the systems.

- For Petrovskii correct systems estimate (2.6) implies that $e^{tA(\sigma)}$ is a multiplier from $\Psi = \mathcal{S}$ to \mathcal{S} . Then we can choose $\tilde{K}(\cdot)$ smooth and under the condition

$$\tilde{K}(\sigma) = \mathcal{O} \left(\frac{1}{(1 + |\sigma|)^{h_1}} \right) \text{ as } |\sigma| \rightarrow \infty, \quad \text{where } h_1 > h + 1/2 + p_0.$$

This $\tilde{K}(\cdot)$ is a multiplier in \mathcal{S} and operators R, R^{-1}, AR^{-1} are bounded in \mathcal{S}' ; the boundedness of R^{-1} and AR^{-1} is a consequence of the closedness of the space \mathcal{S} under multiplication by any power of x .

- For conditionally correct systems the estimate (2.7) implies that for any $a_0 > a$ the matrix-function $e^{tA(\sigma)}$ is a multiplier in the spaces [7]

$$\tilde{\Psi}_1 = \mathcal{S}_{\alpha, A_0} \rightarrow \tilde{\Phi}_1 = \mathcal{S}_{\alpha, A_1},$$

where

$$\alpha = \frac{1}{h}, \quad A_0 = \frac{1}{(hea_0)^{1/h}}, \quad A_1 = \frac{1}{(he(a_0 - a))^{1/h}}, \quad A_0 < A_1 \text{ [7].}$$

We choose $\tilde{K}(\cdot)$ smooth and under the condition

$$\tilde{K}(\sigma) = \mathcal{O}\left(e^{-b|\sigma|^h}\right) \text{ as } |\sigma| \rightarrow \infty, \quad \text{where } b > a,$$

that provides existence of R -semigroup in H . This $\tilde{K}(\cdot)$ is a multiplier in the spaces

$$\tilde{\Phi}_1 \rightarrow \tilde{\Psi}_2 = \mathcal{S}_{\alpha, A_2}, \quad \text{where } \alpha = \frac{1}{h}, \quad A_2 = \frac{1}{(he(a_0 - a + b))^{1/h}}, \quad A_2 < A_0.$$

Since $\tilde{K}^{-1}(\sigma) = \mathcal{O}\left(e^{b|\sigma|^h}\right)$ and $A(\sigma)\tilde{K}^{-1}(\sigma) = o\left(e^{b_1|\sigma|^h}\right)$ as $\sigma \rightarrow \infty$ for arbitrary $b_1 \in (b; a_0 - a + b)$, they are multipliers in spaces

$$\tilde{\Psi}_2 = \mathcal{S}_{\alpha, A_2} \rightarrow \tilde{\Phi}_2 = \mathcal{S}_{\alpha, A_3},$$

where

$$\alpha = \frac{1}{h}, \quad A_3 = \frac{1}{(he(a_0 - a + b - b_1))^{1/h}} > A_1.$$

Since $A_2 < A_0 < A_1 < A_3$ all the mentioned spaces are connected as follows:

$$\tilde{\Psi}_2 \subset \tilde{\Psi}_1 \subset \tilde{\Phi}_1 \subset \tilde{\Phi}_2.$$

By Theorem 3.5, operators R^{-1} and AR^{-1} are bounded in spaces

$$\Phi'_2 = (\mathcal{S}^{\alpha, A_3})' \rightarrow \Psi'_2 = (\mathcal{S}^{\alpha, A_2})',$$

thus the required space is $\Psi' = \Psi'_2 = (\mathcal{S}^{\alpha, A_2})'$.

• For incorrect systems the estimate (2.8) implies the analogous results, so, the required space is $\Psi' = (\mathcal{S}^{\alpha, A})'$, where

$$\alpha = \frac{1}{p_0}, \quad A = \frac{1}{(p_0 e(a_0 - a + b))^{1/p_0}},$$

for arbitrary $a_0 > a$ and $b > a$. □

In conclusion, we give two examples of operators $B(X)$ satisfying the conditions of Theorem 1, that is $B(X) : \mathbb{H}_Q \rightarrow H$ under the condition (2.2), where X is a generalized solution to (2.13). To explain the main ideas of constructing the examples, we present them in one-dimensional spaces.

Let Q be a trace class operator in a separable Hilbert space \mathbb{H} and $\{e_i\}$ be a basis of its eigenvectors in \mathbb{H} : $Qe_i = \sigma_i e_i$, where $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$. For example, in the case of $\mathbb{H} := L^2(\mathbb{R})$, as $\{e_i\}$ may be taken Hermite functions; they are eigenfunctions of the trace class operator $Q = \hat{D}^{-1}$, $\hat{D} := -\frac{d^2}{dx^2} + x^2 + 1$, in $L^2(\mathbb{R})$ corresponding to eigenvalues $\sigma_i = \frac{1}{2^i}$. Let $\{g_i = \sigma_i e_i\}$ and $\{f_j\}$ be orthogonal bases in spaces \mathbb{H}_Q and H , respectively.

For a generalized solution X defined via a (generalized) Fourier series

$$(3.7) \quad \langle \psi, X \rangle = \langle \psi, \sum_j X_j f_j \rangle, \quad \psi \in \Psi,$$

we define the operator $B(X) : \mathbb{H}_Q \rightarrow H$ (generally, depending on X nonlinearly) via a Fourier series too:

$$B(X)g_i = \sum_j b_{ij}(X)f_j, \quad f_j \in H.$$

Since X is a generalized solution, the series (3.7) may be divergent in H , that is $\sum_j X_j^2 = \infty$, and is convergent in Ψ' . Suppose that X is such that $\sum_j X_j^2 \frac{1}{j^{2p}} < \infty$ for some $p \in \mathbb{N}$, and let the elements of the matrix b_{ij} , which define the operator $B(X)$, be as follows:

$$b_{ij}(X) := \sigma_i \frac{\sin X_j}{j^p}.$$

It is not difficult to show that the condition $\sum_i \sigma_i^2 < \infty$ is sufficient for $B(X)$ to be a Hilbert–Schmidt operator from \mathbb{H}_Q to H :

$$\begin{aligned} \|B(X)\|_{\text{HS}}^2 &= \sum_i \|B(X)g_i\|_H^2 = \sum_i \sum_j b_{ij}^2(X) \\ &= \sum_{ij} \sigma_i^2 \frac{\sin^2 X_j}{j^{2p}} \leq C \sum_i \sigma_i^2 \sum_j \frac{X_j^2}{j^{2p}} < \infty. \end{aligned}$$

It follows that in this case Ψ can be taken in such a way that the series (3.7) for X is convergent, that is $\psi = \sum_i \psi_i e_i \in \Psi$ are taken under the condition $\sum_j \psi_j^2 j^{2p} < \infty$.

Now we give an example constructed using the same idea and related to the Fourier transform techniques used in the paper. Let $\mathbb{H} = H = L^2(\mathbb{R})$ and $\tilde{X}(t, \cdot)$, $t \in [0; T]$, be the generalized Fourier transform of X . Suppose that for any $t \in [0; T]$, $\tilde{X}(t, \cdot)$ is a regular generalized function. Define $B(X)$ as the multiplication by the function

$$b_X(x) = \int_{-\infty}^{\infty} e^{-i\sigma x} v(\sigma) s(\tilde{X}(t, \sigma)) d\sigma.$$

Here, in order to be $B(X)$ a Hilbert–Schmidt operator from \mathbb{H}_Q to H , it is sufficient for $B(X)$ to be bounded from \mathbb{H} to H . Indeed,

$$\|B(X)\|_{\text{HS}}^2 = \sum_i \|B(X)g_i\|_H^2 \leq \sum_i C \|g_i\|_{\mathbb{H}}^2 = C \sum_i \|\sigma_i e_i\|_{\mathbb{H}}^2 = C \sum_i \sigma_i^2 < \infty.$$

The introduced $B(X)$ is bounded if the function $b_X(\cdot)$ is bounded; that in its turn is provided by the condition $v(\cdot) s(\tilde{X}(t, \cdot)) \in L^1(\mathbb{R})$.

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