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DISTRIBUTIONALLY CHAOTIC PROPERTIES OF ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In the paper under review, we analyze a class of abstract distributionally chaotic (multi-term) fractional differential equations in Banach spaces, associated with use of the Caputo fractional derivatives.

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1. Introduction and preliminaries

Distributional chaos is still an active field of research in the theory of hypercyclicity. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X is said to be distributionally chaotic iff there are an uncountable set $S \subseteq X$ and a number $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that

$$\overline{Dens}\big(\{s \geq 0 : ||T(s)x - T(s)y|| \geq \sigma\}\big) = 1 \text{ and}$$
$$\overline{Dens}\big(\{s \geq 0 : ||T(s)x - T(s)y|| < \epsilon\}\big) = 1,$$

where the upper density of a set $D \subseteq [0,\infty)$ is defined by

$$\overline{Dens}(D) := \limsup_{t\to+\infty} \frac{m(D\cap[0,t])}{t},$$

with $m(\cdot)$ being the Lebesgue's measure on $[0,\infty)$. If, moreover, we can choose S to be dense in X, then $(T(t))_{t\geq 0}$ is said to be densely distributionally chaotic. As it is well known, the question whether $(T(t))_{t\geq 0}$ is distributionally chaotic or not is closely connected with the existence of distributionally irregular vectors of $(T(t))_{t\geq 0}$, i.e., those elements $x \in X$ such that for each $\sigma > 0$ we have that

$$\overline{Dens}\big(\{s \geq 0 : ||T(s)x|| > \sigma\}\big) = 1 \text{ and } \overline{Dens}\big(\{s \geq 0 : ||T(s)x|| < \sigma\}\big) = 1.$$

Fairly complete information on distributionally chaotic strongly continuous semigroups in Banach spaces can be obtained by consulting [1], [5]-[8] and

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[14]; concerning Li-Yorke chaotic properties of translation semigroups, mention should be made of [36]. Compared with the above, Devaney chaoticity of strongly continuous semigroups in Banach spaces has received much more attention so far; cf. [3]-[4], [12]-[14], [17]-[20], [23]-[25], [35], [28, Chapter 3] and references cited there.

In a joint paper with J. A. Conejero, P. Miana and M. Murillo-Arcila [15], the author has recently analyzed distributionally chaotic properties of unbounded linear operators, strongly continuous semigroups and fractionally integrated C-semigroups in the setting of infinite-dimensional Fréchet spaces; cf. also [10]. The main aim of this paper is to consider the basic distributionally chaotic properties of abstract (multi-term) fractional differential equations. Although our results can be formulated in the setting of infinite-dimensional Fréchet spaces, we shall work in Banach spaces for the sake of simplicity and better exposition.

We assume that X is an infinite-dimensional complex Banach space; the norm of an element $x \in X$ is denoted by $\|x\|$, and the space consisting of all continuous linear mappings from X into X is denoted by $L(X)$. We usually denote a closed linear operator acting on X by A; unless stated otherwise, we shall always assume henceforth that $C \in L(X)$ is an injective operator satisfying $CA \subseteq AC$. The domain, range, kernel space and point spectrum of A are denoted by $D(A)$, $R(A)$, $Kern(A)$ and $\sigma_p(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \Big\{ \lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(X) \Big\}.$$

Suppose now that F is a linear subspace of X. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$.

Given $s \in \mathbb{R}$ in advance, set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The convolution-like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$, $0^\zeta := 0$, $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ ($\zeta > 0$, $t > 0$), $\mathbb{N}_l := \{1, \ldots, l\}$, $\mathbb{N}_l^0 := \{0, 1, \ldots, l\}$ ($l \in \mathbb{N}$) and $g_0(t) :=$ the Dirac δ-distribution. The Laplace transform and its inverse transform will be denoted by $\mathcal{L}$ and $\mathcal{L}^{-1}$, respectively. For more details about various applications of vector-valued Laplace transform, see [2] and [28]. In this paper we shall consider only non-degenerate operator families.

Assume $\alpha > 0$, $m = \lceil \alpha \rceil$ and $\beta > 0$. Recall that the Caputo fractional derivative $\mathbf{D}_t^\alpha u$ ([9], [28]) is defined for those functions $u \in C^{m-1}([0,\infty) : X)$ for which $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m([0,\infty) : X)$; if this is the case, then we have

$$\mathbf{D}_t^\alpha u(t) = \frac{d^m}{dt^m}\left[g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1} \right) \right].$$

Denote by $E_{\alpha,\beta}(z)$ the Mittag-Leffler function $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n + \beta)$, $z \in \mathbb{C}$ ([9]). Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. We shall use the following asymptotic formulae ([9], [28]): If $0 < \alpha < 2$ and $\beta > 0$, then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \varepsilon_{\alpha,\beta}(z),\ |\arg(z)| < \alpha\pi/2, \tag{1.1}$$

and

$$E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z),\ |\arg(-z)| < \pi - \alpha\pi/2, \tag{1.2}$$

where

$$\varepsilon_{\alpha,\beta}(z) = \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O\big(|z|^{-N}\big),\ |z| \to \infty. \tag{1.3}$$

2. Distributionally chaotic properties of certain classes of abstract (multi-term) fractional differential equations

In the first part of this section, we consider distributionally chaotic properties of solutions of the following abstract evolution equation:

$$\mathbf{D}_t^{\alpha} u(t) = Au(t),\ t \geq 0;\ u(0) = x,\ u^{(k)}(0) = 0,\ k = 1, \ldots, m-1, \tag{2.1}$$

where $\alpha \in (0,2) \setminus \{1\}$ and $m = \lceil \alpha \rceil$. A function $u \in C^{m-1}([0,\infty) : E)$ is said to be a (strong) solution of (2.1) if $Au \in C([0,\infty) : E)$, $\int_0^{\cdot} \frac{(\cdot - s)^{m-\alpha-1}}{\Gamma(m-\alpha)} \big[u(s) - x\big]\, ds \in C^m([0,\infty) : E)$ and (2.1) holds. If so, then we can apply the equality [9, (1.21)] in order to see that the function $t \mapsto u(t)$, $t \geq 0$ is a strong solution of the associated Volterra integral equation:

$$u(t) = x + \int_0^t g_\alpha(t-s) Au(s)\, ds,\ t \geq 0; \tag{2.2}$$

cf. [28, Subsection 2.1.1, p. 42] for the notion of a mild (strong, weak) solution of the abstract Volterra equation:

$$u(t) = f(t) + \int_0^t a(t-s) Au(s)\, ds,\ t \geq 0,$$

where $f \in C([0,\infty) : E)$. The second part of section is devoted to the study of distributionally chaotic properties of solutions of the following multi-term problem:

$$\begin{aligned} &\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} c_i \mathbf{D}_t^{\alpha_i} u(t) = A\mathbf{D}_t^{\alpha} u(t), \quad t \geq 0, \\ &u^{(k)}(0) = u_k, \quad k = 0, \ldots, \lceil \alpha_n \rceil - 1, \end{aligned} \tag{2.3}$$

where $n \in \mathbb{N} \setminus \{1\}$, A is a closed linear operator on X, $c_i \in \mathbb{C}$ $(1 \leq i \leq n-1)$, $0 \leq \alpha_1 < \cdots < \alpha_n$ and $0 \leq \alpha < \alpha_n$ ([30]).

We refer the reader to [34] and [28] for some applications of various classes of (a,k)-regularized C-resolvent families in the theory of abstract Volterra integro-differential equations; the basic information on hypercyclic and topologically mixing properties of abstract Volterra integro-differential equations and abstract time-fractional equations can be obtained by consulting [28, Chapter 3]. The following definition, being sufficient for our purposes, is a very special case of the general definition of an (a,k)-regularized C-resolvent family in a sequentially complete locally convex space [29, Definition 2.1]; cf. [16] and [27] for more details concerning the case $\alpha = 1$.

Definition 2.1. Let $\alpha > 0$, and let A be a closed linear operator on X. A strongly continuous operator family $(R_\alpha(t))_{t\geq 0}$ is called an α-times C-regularized resolvent family having A as a subgenerator iff the following holds:

(i) $R_\alpha(t)A \subseteq AR_\alpha(t),\ t \geq 0,\ R_\alpha(0) = C$ and $CA \subseteq AC$,

(ii) $R_\alpha(t)C = CR_\alpha(t),\ t \geq 0$ and

(iii) $R_\alpha(t)x = Cx + \int_0^t g_\alpha(t-s)AR_\alpha(s)x\,ds,\ t \geq 0,\ x \in D(A)$;

$(R_\alpha(t))_{t\geq 0}$ is said to be exponentially bounded iff there exist $M \geq 1$ and $\omega \geq 0$ such that $\|R_\alpha(t)\| \leq Me^{\omega t}$, $t \geq 0$. In the case $C = I$, it is also said that $(R_\alpha(t))_{t\geq 0}$ is an α-times regularized resolvent family with subgenerator A.

The integral generator of $(R_\alpha(t))_{t\geq 0}$ is defined by

$$\hat{A} := \left\{(x,y) \in X \times X : R_\alpha(t)x - Cx = \int_0^t g_\alpha(t-s)R_\alpha(s)y\,ds \text{ for all } t \geq 0\right\},$$

and it is a closed linear operator which is an extension of any subgenerator of $(R_\alpha(t))_{t\geq 0}$. In the sequel, we assume that A is a densely defined subgenerator of an α-times C-regularized resolvent family $(R_\alpha(t))_{t\geq 0}$. Then the following equality holds

$$R_\alpha(t)x = Cx + A\int_0^t g_\alpha(t-s)R_\alpha(s)x\,ds,\ t \geq 0,\ x \in X.$$

We define the solution space $Z_\alpha(A)$ as the set which consists of those vectors $x \in X$ such that $R_\alpha(t)x \in R(C)$, $t \geq 0$ and the mapping $t \mapsto C^{-1}R_\alpha(t)x$, $t \geq 0$ is continuous. Then $R(C) \subseteq Z_\alpha(A)$, and $x \in Z_\alpha(A)$ iff there exists a unique mild solution of the abstract Volterra equation (2.2); if this is the case, the solution is given by $u(t) = C^{-1}R_\alpha(t)x$, $t \geq 0$ (on l. 7, p. 410 of [28], we have made an obvious mistake by stating that the function $u(t) = C^{-1}R_\alpha(t)x$, $t \geq 0$ is a strong solution of (2.1)).

Definition 2.2. Let $\alpha > 0$, let A be a densely defined subgenerator of an α-times C-regularized resolvent family $(R_\alpha(t))_{t\geq 0}$, and let $\tilde{X}$ be a closed linear subspace of X. Then it is said that $(R_\alpha(t))_{t\geq 0}$ is $\tilde{X}$-distributionally chaotic iff there are an uncountable set $S \subseteq Z_\alpha(A) \cap \tilde{X}$ and a number $\sigma > 0$ such that for each $\epsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that

$$\overline{Dens}\Big(\big\{s \geq 0 : \big\|C^{-1}R_\alpha(s)x - C^{-1}R_\alpha(s)y\big\| \geq \sigma\big\}\Big) = 1 \text{ and}$$
$$\overline{Dens}\Big(\big\{s \geq 0 : \big\|C^{-1}R_\alpha(s)x - C^{-1}R_\alpha(s)y\big\| < \epsilon\big\}\Big) = 1.$$

If, moreover, S can be chosen to be dense in $\tilde{X}$, then $(R_\alpha(t))_{t\geq 0}$ is said to be densely $\tilde{X}$-distributionally chaotic. In the case that $\tilde{X} = X$, then it is also said that $(R_\alpha(t))_{t\geq 0}$ is (densely) distributionally chaotic.

For the sequel, observe the following fact: If $r > 0$, $\theta \in (-\pi, \pi] \setminus [-\pi/2, \pi/2]$, $\lambda = re^{i\theta}$ and $\alpha \in (0, 2)$, then

$$\arg\big(-\lambda^\alpha t^\alpha\big) = \begin{cases} \theta\alpha - \pi, & \pi/2 < \theta \leq \pi,\ t > 0, \\ \theta\alpha + \pi, & -\pi < \theta < (-\pi)/2,\ t > 0. \end{cases} \tag{2.4}$$

Set

$$\Omega^1_{0,-} := \Big\{\lambda = re^{i\theta} \in \mathbb{C} \setminus \{0\} : \pi/2 < \theta \leq \pi, \text{ and } \theta < \pi/\alpha$$
$$\text{or } \theta \geq \pi/\alpha \text{ and } \theta < (2\pi/\alpha) - (\pi/2)\Big\},$$

$$\Omega^2_{0,-} := \Big\{\lambda = re^{i\theta} \in \mathbb{C} \setminus \{0\} : -\pi < \theta < (-\pi)/2, \text{ and } \theta > (-\pi/\alpha)$$
$$\text{or } \theta \leq (-\pi)/\alpha \text{ and } \theta > (\pi/2) - (2\pi/\alpha)\Big\},$$

and $\Omega_{0,-} := \Omega^1_{0,-} \cup \Omega^2_{0,-}$. By (2.4), we easily infer that

$$\big|\arg\big(-\lambda^\alpha t^\alpha\big)\big| < \pi - \pi\alpha/2, \quad \lambda \in \Omega_{0,-},$$

and by the asymptotic formulae (1.2)-(1.3), we get that

$$E_\alpha\big(\lambda^\alpha t^\alpha\big) \to 0, \quad t \to \infty,\ \lambda \in \Omega_{0,-}. \tag{2.5}$$

Keeping in mind [15, Theorem 4.1], (2.5) and the asymptotic formula (1.1), it is quite easy to extend the assertion of [15, Corollary 5.7] to fractional resolvent families; for more details, cf. the proofs of above-mentioned corollary and [31, Theorem 2.3], as well as [31, Remark 1(iv)]. Notice only that we cannot expect a certain distributionally chaotic behaviour of the operator $C^{-1}R_\alpha(t_0)$ if $\alpha \neq 1$ $(t_0 > 0)$, and that the notion of distributional chaos is meaningful for fractional differential equation (2.1) of arbitrary order $\alpha \in (0, 2) \setminus \{1\}$, in contrast with the notion of chaos in the sense of Devaney ([31]):

Theorem 2.3. *Let $\alpha \in (0,2) \setminus \{1\}$, and let A be densely defined.*

(i) Suppose that A subgenerates a global α-times C-regularized resolvent family $(R_\alpha(t))_{t\geq 0}$ on a separable space X. Let the following conditions hold:

(a) There exists a dense subset X_0' of X such that $\lim_{t\to\infty} R_\alpha(t)x = 0$, $x \in X_0'$.

(b) There exists $x \in X$ such that $\lim_{t\to\infty} \|R_\alpha(t)x\| = \infty$.

Then $(R_\alpha(t))_{t\geq 0}$ is distributionally chaotic. If, moreover, $R(C)$ is dense in X, then $(R_\alpha(t))_{t\geq 0}$ is densely distributionally chaotic.

(ii) Suppose that A subgenerates a global α-times C-regularized resolvent family $(R_\alpha(t))_{t\geq 0}$ on a separable space X, and there exists an open connected subset Ω of $\mathbb{C}$ which satisfies $\Omega \cap (-\infty, 0] = \emptyset$, $\Omega^\alpha := \{\lambda^\alpha : \lambda \in \Omega\} \subseteq \sigma_p(A)$ and $\Omega \subseteq \Omega_{0,-}$. Let $f : \Omega^\alpha \to E$ be an analytic mapping such that $f(\lambda^\alpha) \in Kern(A - \lambda^\alpha) \setminus \{0\}$, $\lambda \in \Omega$, and let $\tilde{X} := \overline{span\{f(\lambda^\alpha) : \lambda \in \Omega\}}$. If there exists $\lambda_0 \in \mathbb{C}_+$ such that $(\lambda_0)^\alpha \in \sigma_p(A)$ and $\tilde{X} = X$, then the conclusions of part (i) continue to hold.

(iii) Suppose that A subgenerates a global α-times C-regularized resolvent family $(R_\alpha(t))_{t\geq 0}$ on a separable space X, and there exists an open connected subset Ω of $\mathbb{C}$ which satisfies $\Omega \cap (-\infty, 0] = \emptyset$, $\Omega^\alpha := \{\lambda^\alpha : \lambda \in \Omega\} \subseteq \sigma_p(A)$ and $\Omega \subseteq \Omega_{0,-}$. Let $f : \Omega^\alpha \to E$ be an analytic mapping such that $f(\lambda^\alpha) \in Kern(A - \lambda^\alpha) \setminus \{0\}$, $\lambda \in \Omega$. If there exists $\lambda_0 \in \mathbb{C}_+$ such that $(\lambda_0)^\alpha \in \sigma_p(A)$ (denote by $f((\lambda_0)^\alpha)$ the corresponding eigenfunction), $C(\tilde{X}) \subseteq \tilde{X}$ and $f((\lambda_0)^\alpha) \in \tilde{X}$, then the operator $A_{|\tilde{X}}$ subgenerates a global distributionally chaotic α-times $C_{|\tilde{X}}$-regularized resolvent family $(R_\alpha(t)_{|\tilde{X}})_{t\geq 0}$ on the space $\tilde{X}$. Furthermore, if $R(C_{|\tilde{X}})$ is dense in $\tilde{X}$, then $(R_\alpha(t)_{|\tilde{X}})_{t\geq 0}$ is densely distributionally chaotic in the space $\tilde{X}$.

Remark 2.4. (i) It should be observed that Theorem 2.3(ii)-(iii), as well as Theorem 2.6(ii)-(iii) below, can be slightly improved by assuming that there exist $n \in \mathbb{N}$, open connected subsets Ω_i of $\mathbb{C}$ and analytic mappings $f_i : \Omega_i^\alpha \to X$ which satisfy, for every $i = 1, \ldots, n$: $\Omega_i \cap (-\infty, 0] = \emptyset$, $\Omega_i^\alpha \subseteq \sigma_p(A)$, $\Omega_i \subseteq \Omega_{0,-}$ and $f_i(\lambda^\alpha) \in Kern(A - \lambda^\alpha) \setminus \{0\}$, $\lambda \in \Omega_i$ (cf. [12] and [31, Remark 1(iii)]). Set $\hat{X} := \overline{span\{f_i(\lambda^\alpha) : \lambda \in \Omega_i,\ 1 \leq i \leq n\}}$ and assume that Ω_i' is an open connected subset of Ω_i which admits a cluster point in Ω_i for $1 \leq i \leq n$. Then $\hat{X} = \overline{span\{f_i(\lambda^\alpha) : \lambda \in \Omega_i',\ 1 \leq i \leq n\}}$ and the conditions $C(\hat{X}) \subseteq \hat{X}$, $f((\lambda_0)^\alpha) \in \hat{X}$, implies that the operator $A_{|\hat{X}}$ subgenerates a global distributionally chaotic α-times $C_{|\hat{X}}$-regularized resolvent family $(R_\alpha(t)_{|\hat{X}})_{t\geq 0}$ on the space $\hat{X}$. Furthermore, if $R(C_{|\hat{X}})$ is dense in $\hat{X}$, then $(R_\alpha(t)_{|\hat{X}})_{t\geq 0}$ is densely distributionally chaotic in the space $\hat{X}$.

(ii) If $\alpha = 2$, then $(R_\alpha(t))_{t\geq 0}$ is a C-regularized cosine function subgenerated by A. Since $E_2(z^2) = \cosh z$, $z \in \mathbb{C}$, we have $\lim_{t\to+\infty} E_2(\lambda^2 t^2) = \infty$,

$\lambda \in \mathbb{C} \setminus i\mathbb{R}$, so that Theorem 2.3(ii)-(iii) cannot be simply reformulated for the abstract differential equations of second order. On the other hand, it is well known that the operator $\mathcal{A} := \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ is a subgenerator of an $(\alpha+1)$-times integrated $\mathcal{C}$-semigroup $(S_{\alpha+1}(t))_{t\geq 0}$ on $X \times X$, where $\mathcal{C} := \left(\begin{smallmatrix} C & 0 \\ 0 & C \end{smallmatrix}\right)$, provided that the operator A is a subgenerator of an α-times integrated C-cosine function $(C_\alpha(t))_{t\geq 0}$ on X; cf. [27, Proposition 2.1.11] and [28, Subsection 2.1.2]. Keeping in mind our recent results on distributionally chaotic fractionally integrated C-semigroups given in [15, Section 5], it is not difficult to prove that $(S_{\alpha+1}(t))_{t\geq 0}$ is (densely) distributionally chaotic provided that X is separable, as well as that $\rho_C(A) \neq \emptyset$ and there exists an open connected subset Ω of $\mathbb{C}$ such that $\Omega \subseteq \sigma_p(A)$ and $\Omega \cap (-\infty, 0] \neq \emptyset$. The analysis of distributionally chaotic fractionally integrated C-cosine functions is very non-trivial, so we will not discuss this theme within the framework of this paper any longer (for more details concerning hypercyclic and Devaney chaotic integrated C-cosine functions, we refer the reader to [11], [26] and [28, Section 3.2]).

In the remaining part of paper, we analyze distributionally chaotic properties of solutions of the equation (2.3) with $A_j = c_j I$, where $c_j \in \mathbb{C}$, $j \in \mathbb{N}_{n-1}$. Set $m_j := \lceil \alpha_j \rceil$, $1 \leq j \leq n$, $m := m_0 := \lceil \alpha \rceil$, $A_0 := A$, $\alpha_0 := \alpha$, and recall ([30], [28]) that a function $u \in C^{m_n-1}([0,\infty) : E)$ is called a (strong) solution of (2.3) iff $c_i \mathbf{D}_t^{\alpha_i} u \in C([0,\infty) : X)$ for $1 \leq i \leq n-1$, $A\mathbf{D}_t^{\alpha} u \in C([0,\infty) : X)$, $g_{m_n-\alpha_n} * (u - \sum_{k=0}^{m_n-1} u_k g_{k+1}) \in C^{m_n}([0,\infty) : X)$ and (2.3) holds. By a mild solution of (2.3) we mean any function $u \in C([0,\infty) : X)$ such that the following holds:

$$u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot) + \sum_{j=1}^{n-1} c_j g_{\alpha_n-\alpha_j} * \left[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot) \right]$$
$$= A\left(g_{\alpha_n-\alpha} * \left[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot) \right] \right).$$

Given $i \in \mathbb{N}^0_{m_n-1}$ in advance, set $D_i := \{ j \in \mathbb{N}_{n-1} : m_j - 1 \geq i \}$ and $\mathcal{D}_i := \{ j \in \mathbb{N}^0_{n-1} : m_j - 1 \geq i \}$.

Definition 2.5. ([28]) Suppose $C,\ C_1,\ C_2 \in L(X)$ with $C,\ C_2$ being injective. A sequence $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ of strongly continuous operator families in $L(X)$ is called a global

(i) C_1-existence propagation family for (2.3) iff $R_i(0) = g_i(0)C_1$ and the following holds:

$$\Big[R_i(\cdot)x - g_i(\cdot)C_1x\Big] + \sum_{j\in D_i} c_j\Big[g_{\alpha_n-\alpha_j} * \Big(R_i(\cdot)x - g_i(\cdot)C_1x\Big)\Big] + \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} c_j\big(g_{\alpha_n-\alpha_j} * R_i\big)(\cdot)x = \begin{cases} A\big(g_{\alpha_n-\alpha} * R_i\big)(\cdot)x, & m-1<i,\ x\in E, \\ A\Big[g_{\alpha_n-\alpha} * \big(R_i(\cdot)x - g_i(\cdot)C_1x\big)\Big](\cdot), & m-1\geq i,\ x\in E, \end{cases}$$

for any $i = 0, \ldots, m_n - 1$;

(ii) C_2-uniqueness propagation family for (2.3) iff $R_i(0) = g_i(0)C_2$ and the following holds:

$$\Big[R_i(\cdot)x - g_i(\cdot)C_2x\Big] + \sum_{j\in D_i} c_j\Big[g_{\alpha_n-\alpha_j} * \Big(R_i(\cdot)x - g_i(\cdot)C_2x\Big)\Big] + \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} c_j\big(g_{\alpha_n-\alpha_j} * R_i(\cdot)x\big)(\cdot) = \begin{cases} \big(g_{\alpha_n-\alpha} * R_i(\cdot)Ax\big)(\cdot), & m-1<i, \\ g_{\alpha_n-\alpha} * \Big[R_i(\cdot)Ax - g_i(\cdot)C_2Ax\Big](\cdot), & m-1\geq i, \end{cases}$$

for any $i = 0, \ldots, m_n - 1$ and $x \in D(A)$;

(iii) C-resolvent propagation family for (2.3), in short C-propagation family for (2.3), if $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a C-uniqueness propagation family for (2.3), and for every $t \geq 0$ and $i \in \mathbb{N}^0_{m_n-1}$, $R_i(t)A \subseteq AR_i(t)$, $R_i(t)C = CR_i(t)$ and $CA \subseteq AC$.

The notion of analyticity of a global α-times C-regularized resolvent family for (2.1) and a global C-resolvent propagation family for (2.3) will be understood in the general sense of [29, Definition 3.1]. In the case that $C = I$, any C-resolvent propagation family for (2.3) is also called a resolvent propagation family for (2.3), or simply a resolvent propagation family, if there is no risk for confusion. As mentioned earlier, we assume that any single operator family $(R_i(t))_{t\geq 0}$ of the tuple $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is non-degenerate, i.e., that the supposition $R_i(t)x = 0$, $t \geq 0$ implies $x = 0$ ($i \in \mathbb{N}^0_{m_n-1}$). Then we also say that the operator A is a subgenerator of $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$. Recall that the integral generator $\hat{A}$ of $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is defined as the set of those pairs $(x, y) \in X \times X$ such that, for every $i =$

$0,\ldots,m_n-1$ and $t\geq 0$, the following holds:

$$\Big[R_i(\cdot)x-g_i(\cdot)Cx\Big]+\sum_{j=1}^{n-1}c_jg_{\alpha_n-\alpha_j}*\Big[R_i(\cdot)x-\big(k*g_i\big)(\cdot)Cx\Big]$$
$$+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}c_j\big[g_{\alpha_n-\alpha_j+i}*k\big](\cdot)Cx$$
$$=\begin{cases}\big[g_{\alpha_n-\alpha}*R_i\big](\cdot)y, & m-1<i,\\ g_{\alpha_n-\alpha}*\Big[R_i(\cdot)y-\big(k*g_i\big)(\cdot)Cy\Big], & m-1\geq i.\end{cases}$$

From now on, we assume that $C^{-1}AC=A$ is densely defined and subgenerates a global C-resolvent propagation family $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$. Then A is, in fact, the integral generator of $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$. Furthermore, we assume that, for every $i\in\mathbb{N}^0_{m_n-1}$ with $m-1\geq i$, one has $\mathbb{N}_{n-1}\setminus D_i\neq\emptyset$ and $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}|c_j|^2>0$. Then the problem (2.3) has at most one mild (strong) solution.

For each $i\in\mathbb{N}^0_{m_n-1}$ we denote by $Z_i(A)$ (with a little abuse of notation) the set which consists of those vectors $x\in E$ such that $R_i(t)x\in R(C)$, $t\geq 0$ and the mapping $t\mapsto C^{-1}R_i(t)x$, $t\geq 0$ is continuous. Then $R(C)\subseteq Z_i(A)$, and it can be easily proved that $x\in Z_i(A)$ iff there exists a unique mild solution of (2.3) with $u_k=\delta_{k,i}x$, $k\in\mathbb{N}^0_{m_n-1}$; if this is the case, the unique mild solution of (2.3) is given by $u(t;x):=u_i(t;x):=C^{-1}R_i(t)x$, $t\geq 0$. We know [28] that the suppositions $\lambda\in\mathbb{C}$, $x\in E$ and $Ax=\lambda x$ imply that $x\in Z_i(A)$; then the unique strong solution of (2.3) is given by

$$u_i(t;x)=\mathcal{L}^{-1}\left(\frac{z^{-i-1}+\sum_{j\in D_i}c_jz^{-\alpha_n-i-1+\alpha_j}-\chi_{\mathcal{D}_i}(0)\lambda z^{-\alpha_n-i-1+\alpha}}{1+\sum_{j=1}^{n-1}c_jz^{\alpha_j-\alpha_n}-\lambda z^{\alpha-\alpha_n}}\right)(t)x,$$

for any $t\geq 0$ and $i\in\mathbb{N}^0_{m_n-1}$. Set $P_\lambda:=\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}c_j\lambda^{\alpha_j-\alpha}$, $\lambda\in\mathbb{C}\setminus\{0\}$ and

$$F_i(\lambda,t):=\mathcal{L}^{-1}\left(\frac{z^{-i-1}+\sum_{j\in D_i}c_jz^{-\alpha_n-i-1+\alpha_j}-\chi_{\mathcal{D}_i}(0)P_\lambda z^{-\alpha_n-i-1+\alpha}}{1+\sum_{j=1}^{n-1}c_jz^{\alpha_j-\alpha_n}-P_\lambda z^{\alpha-\alpha_n}}\right)(t),$$

for any $t\geq 0$, $i\in\mathbb{N}^0_{m_n-1}$ and $\lambda\in\mathbb{C}\setminus\{0\}$.

Let $i\in\mathbb{N}^0_{m_n-1}$, and let $\tilde{X}$ be a closed linear subspace of X. Then the notion of (dense) ($\tilde{X}$-)distributional chaos of $(R_i(t))_{t\geq 0}$ will be undersood in the sense of Definition 2.2, with $(R_\alpha(t))_{t\geq 0}$ replaced by $(R_i(t))_{t\geq 0}$. It is clear that the notion of ($\tilde{X}$-)distributionally irregular vector can be introduced for any operator family considered in this section so far; this is not of crucial importance for our investigation and we shall skip all relevant details for the sake of brevity.

As explained in [30, Remark 1(5)], the following theorem is a slight extension of Theorem 2.3. The proof is similar and therefore omitted.

Theorem 2.6. *Let X be separable, let $i\in\mathbb{N}^0_{m_n-1}$, let $C^{-1}AC=A$ be densely defined and generate the global C-resolvent propagation family $((R_0(t))_{t\geq 0},\ldots,$ $(R_{m_n-1}(t))_{t\geq 0})$.*

(i) Assume that the following conditions hold:

(a) There exists a dense subset X_0' of X such that $\lim_{t\to\infty} R_i(t)x = 0$, $x \in X_0'$.

(b) There exists $x \in X$ such that $\lim_{t\to\infty} \|R_i(t)x\| = \infty$.

Then $(R_i(t))_{t\geq 0}$ is distributionally chaotic. If, moreover, $R(C)$ is dense in X, then $(R_i(t))_{t\geq 0}$ is densely distributionally chaotic.

(ii) Suppose that A is the integral generator of the global C-resolvent propagation family $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$, $i \in \mathbb{N}^0_{m_n-1}$, Ω is an open connected subset of $\mathbb{C}$, $\Omega \cap (-\infty, 0] = \emptyset$ and $P_\Omega := \{P_\lambda : \lambda \in \Omega\} \subseteq \sigma_p(A)$. Let $f : P_\Omega \to X$ be an analytic mapping such that $f(P_\lambda) \in Kern(P_\lambda - A) \setminus \{0\}$, $\lambda \in \Omega$ and let $\tilde{X} := \overline{span\{f(P_\lambda) : \lambda \in \Omega\}}$. Suppose $\lambda_0 \in \mathbb{C}$, $P_{\lambda_0} \in \sigma_p(A)$, $\tilde{X} = X$,

$$\lim_{t\to+\infty} \bigl|F_i(\lambda_0, t)\bigr| = +\infty \text{ and } \lim_{t\to+\infty} F_i(\lambda, t) = 0, \ \lambda \in \Omega.$$

Then $(R_i(t))_{t\geq 0}$ is distributionally chaotic; if, moreover, $R(C)$ is dense in X, then $(R_i(t))_{t\geq 0}$ is densely distributionally chaotic.

(iii) Let the assumptions of (ii) hold, and let $\tilde{X} \neq X$. Denote by $f(P_{\lambda_0})$ the eigenfunction corresponding to the eigenvalue P_{λ_0}. If $C(\tilde{X}) \subseteq \tilde{X}$ and $f(P_{\lambda_0}) \in \tilde{X}$, then the operator $A_{|\tilde{X}}$ is the densely defined integral generator of the $C_{|\tilde{X}}$-resolvent propagation family $((R_0(t)_{|\tilde{X}})_{t\geq 0}, \ldots,$ $(R_{m_n-1}(t)_{|\tilde{X}})_{t\geq 0})$ in the Banach space $\tilde{X}$, $C^{-1}_{|\tilde{X}} A_{|\tilde{X}} C_{|\tilde{X}} = A_{|\tilde{X}}$ and the operator family $((R_i(t)_{|\tilde{X}})_{t\geq 0}$ is distributionally chaotic in the space $\tilde{X}$. The additional assumption $\overline{C(\tilde{X})} = \tilde{X}$ implies that $(R_i(t)_{|\tilde{X}})_{t\geq 0}$ is densely distributionally chaotic.

Remark 2.7. The conditions $f((\lambda_0)^\alpha) \in \tilde{X}$ and $f(P_{\lambda_0}) \in \tilde{X}$, cf. the formulations of Theorem 2.3(iii) and Theorem 2.6(iii), have been considered in [15] for the abstract differential equations of first order. The conclusions clarified in [15] can be simply reformulated for the abstract fractional differential equations of the form (2.1)-(2.3).

Plenty of various examples from [30]-[31] can be used for the illustration of our abstract theoretical results. We shall quote just one such example.

Example 2.8. ([24], [30]) It is worth noting that Theorem 2.3 and Theorem 2.6 can be successfully applied in the analysis of a large class of abstract multi-term fractional differential equations on the symmetric spaces of non-compact type, Damek-Ricci or Heckman-Opdam root spaces ([3], [24], [35]). Consider, for example, the situation in which the assumptions of [24, Theorem 3.1(a)] hold: X is a symmetric space of non-compact type and rank one, $p > 2$, the parabolic domain P_p and the positive real number c_p possess the same meaning as in [24]. Let $\Delta^\natural_{X,p}$ denote the corresponding Laplace-Beltrami operator, and let $P(z) =$

$\sum_{j=0}^{n} a_j z^j$, $z \in \mathbb{C}$ be a non-constant complex polynomial with $a_n > 0$. We start with the analysis of following case: $\zeta \in (1,2)$, $\pi - n \arctan \frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2} > 0$ and

$$\theta \in \left(n \arctan \frac{|p-2|}{2\sqrt{p-1}} + \zeta\frac{\pi}{2} - \pi, \pi - n \arctan \frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2} \right).$$

Then the operator $-e^{i\theta}P(\Delta^{\natural}_{X,p})$ is the integral generator of an exponentially bounded, analytic ζ-times regularized resolvent family $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ of angle $\frac{1}{\zeta}(\pi - n \arctan \frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2} - |\theta|)$. Keeping in mind that $\mathrm{int}(P_p) \subseteq \sigma_p(\Delta^{\natural}_{X,p})$, the condition

$$-e^{i\theta}P\big(\mathrm{int}(P_p)\big) \ \cap \ \big\{te^{\pm i\zeta\frac{\pi}{2}} : t \geq 0\big\} \ \neq \emptyset$$

implies by Theorem 2.3(ii) that $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ is densely distributionally chaotic; we already know that $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ is topologically mixing. Suppose now $n = 2$, $0 < a < 2$, $\alpha_2 = 2a$, $\alpha_1 = 0$, $\alpha = a$, $c_1 > 0$, $i = 0$ and $|\theta| < \min(\frac{\pi}{2} - n \arctan \frac{|p-2|}{2\sqrt{p-1}}, \frac{\pi}{2} - n \arctan \frac{|p-2|}{2\sqrt{p-1}} - \frac{\pi}{2}a)$. Then $-e^{i\theta}P(\Delta^{\natural}_{X,p})$ generates an exponentially bounded, analytic resolvent propagation family $((R_{\theta,P,0}(t))_{t\geq 0}, \ldots, (R_{\theta,P,\lceil 2a\rceil -1}(t))_{t\geq 0})$ of angle $\min(\frac{\pi - n \arctan \frac{|p-2|}{2\sqrt{p-1}} - |\theta|}{a} - \frac{\pi}{2}, \frac{\pi}{2})$. Moreover,

$$\begin{aligned} F_0(\lambda, t) =& \frac{\lambda^a t^{-a}}{\lambda^{2a} - c_1}\Big(E_{a,2-a}\big(\lambda^a t^a\big) - E_{a,2-a}\big(c_1\lambda^{-a}t^a\big)\Big) \\ &+ \frac{\lambda^a}{\lambda^{2a} - c_1}\Big[\lambda^a E_a\big(\lambda^a t^a\big) + (a-1)\lambda^a E_{a,2}\big(\lambda^a t^a\big) \\ &- c_1\lambda^{-a}E_a\big(c_1\lambda^{-a}t^a\big) - (a-1)c_1\lambda^{-a}E_{a,2}\big(c_1\lambda^{-a}t^a\big)\Big] \\ &+ \big(\lambda^a + c_1\lambda^{-a}\big)\frac{\lambda^a}{\lambda^{2a} - c_1}\Big(E_a\big(\lambda^a t^a\big) - E_a\big(c_1\lambda^{-a}t^a\big)\Big), \quad t > 0; \end{aligned} \tag{2.6}$$

cf. [28, p. 418]. Using the asymptotic expansion formulae (1.1)-(1.3) and (2.6), it can be simply verified that the condition

$$-e^{i\theta}P\big(\mathrm{int}(P_p)\big) \ \cap \ \Big\{\big(it\big)^a + c_1\big(it\big)^{-a} : t \in \mathbb{R} \setminus \{0\}\Big\} \ \neq \emptyset$$

implies that $(R_{\theta,P,0}(t))_{t\geq 0}$ is both densely distributionally chaotic and topologically mixing. Observe, finally, that we can consider the case $\zeta \in (0,1)$ here.

Concerning the invariance of hypercyclic and topologically mixing properties under the action of subordination principles ([9], [27]-[29], [34]), it has been recently observed in [28, Remark 3.3.16] that the unilateral backward shifts have some advantages over other operators used in the theory of hypercyclicity. With the exception of a relatively small class of abstract PDEs involving unilateral backward shifts, distributionally chaotic properties cannot be simply inherited after application of subordination principles.

We close the paper with the observation that it would be very tempting to say something relevant and noteworthy about distributionally chaotic properties of abstract degenerate differential equations (cf. [21]-[22] and [32]-[33]).

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