Novi Sad J. Math.
Vol. 45, No. 2, 2015, 187-199

INITIAL IDEAL OF BINOMIAL EDGE IDEAL IN DEGREE 2

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Abstract. We study the initial ideal of binomial edge ideal in degree 2 ($[in_<(J_G)]_2$), associated to a graph G. We computed dimension, depth, Castelnuovo-Mumford regularity, Hilbert function and Betti numbers of $[in_<(J_G)]_2$ for some classes of graphs.

AMS Mathematics Subject Classification (2010): 05E40, 16E30

Key words and phrases: Edge ideals; Betti numbers; Binomial edge ideal

1. Introduction

Let K denote a field. Let G denote a connected, simple and undirected graph over the vertices labeled by $[n] = \{1, 2, \ldots, n\}$. The binomial edge ideal $J_G \subseteq S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is an ideal generated by all binomials $x_iy_j - x_jy_i$, $i < j$, such that $\{i, j\}$ is an edge of G. It was introduced in [5] and independently at the same time in [7]. It is a natural generalization of the notion of monomial edge ideal which is introduced by Villarreal in [12]. Monomial edge ideal is an ideal generated by monomials x_ix_j, $i < j$, such that $\{i, j\}$ is an edge of G. The algebraic properties of binomial edge ideals in terms of combinatorial properties of graphs (and vice versa) were investigated by many authors in [3], [5], [6], [7], [8], [9], [10],[11], [13], [14], [15] and [16]. The Cohen-Macaulay property of binomial edge ideals was studied in [3], [8] and [9]. As a generalization of the Cohen-Macaulay property, the first author has studied approximately Cohen-Macaulay property as well as sequentially Cohen-Macaulay property in [13] and [14], respectively.

In the present paper we investigate the initial ideal of binomial edge ideal in degree 2 ($[in_<(J_G)]_2$), associated to the graph G. We compute dimension, depth, Castelnuovo-Mumford regularity, Hilbert function and Betti numbers of $[in_<(J_G)]_2$ for the complete graph, complete bipartite graph, cycle and kite. We find dimension and depth of $[in_<(J_G)]_2$ in case of an arbitrary tree.

The paper is organized as follows: In Section 2, we introduced some notations and give results that we need in the rest of the paper. In particular we give a short summary on minimal free resolutions. Section 3 is devoted to the algebraic properties of $[in_<(J_G)]_2$ where G is a complete graph. In Section 4, the algebraic properties of $[in_<(J_G)]_2$ of complete bipartite graphs are discussed. In Section 5, we do the same for the classes of all cycles and kites as we did for the complete graphs in Section 3. In the last section, we compute dimension and depth of $[in_<(J_G)]_2$ where G is a tree.

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2. Preliminaries

In this section we will introduce the notation used in the article. Moreover, we summarize a few auxiliary results that we need.

Edge ideals are the simplest polynomial ideals that can be associated to the graphs.

Consider a connected undirected graph G on n vertices labeled by $[n] = \{1, 2, \ldots, n\}$ and let $S = K[x_1, x_2, \ldots, x_n]$ denote the polynomial ring in the n variables where K is an arbitrary field. The **edge ideal** I_G associated to the graph G is the ideal of S generated by the set of square-free monomials $x_i x_j$ such that $\{i, j\}$ is an edge of G, that is $I_G = (x_i x_j : \{i, j\} \in E(G))$, where $E(G)$ is the edge set of G.

Here we set $I_G = (0)$ if all vertices of G are isolated.

In order to understand the minimal primes of these edge ideals, the concept of minimal vertex cover is important.

Definition 2.1. Let G be a graph with vertex set V. A subset $M \subset V$ is called a **minimal vertex cover** for G if the following conditions are satisfied:

(a) Every edge of G is incident with some vertex in M.

(b) There is no proper subset of M with the first property.

The set M satisfying the condition (a) only is called a **vertex cover** of G. We regard M an empty set if all the vertices of G are isolated.

Now we will introduce a result which establishes one to one correspondence between the minimal vertex covers of a graph and the minimal primes of the corresponding edge ideal.

Proposition 2.2. *Let $S = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field K and G a graph with vertex set V. Let I be an ideal of polynomial ring S generated by $M = \{x_{i_1}, \ldots, x_{i_t}\}$, then the following conditions are equivalent:*

(a) I is a minimal prime ideal of I_G.

(b) M is a minimal vertex cover of G.

Proof. For the proof see [12, Proposition 6.1.16]. □

Example 2.3. We consider a simple graph on 5 vertices as depicted in Figure 1. It is easy to see that $\{2, 4\}$, $\{1, 3, 4\}$ and $\{1, 3, 5\}$ are the only minimal vertex covers of G. Therefore by Proposition 2.2 we have $I_G = (x_2, x_4) \cap (x_1, x_3, x_4) \cap (x_1, x_3, x_5)$.

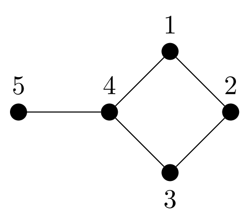

FIGURE 1

We denote by G a connected undirected graph on n vertices labeled by $[n] = \{1, 2, \ldots, n\}$. For an arbitrary field K let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the polynomial ring in the $2n$ variables. To the graph G one can associate an ideal $J_G \subset S$ generated by all binomials $x_i y_j - x_j y_i$ for $i < j$ such that $\{i, j\}$ forms an edge of G. This ideal J_G is called the **binomial edge ideal** associated to the graph G. This construction was invented by Herzog et al. in [5] and independently found in [7]. At first let us recall some of their definitions.

Definition 2.4. Fix the previous notation. For a set $T \subset [n]$ let $\tilde{G}_T$ denote the complete graph on the vertex set T. Moreover let $G_{[n]\setminus T}$ denote the graph obtained by deleting all vertices of G that belong to T.

Let $c = c(T)$ denote the number of connected components of $G_{[n]\setminus T}$. Let $G_1, \ldots, G_c$ denote the connected components of $G_{[n]\setminus T}$. Then define

$$P_T(G) = (\cup_{i\in T}\{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{C(T)}}),$$

where $\tilde{G}_i, i = 1, \ldots, c$, denotes the complete graph on the vertex set of the connected component $G_i, i = 1, \ldots, c$.

The following result is important for the understanding of the binomial edge ideal of G.

Lemma 2.5. *With the previous notation the following holds:*

(a) $P_T(G) \subset S$ *is a prime ideal of height* $n - c + |T|$, *where* $|T|$ *denotes the number of elements of* T.

(b) $J_G = \cap_{T\subseteq[n]} P_T(G)$.

(c) $J_G \subset P_T(G)$ *is a minimal prime if and only if either* $T = \emptyset$ *or* $T \neq \emptyset$ *and* $c(T \setminus \{i\}) < c(T)$ *for each* $i \in T$.

Proof. For the proof we refer to Section 3 of the paper [5]. □

Therefore J_G is the intersection of prime ideals. That is, S/J_G is a reduced ring. Moreover, we remark that J_G is an ideal generated by quadrics and therefore homogeneous, so that S/J_G is a graded ring with natural grading induced by the $\mathbb{N}$-grading of S.
Let M be a graded finitely generated S-module. By the Hilbert syzygy Theorem, M has a finite minimal graded free resolution:

$$F_\bullet : 0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0$$

where $F_i = \bigoplus_j S(-d_{ij})^{\beta_{ij}}$ for $i \geq 0$ and p is called the **projective dimension** of M. The numbers β_{ij} are uniquely determined by M i.e. $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^S(K, M)_{i+j}, i, j \in \mathbb{Z}$, as **graded Betti numbers** of M. We can also define **Castelnuovo-Mumford regularity** $\operatorname{reg} M = \max\{j \in \mathbb{Z} | \beta_{i,j}(M) \neq$

$0\}$. The **Betti table** looks as in the following:

	0	1	$\cdots$	p
0	$\beta_{0,0}$	$\beta_{1,0}$	$\cdots$	$\beta_{p,0}$
1	$\beta_{0,1}$	$\beta_{1,1}$	$\cdots$	$\beta_{p,1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
r	$\beta_{0,r}$	$\beta_{1,r}$	$\cdots$	$\beta_{p,r}$

Note that all the $\beta_{i,j}$ outside of the Betti table are zero. For more details and related facts we refer the book of Bruns and Herzog [1]. The following result in [1] is important for us.

Lemma 2.6. *Let $I \subset S$ be a graded ideal with $pd(S/I) = p$ and $regS/I = r$ then Hilbert series of S/I can be computed from the graded Betti numbers as follows:*

$$H(S/I, t) = \frac{\sum_{j=0}^{r} \sum_{i=0}^{p} (-1)^i \beta_{i,j}(S/I) t^{i+j}}{(1-t)^{2n}}.$$

Let J_G be the binomial edge ideal of a graph G, then **the initial ideal of binomial edge ideal in degree 2** is defined as $[in_<(J_G)]_2 = (x_i y_j : \{i, j\} \in E(G), i < j)$.

It can easily be seen that the ideal $[in_<(J_G)]_2$ associated to the graph G is the monomial edge ideal of a bipartite graph on $2n$ vertices with vertex sets $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_n\}$.

Definition 2.7. A **labeling** of G is a bijection $V(G) \simeq [n] = \{1, \ldots, n\}$, and given a labeling, we typically assume $V(G) = [n]$. A labeling is said to be **closed labeling** if whenever we have distinct edges $\{j, i\}, \{i, k\} \in E(G)$ with either $j > i < k$ or $j < i > k$, then $\{j, k\} \in E(G)$. Finally, a graph is said to be **closed graph** if it has a closed labeling.

For further investigations we need the following theorem from [5].

Theorem 2.8. *Let $J_G \subset S$ be the binomial edge ideal of a graph G on vertex set $[n]$, then J_G has quadratic Gröbner basis if and only if G is a closed graph.*

Proof. For the proof see [2, Theorem 3.4]. □

It is clear from the above theorem that in case of a closed graph G, we have $[in_<(J_G)]_2 = in_<(J_G)$.
Now we address the question why the initial ideal of binomial edge ideal in degree 2 is important to us. For this there are two interesting results as follows:

Lemma 2.9. *For any graded ideal $I \subset S$, we have*

$$\beta_{i,j}(I) \leq \beta_{i,j}(in_<(I)).$$

for all i and j.

Proof. For the proof we refer to the book [4, Corollary 3.3.3]. □

Lemma 2.10. *Let $I \subset S$ be a graded ideal. Then, for all k and for all $j \leq k$, we have*

$$\beta_{i,j}(I) = \beta_{i,j}(I_{\leq k}).$$

Proof. For the proof we refer to the book [4, Lemma 8.2.12]. □

In these results if we take $I = in_<(J_G)$ and $k = 2$ then we have $\beta_{i,j}(in_<(J_G))$ $=\beta_{i,j}(in_<(J_G)_{\leq 2})$ for $j < 2$ and we get $\beta_{i,0}(in_<(J_G))$ and $\beta_{i,1}(in_<(J_G))$ for all i. So by knowing these Betti numbers for $in_<(J_G)$ we get bounds for Betti numbers of original binomial edge ideal in lower strengths.

3. The initial ideal of binomial edge ideal in degree 2 of a complete graph

The **complete graph** K_n has all possible edges. In this section we study $[in_<(J_{K_n})]_2$. We compute primary decomposition, dimension, depth, Hilbert series and Betti numbers. Since K_n is a closed graph, therefore $[in_<(J_{K_n})]_2 = in_<(J_{K_n})$.

Theorem 3.1. *Let K_n be the complete graph on n vertices and $in_<(J_{K_n}) = (x_i y_j : 1 \leq i < j \leq n)$ be the initial ideal of binomial edge ideal in degree 2 associated with K_n, then we have:*

(a) $in_<(J_{K_n}) = \bigcap_{i=0}^{n-1} I_{n,i}$ *is the minimal primary decomposition of* $in_<(J_{K_n})$, *where* $I_{n,i} = (x_1, \ldots, x_i, y_{i+2}, \ldots, y_n)$.

(b) $S/in_<(J_{K_n})$ *is Cohen-Macaulay of dimension* $n + 1$.

(c) The Hilbert series of $S/in_<(J_{K_n})$ *is* $H(S/in_<(J_{K_n}), t) = \frac{1}{(1-t)^{n+1}}(1 + (n-1)t)$.

(d) $\mathrm{reg}(S/in_<(J_{K_n})) = 1$ *and* $S/in_<(J_{K_n})$ *has a linear resolution.*

(e) The Betti numbers of $S/in_<(J_{K_n})$ *are* $b_i = i\binom{n}{i+1}$ *for* $i = 1, \ldots, n-1$.

Proof. **(a)** In order to find the primary decomposition of $in_<(J_{K_n})$, we will use Proposition 2.2. Let M be the minimal vertex cover of $in_<(J_{K_n})$. We have to show that each vertex cover M of $in_<(J_{K_n})$ is of the form $\{x_1, \ldots, x_i, y_{i+2}, \ldots, y_n\}$ for all $0 \leq i \leq n-1$. For $i = 0$, $M = \{y_2, \ldots, y_n\}$ and for $i = n-1$, $M = \{x_1, \ldots, x_{n-1}\}$ so there is nothing to prove. Now the only thing which we have to show is $x_j \notin M$ for all $j \geq i+2$.
If $x_j \in M$ then y_{j+1} should be in M otherwise the edge $\{x_{j-1}, y_{j+1}\}$ will not be covered. In this way the vertex cover M obtained will not be minimal because $M \setminus \{x_j\}$ is minimal vertex cover contained in M.
(b) Since $\underline{\mathbf{x}} = y_1, x_1 - y_2, x_2 - y_3, \ldots, x_{n-1} - y_n, x_n$ is a system of parameters

for the complete graph, so $S/(in_<(J_{K_n}),\underline{\mathbf{x}}) \cong K[x_1,\ldots,x_{n-1}]/(x_ix_j, 1 \le i \le j \le n-1)$. Now the length is $l(S/(in_<(J_{K_n}),\underline{\mathbf{x}})) = 1+n-1 = n$ and degree is $\deg(S/in_<(J_{K_n})) = n$ by (a). Which proves the statement in (b).
(c) As binomial edge ideal J_{K_n} of a complete graph on n vertices is graded so $H(S/J_{K_n},t) = H(S/in_<(J_{K_n}),t)$. Thus we get our required Hilbert function.
(d) As $1 = reg(S/(in_<(J_{K_n}),\underline{\mathbf{x}})) = reg(S/in_<(J_{K_n}))$, where $\underline{\mathbf{x}}$ is a regular sequence and hence $S/in_<(J_{K_n})$ has a linear resolution.
(e) The graded minimal free resolution of $S/in_<(J_{K_n})$ is of the form

$$0 \to F_{n-1}^{b_{n-1}} \to \cdots \to F_2^{b_2} \to F_1^{b_1} \to S \to S/in_<(J_{K_n}) \to 0$$

Where $F_i = S^{b_i}(-i-1)$ and by using Lemma 2.6 and comparing it with the Hilbert series in (c), we get the required Betti numbers. □

4. The initial ideal of binomial edge ideal in degree 2 of a complete bipartite graph

A graph is said to be **bipartite graph** if the vertex set is decomposed into two disjoint sets in such a way there are no two vertices in a same set which are adjacent. **Complete bipartite graph** is a bipartite graph G in which for any two vertices $v_1 \in V_1$ and $v_2 \in V_2$, $\{v_1, v_2\}$ is an edge in G. If $|V_1| = n$ and $|V_2| = m$ then it is usually denoted by $K_{n,m}$. $K_{1,m}$ is known as **m-star graph**. In this section we study $[in_<(J_{m,n})]_2$ of complete bipartite graph. We compute its primary decomposition, dimension, depth, Hilbert series and Betti numbers.

Theorem 4.1. *Let $J_{m,n}$ be a complete bipartite graph on $m+n$ vertices with $m \ge n$. Let $[in_<(J_{m,n})]_2$ be the initial ideal of binomial edge ideal in degree 2 associated with $J_{m,n}$ then we have:*

(a) $[in_<(J_{m,n})]_2 = (x_1,x_2,\ldots,x_m) \cap (y_{m+1},y_{m+2},\ldots,y_{m+n})$ *is the minimal primary decomposition of* $[in_<(J_{m,n})]_2$.

(b) $\dim S/[in_<(J_{m,n})]_2 = 2m+n$ *and* $\operatorname{depth} S/[in_<(J_{m,n})]_2 = m+n+1$.

(c) The Hilbert series of $S/[in_<(J_{m,n})]_2$ *is*

$$H(S/[in_<(J_{m,n})]_2,t) = \frac{(1-t)^{m-n}+1-(1-t)^m}{(1-t)^{2m+n}}.$$

(d) $\operatorname{reg}(S/[in_<(J_{m,n})]_2) = 1$ *and* $S/[in_<(J_{m,n})]_2$ *has a linear resolution.*

(e) The Betti numbers of $S/[in_<(J_{m,n})]_2$ *are* $b_{0,0} = 1$ *and* $b_{i,1} = \binom{m+n}{i+1} - \binom{m}{i+1} - \binom{n}{i+1}$.

Proof. **(a)** It is easy to see that

$$[in_<(J_{m,n})]_2 = (x_1,x_2,\ldots,x_m)(y_{m+1},\ldots,y_{m+n}),$$

which, being the product of two prime ideals in different sets of variables gives the primary decomposition in (a).
(b) From primary decomposition in (a), there exists a short exact sequence

$$0 \to S/[in_<(J_{m,n})]_2 \to S/(x_1,\dots,x_m) \oplus S/(y_{m+1},\dots,y_{m+n}) \to S/(x_1,\dots,x_m,y_{m+1},\dots,y_{m+n}) \to 0 \tag{4.1}$$

We have the following isomorphisms.

(1) $S/(x_1,\dots,x_m) \cong K[x_{m+1},\dots,x_{m+n},y_1,\dots,y_{m+n}]$,

(2) $S/(y_{m+1},\dots,y_{m+n}) \cong K[x_1,\dots,x_{m+n},y_1,\dots,y_m]$ and

(3) $S/(x_1,\dots,x_m,y_{m+1},\dots,y_{m+n}) \cong K[x_{m+1},\dots,x_{m+n},y_1,\dots,y_m]$.

The isomorphisms $(1),(2)$ and (3) are Cohen-Macaulay of dimension $m+2n, 2m+n$ and $m+n$, respectively. Since $m \geq n$, so $2m+n \geq m+2n$. Applying Local Cohomology to the exact sequence 4.1 induces the following isomorphisms.
$H^{m+n+1}(S/[in_<(J_{m,n})]_2) \cong H^{m+n}(S/(x_1,\dots,x_m,y_{m+1},\dots,y_{m+n}))$,
$H^{m+2n}(S/[in_<(J_{m,n})]_2) \cong H^{m+2n}(S/(x_1,\dots,x_m))$ and
$H^{2m+n}(S/[in_<(J_{m,n})]_2) \cong H^{2m+n}(S/(y_{m+1},\dots,y_{m+n}))$.
Thus we get the required result.
(c) It follows from the exact sequence 4.1.
(d) The behavior of regularity on the exact sequence 4.1, implies $\operatorname{reg}(S/[in_<(J_{m,n})]_2) \leq 1$. Since $S/[in_<(J_{m,n})]_2$ is not a polynomial ring so $\operatorname{reg}(S/[in_<(J_{m,n})]_2) = 1$ and $S/[in_<(J_{m,n})]_2$ has a linear resolution.
(e) The graded minimal free resolution of $S/[in_<(J_{m,n})]_2$ is of the form

$$0 \to S(-m-n)^{b_{m+n-1,1}} \to \cdots \to S(-3)^{b_{2,1}} \to S(-2)^{b_{1,1}} \to S \to S/[in_<(J_{m,n})]_2 \to 0$$

By using Lemma 2.6 and comparing it with the Hilbert series in (c), we get the required result. □

5. The Initial Ideals of Binomial Edge Ideals in Degree 2 of Cycle and Kite Graphs

A **cycle** is a connected graph in which all vertices are of degree 2. In particular, for $n=3$ it is the triangle and $n=4$ it is the square. A graph is said to be a **kite graph** if a line is attached to one of the vertex of a cycle.
In this section we study the initial ideals of binomial edge ideals in degree 2 of cycle and kite graphs. We compute dimension, depth, Hilbert series and Betti numbers in each case. As a technical tool we need the following lemma.

Lemma 5.1. *Let M denote a finitely generated graded S-module. Let $\underline{f} = f_1,\dots,f_l$ denote an M-regular sequence of forms of degree 1. Then*

$$\operatorname{Tor}_i^S(K, M/\underline{f}M) \cong \oplus_{j=0}^{l} \operatorname{Tor}_{i-j}^S(K,M)^{\binom{l}{j}}(-j).$$

Proof. For the proof of the statement let $l = 1$ and $f = f_1$. Then the short exact sequence

$0 \to M(-1) \xrightarrow{f} M \to M/fM \to 0$ provides an isomorphism

$$\mathrm{Tor}_i^S(K, M/fM) \cong \mathrm{Tor}_i^S(K, M) \oplus \mathrm{Tor}_{i-1}^S(K, M)(-1)$$

for all $i \in \mathbb{Z}$. The general case follows from induction on i.

□

In the present section we define an ideal $I_{l,k} = (x_l y_{l+1}, \dots, x_k y_{k+1} : 1 \le l < k \le n-1)$. It is easy to see that $S/I_{l,k} \cong S/I_{l^*,k^*}$ if and only if $k - l = k^* - l^*$. Note that $I_{1,n-1} = [in_<(J_L)]_2 \cong in_<(J_L)$ where J_L is the binomial edge ideal of a line on n vertices which is also a closed graph. The ideal $I_{l,k}$ is a complete intersection, therefore by Koszul complex we have:

$$Tor_i^S(K, S/I_{l,k}) = K^{\binom{k-l+1}{i}}(-2i); 0 \le i \le k - l + 1.$$

Corollary 5.2. *With the previous notations, let z_1 and z_2 be two regular elements on $S/I_{l,k}$ then we have:*

$$Tor_i^S(K, S/I_{l,k}, z_1, z_2) = \begin{cases} K(0), & \text{if } i = 0; \\ K^{k-l}(-2) \oplus K^2(0), & \text{if } i = 1; \\ K^{\binom{k-l}{2}}(-4) \oplus K^{2(k-l+1)}(-3) \oplus K(-2), & \text{if } i = 2; \\ K^{\binom{k-l}{i}}(-2i) \oplus K^{2\binom{k-l}{i-1}}(-2i+1) \oplus K^{\binom{k-l}{i-2}}(-2i+2), & \text{if } i > 2. \end{cases}$$

Proof. Since $Tor_i^S(K, S/I_{l,k}) = K^{\binom{k-l+1}{i}}(-2i); 0 \le i \le k - l + 1$ where $I_{l,k}$ is the ideal corresponding to a line of length $k - l + 1$. Now if z is a regular element then, by using Lemma 5.1, we have:

$$Tor_i^S(K, S/I_{l,k}, z) = Tor_i^S(K, S/I_{l,k}) \oplus Tor_{i-1}^S(K, S/I_{l,k})(-1).$$

From this we get the following:

$$Tor_i^S(K, S/I_{l,k}, z) = \begin{cases} K(0), & \text{if } i = 0; \\ K^{k-l}(-2) \oplus K(-1), & \text{if } i = 1; \\ K^{\binom{k-l}{i}}(-2i) \oplus K^{\binom{k-l}{i}}(-2i+1), & \text{if } i \ge 2. \end{cases}$$

Now if z_1 and z_2 are two regular elements then again, by using Lemma 5.1, we get:

$$Tor_i^S(K, S/I_{l,k}, z_1, z_2) = Tor_i^S(K, S/I_{l,k}) \oplus 2Tor_{i-1}^S(K, S/I_{l,k})(-1) \oplus Tor_{i-1}^S(K, S/I_{l,k})(-2).$$

Thus we obtain the required result. □

Theorem 5.3. *Let C be a graph of a cycle on n vertices. Let $[in_<(J_C)]_2$ be the initial ideal of binomial edge ideal in degree 2 associated with C, then we have:*

(a) *$S/[in_<(J_C)]_2$ is Cohen-Macaulay of dimension $n+1$.*

(b) *The Hilbert series of $S/[in_<(J_C)]_2$ is*

$$H(S/[in_<(J_C)]_2, t) = \frac{2(1+t)^{n-2} - (1+t)^{n-3}}{(1-t)^{n+1}}.$$

(c) $\operatorname{reg}(S/[in_<(J_C)]_2) = n-2$.

(d) *We have the following Betti numbers for $S/[in_<(J_C)]_2$.*

$$\beta_{i,j} = \begin{cases} 2\binom{n-2}{i} + \binom{n-3}{i-1} - \binom{n-3}{i}, & \text{if } \ j = i = 0, \dots, n-2 \ ; \\ 2\binom{n-3}{i-2}, & \text{if } \ j = i-1 = 2, \dots, n-2 \ ; \\ 0, & \text{otherwise} \ . \end{cases}$$

Proof. **(a)** Note that

$$[in_<(J_C)]_2 = (x_1, I_{2,n-1}) \cap (y_n, I_{1,n-2}).$$

From above decomposition it is easily seen that $\dim S/[in_<(J_C)]_2 = n+1$. Also there exists a short exact sequence

(5.1)

$$0 \to S/[in_<(J_C)]_2 \to S/(x_1, I_{2,n-1}) \oplus S/(y_n, I_{1,n-2}) \to S/(x_1, y_n, I_{2,n-2}) \to 0$$

where $(x_1, y_n, I_{2,n-1}, I_{1,n-2}) = (x_1, y_n, I_{2,n-2})$.
Now $S/(x_1, I_{2,n-1}), S/(y_n, I_{1,n-2})$ and $S/(x_1, y_n, I_{2,n-2})$ all are Cohen-Macaulay of dimension $n+1$. By applying Local Cohomology to the exact sequence 5.1, we get $\operatorname{depth} S/[in_<(J_C)]_2 = n+1$, which completes the proof of (a).
(b) It follows from the exact sequence 5.1.
(c) Since $S/[in_<(J_C)]_2$ is Cohen-Macaulay, so the claim follows from (b).
(d) As we have $[in_<(J_C)]_2 = (x_1y_n, I_{1,n-1})$. So there exists a short exact sequence

$$0 \to S/(I_{1,n-1} : x_1y_n)(-2) \stackrel{x_1y_n}{\to} S/I_{1,n-1} \to S/[in_<(J_C)]_2 \to 0.$$

Now $I_{1,n-1} : x_1y_n = (I_{1,n-1} : y_n) : x_1 = (y_2, x_{n-1}, I_{2,n-2})$. So the above short exact sequence is the same as

$$0 \to S/(y_2, x_{n-1}, I_{2,n-2})(-2) \to S/I_{1,n-1} \to S/[in_<(J_C)]_2 \to 0. \tag{5.2}$$

Now by applying Tor to the exact sequence 5.2, we get the following Tor sequence

$$\cdots \to Tor_i^S(K, S/(y_2, x_{n-1}, I_{2,n-2}))(-2) \to Tor_i^S(K, S/I_{1,n-1}) \to$$
$$Tor_i^S(K, S/[in_<(J_C)]_2) \to Tor_{i-1}^S(K, S/(y_2, x_{n-1}, I_{2,n-2}))(-2)$$
$$\to Tor_{i-1}^S(K, S/I_{1,n-1}) \to \dots$$

Using Corollary 5.2, we get

$$\cdots \to K^{\binom{n-3}{i}}(-2i-2)\oplus K^{2\binom{n-3}{i-1}}(-2i-1)\oplus K^{\binom{n-3}{i-2}}(-2i) \to K^{\binom{n-1}{i}}(-2i) \to$$
$$Tor_i^S(K, S/[in_<(J_C)]_2) \to K^{\binom{n-3}{i-1}}(-2i)\oplus K^{2\binom{n-3}{i-2}}(-2i+1)\oplus K^{\binom{n-3}{i-3}}(-2i+2)$$
$$\to K^{\binom{n-1}{i-1}}(-2i+2) \to \dots$$

Which shows that

$$Tor_i^S(K, S/[in_<(J_C)]_2) = K^{u_i}(-2i)\oplus K^{2\binom{n-3}{i-2}}(-2i+1)\oplus K^{v_i}(-2i+2)$$

Now by applying Tor to the exact sequence 5.1, we get the following Tor sequence

$$\cdots \to Tor_{i+1}^S(K, S/(x_1, I_{2,n-1}))\oplus Tor_{i+1}^S(K, S/(y_n, I_{1,n-2})) \to$$
$$Tor_{i+1}^S(K, S/(x_1, y_n, I_{2,n-2})) \to Tor_i^S(K, S/[in_<(J_C)]_2) \to$$
$$Tor_i^S(K, S/(x_1, I_{2,n-1}))\oplus Tor_i^S(K, S/(y_n, I_{1,n-2})) \to$$
$$Tor_i^S(K, S/(x_1, y_n, I_{2,n-2})) \to \dots$$

Again using Corollary 5.2, we get

$$\cdots \to K^{2\binom{n-2}{i+1}}(-2i-2)\oplus K^{2\binom{n-2}{i-1}}(-2i-1) \to \tag{5.3}$$
$$K^{\binom{n-3}{i+1}}(-2i-2)\oplus K^{2\binom{n-3}{i}}(-2i-1)\oplus K^{\binom{n-3}{i-1}}(-2i) \to$$
$$Tor_i^S(K, S/[in_<(J_C)]_2) \to K^{2\binom{n-2}{i}}(-2i)\oplus K^{2\binom{n-2}{i-1}}(-2i+1) \to$$
$$K^{\binom{n-3}{i}}(-2i)\oplus K^{2\binom{n-3}{i-1}}(-2i+1)\oplus K^{\binom{n-3}{i-2}}(-2i+2) \to \dots$$

From this Tor sequence we have

$$K^{v_i}(-2i+2) = 0.$$

Which implies

$$Tor_i^S(K, S/[in_<(J_C)]_2) = K^{u_i}(-2i)\oplus K^{2\binom{n-3}{i-2}}(-2i+1).$$

By restricting the sequence 5.3 in degree $2i$, we get the following exact sequence.

$$0 \to K^{\binom{n-3}{i-1}}(-2i) \to K^{u_i}(-2i) \to K^{2\binom{n-2}{i}}(-2i) \to K^{\binom{n-3}{i}}(-2i) \to 0.$$

So $u_i = 2\binom{n-2}{i} + \binom{n-3}{i-1} - \binom{n-3}{i}$. □

Theorem 5.4. *Let K be a graph of a kite on n vertices. Let $[in_<(J_K)]_2$ be the initial ideal of binomial edge ideal in degree 2 associated with K, then we have:*

(a) $[in_<(J_K)]_2$ is Cohen Macaulay of dimension $n+1$.

(b) The Hilbert series of $S/[in_<(J_K)]_2$ *is*

$$H(S/[in_<(J_K)]_2, t) = \frac{2(1+t)^{n-2} - (1+t)^{n-3}}{(1-t)^{n+1}}.$$

(c) $\operatorname{reg}(S/[in_<(J_K)]_2) = n - 2.$

(d) We have the following Betti numbers for $S/[in_<(J_K)]_2$*.*

$$\beta_{i,j} = \begin{cases} 2\binom{n-2}{i} + \binom{n-3}{i-1} - \binom{n-3}{i}, & \textit{if} \quad j = i = 0, \ldots, n-2 \ ; \\ 2\binom{n-3}{i-2}, & \textit{if} \quad j = i-1 = 2, \ldots, n-2 \ ; \\ 0, & \quad \textit{otherwise} \ . \end{cases}$$

Proof. The proof for the kite graph is same as for the cycle graph. □

6. The Initial Ideal of Binomial Edge Ideal in Degree 2 of Tree Graph

The **tree** is a graph in which any two vertices are connected by exactly one path. For the initial ideal of Binomial Edge Ideal in Degree 2 of a tree graph, we fix a labeling of tree we call it as **grapes labeling**. The labeling is as follows:
First we hang the tree just like a bunch of grapes by any of the vertex having degree greater than 1. We labeled this vertex by 1 and named this vertex as a father vertex and all the vertices connected with the father vertex by an edge are called its children and labeled by 2, 3 and so on. Now in the next step consider all the children vertices as father vertices and label their children vertices as before and continue this process until the whole tree is labeled.

Example 6.1. Lets apply grapes labeling on the following tree shown in Figure 2. The father vertex is labeled as 1 who has three children labeled as 2, 3 and 4. Now in next step the father vertex labeled with 2 has two children and father vertex labeled with 4 has one child that are labeled as 5, 6 and 7 respectively.

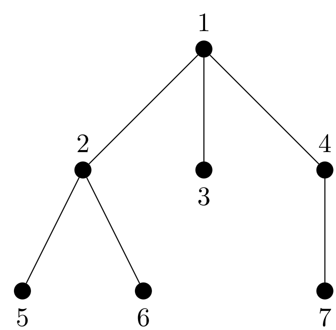

FIGURE 2

In this section we study $[in_<(J_T)]_2$ with the above mentioned labeling. We compute dimension and depth for an arbitrary tree.

Theorem 6.2. *Let T be a graph of a tree on n vertices. Let $[in_<(J_T)]_2$ be the initial ideal of binomial edge ideal in degree 2 associated with T. Let t denote the number of vertices of degree 1. Then*

(a) $\dim S/[in_<(J_T)]_2 = n + t$.

(b) $\operatorname{depth} S/[in_<(J_T)]_2 = n + 1$.

Proof. (a) As $[in_<(J_T)]_2$ is the monomial edge ideal of a bipartite graph on $2n$ vertices with vertex sets $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_n\}$. It can easily be seen that each connected component of this bipartite graph is i-star graph having dimension i. Now in a tree if total number of father vertices of children i are c_i then in corresponding bipartite graph we have c_i number of components of i-star graph. So total dimension of i-star graphs is ic_i. As our graph is simple so y_1 in the corresponding bipartite graph is an isolated vertex thus $\sum_{i>0} ic_i = n - 1$. On the other hand if j is a vertex of degree 1 then the number $t+1$ should be added to get the required formula because the variables y_1 and x_j are not contributing in the generators of $[in_<(J_T)]_2$.

(b) Each i-star graph in the corresponding bipartite graph has depth 1 therefore $\sum_{i>0} c_i = n-t$. Similarly $t+1$ should be added as above to complete the proof. ☐

Remark 6.3. If $n > 1$, $S/[in_<(J_T)]_2$ is never Cohen-Macaulay since $t \geq 2$.

Acknowledgement

The authors are grateful to the reviewer for suggestions to improve the presentation of the manuscript.

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Received by the editors November 19, 2014