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NEARLY EINSTEIN MANIFOLDS

Swapan Kumar Saha[1]

Abstract. The object of this paper is to define and study a new type of non-flat Riemannian manifolds called nearly Einstein manifolds. The notion of this nearly Einstein manifold has been established by an example and an existence theorem. Some geometric properties are obtained.

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1. Introduction

Generalizing the Einstein manifold Prof. M. C. Chaki and R. K. Maity introduced and studied quasi Einstein manifold. The aim of this paper is to define and study a type of non-flat Riemannian manifold called nearly Einstein manifold. This manifold is defined in the next section. Such an n-dimensional manifold shall be denoted by the symbol $(NE)_n$. In existence of nearly Einstein manifold it is shown that every Einstein manifold is a nearly Einstein manifold. But it is not true conversely. So it is meaningful to study the nearly Einstein manifold.

In this paper it is shown that in a $(NE)_n$, the associated scalar is $\frac{1}{n}|S|^2$, where $|S|$ is the length of the Ricci tensor S and in an Einstein $(NE)_n$, the length of the Ricci tensor is $\frac{r}{\sqrt{n}}$, where r is the scalar curvature of the manifold. In a $(NE)_n$, the Ricci tensor L of type (1,1) has two eigenvalues, namely, $\sqrt{\lambda}$ and - $\sqrt{\lambda}$, where λ is the associated scalar defined by (2.1) and the scalar curvature is zero if and only if it is even dimensional. It is shown that in a quasi Einstein $(NE)_n$, the Ricci curvature in the direction of U defined by (2.5) is $\frac{n(\lambda-a^2)-ab}{b}$ and it is shown that in a Ricci recurrent $(NE)_n$, $\frac{2\lambda}{r}$, $r \neq 0$, is an eigenvalue of the Ricci tensor L of type (1,1) corresponding to the eigenvector which is the vector of recurrence. It is proved that a conharmonically flat manifold is a $(NE)_n$ if and only if it is a Ricci semi symmetric manifold. Next an example of nearly Einstein manifold has been constructed in local coordinates. Finally, it is shown that if in a $(NE)_4$ perfect fluid space time in which Einstein equation without cosmological constant holds and the energy momentum tensor obeys the time like convergence condition, then such a space time contains pure matter and in this case isotropic pressure is $\sqrt{\frac{\lambda}{3K^2}}$ and energy density is $\sqrt{\frac{3\lambda}{K^2}}$.

[1]Life Member, Calcutta Mathematical Society. Address: 18/348, Kumar lane ,Chinsurah, Hooghly-712101, West Bengal, India, e-mail: sksaha30@gmail.com

2. Definitions

In this section we first define a nearly Einstein manifold.

Definition 2.1. A non-flat Riemannian manifold $(M^n, g), n > 2$, is called a nearly Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$S(LX, Y) = \lambda g(X, Y) \text{ for all vector fields } X, Y. \tag{2.1}$$

where λ is a non-zero scalar called the associated scalar and L is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S of type (0,2) defined by

$$g(LX, Y) = S(X, Y) \text{ for all vector fields } X, Y. \tag{2.2}$$

Such an n-dimensional manifold shall be denoted by the symbol $(NE)_n$.

Some definitions are stated below. These will be used in the sequel.

Definition 2.2 ([1])**.** A Riemannian Manifold $(M^n, g), n \geq 2$, is called an Einstein manifold if the Ricci tensor S of type (0,2) satisfies the following condition

$$S(X, Y) = \frac{r}{n} g(X, Y) \text{ for every vector field } X, Y, \tag{2.3}$$

where r is the scalar curvature of the manifold.

Einstein manifolds play an important role in Riemannian geometry as well as in general theory of relativity.

In a paper in 2000, M.C. Chaki and R.K. Maity generalized the Einstein manifold as follows:

Definition 2.3 ([2])**.** A non-flat Riemannian manifold $(M^n, g), n > 2$, is called a quasi Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \text{ for all vector fields } X, Y, \tag{2.4}$$

where a and b are scalars and $b \neq 0$ and A is an associated 1-form defined by

$$g(X, U) = A(X), \tag{2.5}$$

U is a unit vector field called the generator of the manifold. Since then works on quasi Einstein manifolds and its generalizations are going on. Some of them are [3, 8, 9, 10, 11].

The Ricci recurrent manifold is defined as follows:

Definition 2.4 ([15])**.** A Riemannian manifold (M^n, g), $n > 2$, is said to be Ricci recurrent if its Ricci tensor S of type (0,2) is not proportional to the metric tensor g and satisfies the condition

$$(\nabla_X S)(Y, Z) = B(X)S(Y, Z), \text{ for all vector fields X,Y,Z,} \tag{2.6}$$

where ∇ is the operator of covariant differentiation with respect to the metric tensor g and B is a non-zero 1-form defined by $g(X, V) = B(X)$. The Ricci recurrent manifolds and its generalizations were studied in [5, 6, 7, 15] and in many other papers.

The conharmonically flat manifold is defined as follows:

Definition 2.5 ([12, 13])**.** Let $\widehat{C}$ and R be the conharmonic curvature tensor and Riemannian curvature tensor respectively, then

$$\widehat{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)LX - g(X,Z)LY + S(Y,Z)X - S(X,Z)Y\}. \tag{2.7}$$

A non-flat Riemannian manifold (M^n, g), $n > 2$, is called conharmonically flat if

$$\widehat{C}(X,Y)Z = 0. \tag{2.8}$$

From (2.7) and (2.8) we get

$$R(X,Y)Z = \frac{1}{n-2}\{g(Y,Z)LX - g(X,Z)LY + S(Y,Z)X - S(X,Z)Y\}. \tag{2.9}$$

The Ricci semi symmetric manifold is defined as follows:

Definition 2.6 ([17])**.** A Riemannian manifold (M^n, g), $n > 2$, is called Ricci semi symmetric if its Ricci tensor S of type (0,2) satisfies the condition

$$[R(X,Y).S](Z,W) = 0 \text{ for all vector fields X,Y,Z,W.} \tag{2.10}$$

3. Main results

To show the existence of a nearly Einstein manifold we prove the following theorem:

Theorem 3.1. *Every Einstein manifold is a nearly Einstein manifold.*

Proof. Putting LX for X in (2.3) we get

$$S(LX, Y) = \frac{r}{n} S(X, Y). \tag{3.1}$$

From (3.1) and (2.3) we get

$$S(LX, Y) = \frac{r^2}{n^2} g(X, Y), \tag{3.2}$$

which shows that the manifold is a nearly Einstein manifold with associated scalar $\frac{r^2}{n^2}$. But the converse implication is not true. □

Some properties of the associated scalar and the scalar curvature of $(NE)_n$ are shown in the following theorems:

Theorem 3.2. *In a $(NE)_n$, the associated scalar is $\frac{1}{n}|S|^2$, where $|S|$ is the length of the Ricci tensor S.*

Proof. Putting $X = Y = e_i$ in (2.1), where $\{e_i\}$, $i = 1, 2, ..., n$ is an orthonormal basis of the tangent space at each point and i is summed for $1 \leq i \leq n$, we get

$$|S|^2 = \lambda n, \tag{3.3}$$

where

$$|S| = \sqrt{S(Le_i, e_i)}$$

is the length of the Ricci tensor S. Hence the theorem.

□

Theorem 3.3. *In an Einstein $(NE)_n$, the length of the Ricci tensor is $\frac{1}{\sqrt{n}}r$.*

Proof. If a $(NE)_n$ is an Einstein manifold, then we get from (2.3) and (2.1)

$$\lambda = \frac{r^2}{n^2}. \tag{3.4}$$

From (3.3) and (3.4) we get

$$|S| = \frac{1}{\sqrt{n}}r. \tag{3.5}$$

Hence we get the above theorem.

□

Theorem 3.4. *In a $(NE)_n$, the Ricci tensor L of type (1,1) has two eigenvalues, namely, $\sqrt{\lambda}$ and - $\sqrt{\lambda}$. The scalar curvature is zero if and only if it is even dimensional.*

Proof. Let ρ be the eigenvalue of the Ricci tensor L of type (1,1) corresponding to any vector field X, then

$$LX = \rho X. \tag{3.6}$$

From (3.6), (2.1) and (2.2) we get

$$(\rho^2 - \lambda)X = 0,$$

for all X. This shows that the Ricci tensor L of type (1,1) has two eigenvalues, namely $\sqrt{\lambda}$, - $\sqrt{\lambda}$. Again let the multiplicity of $\sqrt{\lambda}$ be m and the multiplicity of - $\sqrt{\lambda}$ be $n - m$. Since the scalar curvature is the trace of L, we have

$$r = m\sqrt{\lambda} - (n - m)\sqrt{\lambda} = (2m - n)\sqrt{\lambda}. \tag{3.7}$$

Since $\lambda \neq 0$, the scalar curvature vanishes if and only if the manifold is even dimensional. This proves the theorem.

□

Considering quasi Einstein nearly Einstein manifold we obtain the following theorem:

Theorem 3.5. *In a quasi Einstein $(NE)_n$, the Ricci curvature in the direction of U defined by* (2.5) *is* $\frac{n(\lambda-a^2)-ab}{b}$

Proof. Putting LX for X in (2.4) we get

$$S(LX,Y)=a^2g(X,Y)+abA(X)A(Y)+bA(LX)A(Y). \tag{3.8}$$

From (2.1) and (3.8) we get

$$\lambda g(X,Y)=a^2g(X,Y)+abA(X)A(Y)+bA(LX)A(Y). \tag{3.9}$$

Putting $X=Y=e_i$ in (3.9), where $\{e_i\}$, $i=1,2,...,n$ is an orthonormal basis of the tangent space at each point and i is summed for $1\le i\le n$, we get

$$S(U,U)=\frac{n(\lambda-a^2)-ab}{b}. \tag{3.10}$$

Since U is a unit vector field, $g(U,U)=1$, the Ricci curvature $\frac{S(U,U)}{g(U,U)}$ in the direction of U is $\frac{n(\lambda-a^2)-ab}{b}$. Hence the theorem.

□

Considering Ricci recurrent nearly Einstein manifold we obtain the following theorem:

Theorem 3.6. *In a Ricci recurrent $(NE)_n$, $\frac{2\lambda}{r}$, $r\neq 0$, is an eigenvalue of the Ricci tensor L of type (1,1) corresponding to the eigenvector which is a vector of recurrence.*

Proof. Contracting (2.6) we get

$$(divL)(X)=B(LX). \tag{3.11}$$

Again contracting (2.6) we get

$$X.r=B(X)r. \tag{3.12}$$

Now since $(divL)(X)=\frac{1}{2}X.r$, we get from (3.12)

$$(divL)(X)=\frac{1}{2}B(X)r. \tag{3.13}$$

Putting LX for X in (3.11) using (2.1) and (2.2) we get

$$(divL)(LX)=B(L^2X)=B(\lambda X)=\lambda B(X). \tag{3.14}$$

Putting LX for X in (3.13) we get

$$(divL)(LX)=\frac{r}{2}B(LX). \tag{3.15}$$

From (3.14) and (3.15) we get

$$LV = \frac{2\lambda}{r}V, \tag{3.16}$$

for all X. From (3.16) we conclude that $\frac{2\lambda}{r}, r \neq 0$, is an eigenvalue of the Ricci tensor L of type (1,1) corresponding to the eigenvector V which is a vector of recurrence. This completes the proof. □

Considering conharmonically flat Ricci semi symmetric manifold in a nearly Einstein manifold we obtain the following theorem:

Theorem 3.7. *Every conharmonically flat Ricci semi symmetric manifold is a* $(NE)_n$.

Proof. From (2.10) and the Ricci identity we get

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0. \tag{3.17}$$

From (2.9) and (3.17) we get

$$g(Y,Z)S(LX,W) - g(X,Z)S(LY,W) + g(Y,W)S(LX,Z) - g(X,W)S(LY,Z) = 0. \tag{3.18}$$

Putting $Y = Z = e_i$ in (3.18), where $\{e_i\}$, $i = 1, 2, ..., n$ is an orthonormal basis of the tangent space at each point and summing for $1 \leq i \leq n$, we get

$$S(LX,W) = \frac{1}{n}|S|^2 g(X,W),$$

which shows that this manifold is a $(NE)_n$. Hence the theorem.

□

Now we shall prove the converse part of the Theorem 3.7. We can state it as follows:

Theorem 3.8. *A conharmonically flat* $(NE)_n$ *is a Ricci semi symmetric manifold.*

Proof. We suppose that the condition (2.1) holds in a conharmonically flat manifold. We have from (2.9)

$$\begin{aligned} &[R(X,Y).S](Z,W) \\ &= -[S(R(X,Y)Z,W) + S(Z,R(X,Y)W)] \\ &= -\frac{1}{n-2}[g(Y,Z)S(LX,W) - g(X,Z)S(LY,W) \\ &\quad + g(Y,W)S(LX,Z) - g(X,W)S(LY,Z)] \\ &= 0 \quad \text{[by (2.1)]} \end{aligned} \tag{3.19}$$

Thus we see that a conharmonically flat $(NE)_n$ is a Ricci semi symmetric manifold. Hence the theorem. □

From Theorem 3.7 and Theorem 3.8 we can state the following:

Theorem 3.9. *A conharmonically flat manifold* (M^n, g)*,* $n > 2$*, is a* $(NE)_n$ *if and only if it is a Ricci semi symmetric manifold.*

An example of $(NE)_n$: We construct a manifold (M^3, g) whose metric in local coordinates (x^1, x^2, x^3) is

$$ds^2 = e^{x^1+x^2}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2. \tag{3.20}$$

From (3.20) we get the non-zero components of the metric tensors g_{ij} and g^{ij} as follows:

$$g_{11} = e^{x^1+x^2}, g_{12} = g_{21} = 1, g_{33} = 1 \tag{3.21}$$

and

$$g^{12} = g^{21} = 1, g^{22} = -e^{x^1+x^2}, g^{33} = 1. \tag{3.22}$$

Calculating the Christoffel symbols $\Gamma^{\ i}_{jk}$ we find that such non-zero symbols are as follows:

$$\begin{aligned} \Gamma^1_{11} &= -\frac{1}{2}e^{x^1+x^2}, \\ \Gamma^2_{11} &= \frac{1}{2}e^{x^1+x^2} + \frac{1}{2}e^{2(x^1+x^2)}, \\ \Gamma^2_{12} &= \frac{1}{2}e^{x^1+x^2}. \end{aligned} \tag{3.23}$$

Let R_{ij} and R^i_j be the components in local coordinates of S and L, respectively. Calculating R_{ij} and R^i_j we find that its non-zero components are as follows:

$$\begin{aligned} R_{11} &= -\frac{1}{2}e^{2(x^1+x^2)}, \\ R_{12} = R_{21} &= -\frac{1}{2}e^{x^1+x^2}, \\ R^1_1 = R^2_2 &= -\frac{1}{2}e^{x^1+x^2}, \end{aligned} \tag{3.24}$$

The scalar curvature r is obtained as follows:

$$r = -e^{x^1+x^2} \neq 0.$$

From the above we can verify that

$$R_{ij}R^j_k = \lambda g_{ik}$$

where

$$\lambda = \frac{1}{4}e^{2(x^1+x^2)}$$

i.e. in the index free notation, the defining equation of $(NE)_n$,

$$S(LX,Y) = \lambda g(X,Y).$$

Thus we verify that the constructed (M^3, g) is a nearly Einstein manifold. From (3.21)and (3.24) it can be verified that (M^3, g) can not be an Einstein since $r \neq 0$. Thus the above constructed example is an example of a nearly Einstein manifold which is not an Einstein manifold.

Now we consider an application of $(NE)_n$ in a general relativistic spacetime (M^4, g) and prove the following theorem:

Theorem 3.10. *If in a $(NE)_4$ perfect fluid spacetime in which the Einstein equation without cosmological constant holds and the energy momentum tensor obeys the timelike convergence condition, then such a spacetime contains pure matter and in this case isotropic pressure is $\sqrt{\frac{\lambda}{3K^2}}$ and energy density is $\sqrt{\frac{3\lambda}{K^2}}$.*

Proof. Let a semi Riemannian $(NE)_4$ be a general relativistic spacetime (M^4, g) where g is a Lorentz metric with signature (+,+,+,-). We know from [14, 16] that if the Ricci tensor S of type (0,2) of the spacetime satisfies the condition

$$S(X,X) > 0, \tag{3.25}$$

for every timelike vector field X, then (3.25) is called the timelike convergence condition. In this section we consider a perfect fluid spacetime $(NE)_4$ with unit time like velocity vector field U. Then we have

$$g(U,U) = -1. \tag{3.26}$$

Let $\{e_i\}$, $i = 1, 2, 3, 4$, be an orthonormal basis of the frame field at a point of the spacetime and contracting (2.1) over X and Y, we obtain

$$S(Le_i, e_i) = 4\lambda. \tag{3.27}$$

The sources of any gravitational field (matter and energy) are represented in relativity by a type of (0,2) symmetric tensor T called the energy momentum tensor [14]. T is given by

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y), \tag{3.28}$$

where σ and p are the energy density and the isotropic pressure of the fluid respectively, while A is defined by

$$g(X,U) = A(X), \tag{3.29}$$

and we suppose that T obeys time like convergence condition. The Einstein equation without cosmological constant [14, 4] can be written as

$$S(X,Y) - \frac{1}{2}rg(X,Y) = KT(X,Y), \tag{3.30}$$

where K is the gravitational constant. From (3.28) and (3.30) we get

$$S(X,Y) - \frac{1}{2}rg(X,Y) = K[(\sigma + p)A(X)A(Y) + pg(X,Y)]. \tag{3.31}$$

Taking frame field contracting (3.31) over X and Y we obtain

$$r = K(\sigma - 3p). \tag{3.32}$$

Putting $X = Y = U$ in (3.31) and using (3.26) and (3.32) we get

$$S(U,U) = \frac{K}{2}(\sigma + 3p). \tag{3.33}$$

Putting LX for X in (3.31) and taking the frame field and contracting over X and Y and using (3.26), (3.27), (3.32) and (3.33) we get

$$4\lambda = K^2(\sigma^2 + 3p^2) > 0, \tag{3.34}$$

since λ is non-zero. Since the spacetime is even dimensional, by Theorem 3.4 we get the scalar curvature of $(NE)_4$ spacetime is zero. Hence from (3.32) we get

$$\sigma = 3p. \tag{3.35}$$

From (3.33) and (3.35) we get

$$S(U,U) = K\sigma > 0, \tag{3.36}$$

by (3.25) i.e. $\sigma > 0$ which implies that this $(NE)_4$ spacetime contains pure matter. In this case, isotropic pressure p and energy density σ are given by $p^2 = \frac{\lambda}{3K^2}$ and $\sigma^2 = \frac{3\lambda}{K^2}$, respectively. This completes the proof.

□

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