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# REGULAR Γ−INCLINE AND FIELD Γ−SEMIRING

**M. Murali Krishna Rao[1] and B. Venkateswarlu[2]**

**Abstract.** We introduce the notion of Γ−incline as a generalization of incline, which is an algebraic structure with an additional poset structure. We introduce the notions of regular Γ−incline, integral Γ−incline, field Γ−incline, field Γ−semiring, simple Γ−semiring and pre-integral Γ−semiring. We study their properties and relations between them. We prove that if $M$ is a linearly ordered regular Γ−incline, then $M$ is a commutative Γ−incline and $M$ is a field Γ−semiring with an additional property if and only if $M$ is an integral, simple and commutative Γ−semiring.

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## 1. Introduction

The notion of a semiring was introduced by H. S. Vandiver [15] in 1934. Semiring is a well known universal algebra. Semirings have been used for studying optimization theory, graph theory, matrices, determinants, theory of automata, coding theory, analysis of computer programs, etc.

The notion of a Γ-ring was introduced by N. Nobusawa [13] as a generalization of the rings in 1964. M. K. Sen [14] introduced the notion of Γ- semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [7] in 1932, Lister [8] introduced ternary ring. Dutta & Kar [4] introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. In 1995, M. Murali Krishna Rao [11] introduced the notion of Γ- semiring which is a generalization of Γ- ring, ternary semiring and semiring. After the paper [11] was published, many mathematicians obtained interesting results on Γ-semirings.

The concept of an incline was first introduced by Z. Q. Cao [3] in 1984. Inclines are additively idempotent semirings in which products are less than or equal to either factor. Products reduce the values of quantities and make them go down therefore the structures were named inclines. Idempotent semirings and Kleene algebras have recently been established as fundamental structures in computer sciences. An incline is a generalization of Boolean algebra, fuzzy algebra and distributive lattice and incline is a special type of semiring. An incline has both the semiring structure and the poset structure. Every distributive lattice and every Boolean algebra is an incline but an incline need not

[1]Department of Mathematics, GIT, GITAM University, Visakhapatnam-530 045, Andhra Pradesh, India. e-mail: mmarapureddy@gmail.com

[2]Department of Mathematics, GIT, GITAM University, Visakhapatnam-530 045, Andhra Pradesh, India. e-mail: bvlmaths@gmail.com

be a distributive lattice. The set of all idempotent elements in an incline is a distributive lattice. Von Neumann [12] introduced the concept of regular elements in a ring. A. R. Meenakshi and S. Anbalagan [9] studied regular elements in an incline and proved that a regular commutative incline is a distributive lattice. An incline is a more general algebraic structure than a distributive lattice. Ahn, Jun and Kim [1] have proved that "Every prime ideal of the incline $M$ is equivalent to an irreducible ideal of $M$". In an incline every ideal is a k-ideal. A. R. Meenakshi and N. Jayabalan [10] have proved that every prime ideal in the incline $M$ is an irreducible ideal of $M$ and every maximal ideal in the incline $M$ is an irreducible ideal. W. Yao and S. C. Han [16] studied the relations between ideals, filters and congruences in inclines and it is shown that there is a one to one correspondence between the set of ideals and the set of all regular congruences. Z. Q. Cao et al. [2] studied the incline and its applications. Kim and Rowsh [5, 6] have studied matrices over an incline. Many research scholars have researched the theory of incline matrices. Few research scholars studied the algebraic structure of inclines. Inclines and matrices over inclines are useful tools in diverse areas such as automata theory, design of switching circuits, graph theory, information systems, modeling, decision making, dynamical programming, control theory, classical and nonclassical path finding problems in graphs, fuzzy set theory, data analysis, medical diagnosis, nervous system, probable reasoning, physical measurement and so on.

In this paper, we introduce the notion of $\Gamma-$incline as a generalization of the incline, which is simultaneously an algebraic structure and a poset structure. We introduce the notions of regular $\Gamma-$incline, idempotent $\Gamma-$incline, integral $\Gamma-$incline, pre-integral $\Gamma-$incline, mono $\Gamma-$incline, field $\Gamma-$incline and study the relations between them and their properties. Also, we introduce the notion of field $\Gamma-$semiring, simple $\Gamma-$semiring, pre-integral $\Gamma-$semiring and integral $\Gamma-$semiring and we study the relations between them and their properties.

## 2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

**Definition 2.1.** [2] A set $S$ together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectively) will be called a semiring, provided

(i) addition is a commutative operation.

(ii) multiplication distributes over addition both from the left and from the right.

(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in S$.

**Definition 2.2.** [3] A commutative incline $M$ with additive identity 0 and multiplicative identity 1 is a non-empty set $M$ with operations addition $(+)$ and multiplication $(.)$ defined on $M \times M \to M$ satisfying the following conditions for all $x, y, z \in M$

(i) $x + y = y + x$

(ii) $x + x = x$

(iii) $x + xy = x$

(iv) $x + (y + z) = (x + y) + z$

(v) $x(yz) = x(yz)$

(vi) $x(y + z) = xy + xz$

(vii) $(x + y)z = xz + yz$

(viii) $x1 = 1x = x$

(ix) $x + 0 = 0 + x = x$

(x) $xy = yx$

**Definition 2.3.** [13] Let $M$ and $\Gamma$ be additive abelian groups. If there exists a mapping $M \times \Gamma \times M \to M$ (images of $(x, \alpha, y)$ will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following conditions for all $x, y, z \in M, \alpha, \beta \in \Gamma$

(i) $x\alpha(y\beta z) = (x\alpha y)\beta z$

(ii) $x\alpha(y + z) = x\alpha y + x\alpha z$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $(x + y)\alpha z = x\alpha z + y\alpha z$.

Then $M$ is called a $\Gamma-$ ring.

**Definition 2.4.** [11] Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call $M$ a $\Gamma-$semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (images of $(x, \alpha, y)$ will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring $R$ is a $\Gamma-$semiring with $\Gamma = R$ and ternary operation $x\gamma y$ defined as the usual semiring multiplication.

We illustrate the definition of $\Gamma-$semiring by the following example.

**Example 2.5.** Let $S$ be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices whose entries are from $S$. Then $M_{p,q}(S)$ is a $\Gamma-$semiring with $\Gamma = M_{p,q}(S)$ and the ternary operation defined by the usual matrix multiplication as $x\alpha y = x(\alpha^t)y$, where $\alpha^t$ denotes the transpose of the matrix $\alpha$; for all $x, y$ and $\alpha \in M_{p,q}(S)$.

**Definition 2.6.** [11] A $\Gamma-$semiring $M$ is said to have a zero element if there exist an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M, \alpha \in \Gamma$.

**Definition 2.7.** [11] A $\Gamma-$semiring $M$ is said to be a commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

**Definition 2.8.** [11] Let $M$ be a $\Gamma-$semiring. An element $a \in M$ is said to be $\alpha$ idempotent if $a = a\alpha a$; an element of $M$ is said to be an idempotent of $M$ if it is $\alpha$ idempotent for some $\alpha \in \Gamma$.

**Definition 2.9.** [11] Let $M$ be a $\Gamma-$semiring. If every element of $M$ is an idempotent of $M$, $M$ is said to be an idempotent $\Gamma-$semiring.

**Definition 2.10.** [11] Let $M$ be a $\Gamma-$semiring. An element $a \in M$ is said to be a regular element of $M$ if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

**Definition 2.11.** [11] Let $M$ be a $\Gamma-$semiring. If every element of $M$ is regular, then $M$ is said to be a regular $\Gamma-$semiring.

**Definition 2.12.** [11] A non-empty subset $A$ of the $\Gamma-$semiring $M$ is called a $\Gamma-$subsemiring $M$ if $(A, +)$ is a subsemigroup of $(M, +)$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$.

**Definition 2.13.** [11] An additive subsemigroup $I$ of a $\Gamma-$semiring $M$ is said to be a left (right) ideal of $M$ if $M\Gamma I \subseteq I$ $(I\Gamma M \subseteq I)$. If $I$ is both a left and right ideal then $I$ is called an ideal of $\Gamma-$semiring $M$.

## 3. Regular $\Gamma-$incline

In this section, we introduce the notions of $\Gamma-$incline, zero divisor, unity element, invertible element, regular element and idempotent element in $\Gamma-$ incline and also the concepts of regular $\Gamma-$incline, integral $\Gamma-$incline, idempotent $\Gamma-$incline and field $\Gamma-$incline. We study their properties and relations between them.

**Definition 3.1.** Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. $M$ is called a $\Gamma-$incline if there exists a mapping $M \times \Gamma \times M \to M$ (images of $(x, \alpha, y)$ will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$

(v) $x + x = x$

(vi) $x + x\alpha y = x$

(vii) $y + x\alpha y = y$.

Every incline $M$ is a $\Gamma-$incline with $\Gamma = M$ and ternary operation $x\gamma y$ is defined as the usual incline multiplication. In a $\Gamma-$incline define the order relation such that for all $x, y \in M, y \leq x$ if and only if $y + x = x$. Obviously $\leq$ is a partial order relation over $M$.

**Definition 3.2.** An element 0 of $\Gamma-$incline $M$ is said to be the zero element if $0\alpha x = x\alpha 0 = 0$ for all $x \in M, \alpha \in \Gamma$.

**Definition 3.3.** A $\Gamma-$incline $M$ is said to be a commutative $\Gamma-$incline if $x\alpha y = y\alpha x$ for all $x, y \in M,\ \alpha \in \Gamma$.

**Example 3.4.** Let $M = [0,\ 1]$ and $\Gamma = N$ and let $+$ and the ternary operation be defined as $x + y = max\{x, y\},\ x\gamma y = min\{x, \gamma, y\}$ for all $x, y \in M, \gamma \in \Gamma$ then $M$ is a $\Gamma-$ incline.

**Definition 3.5.** A $\Gamma-$subincline $I$ of a $\Gamma-$incline $M$ is a non-empty subset of $M$ which is closed under the $\Gamma-$incline operations.

**Definition 3.6.** Let $M$ be a $\Gamma-$incline. An element $a \in M$ is said to be $\alpha$ idempotent if $a = a\alpha a$; an element of $M$ is said to be an idempotent of $M$ if it is $\alpha$ idempotent for some $\alpha \in \Gamma$.

**Definition 3.7.** Let $M$ be a $\Gamma-$incline. If every element of $M$ is an idempotent, $M$ is said to be an idempotent $\Gamma-$incline.

**Definition 3.8.** Let $M$ be a $\Gamma-$incline. An element $a \in M$ is said to be a regular element of $M$ if there exist $x \in M,\ \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. The existing $x$ is called generalized inverse of $a$, or $g-$inverse of $a$, or $1-$inverse of $a$. $a[1]$ denotes the set of all $g-$inverses of $a$.

**Definition 3.9.** Let $M$ be a $\Gamma-$ incline. If every element of $M$ is regular, $M$ is said to be a regular $\Gamma-$incline.

**Definition 3.10.** Let $M$ be a $\Gamma-$incline. If $x \leq y$ for all $y \in M$ then $x$ is called the least element of $M$ and denoted as 0. If $x \geq y$ for all $y \in M$ then $x$ is called the greatest element of $M$ and denoted as 1.

**Definition 3.11.** Let $M$ be a $\Gamma-$incline, $a, x \in M$. If $a$ is $1-$inverse of $x$, $x$ is called a $2-$inverse of $a$. An element $a \in M$ is called anti regular if there exists $x \in M$ which is a 2-inverse of $a$. $a[2]$ denotes the set of all 2-inverses of $a$.

**Definition 3.12.** Let $M$ be a $\Gamma-$incline, $a, x \in M$. If $x$ is a $1-$inverse and $2-$inverse of $a$, then $x$ is called a $1, 2-$inverse of $a$. $a[1, 2]$ denotes the set of all $1, 2$ inverses of $a$.

**Definition 3.13.** A $\Gamma-$incline $M$ is said to be linearly ordered if for all $x, y \in M$, we have $x \leq y$ or $y \leq x$, where $\leq$ is the incline order relation.

**Definition 3.14.** Let $M$ be a $\Gamma-$incline. An element $1 \in M$ is said to be an unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

**Definition 3.15.** In a $\Gamma-$incline $M$ with unity 1, an element $a \in M$ is said to be an invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

**Definition 3.16.** A nonzero element $a$ in a $\Gamma-$incline $M$ is said to be a zero divisor if there exist a nonzero element $b \in M$ and $\alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

**Definition 3.17.** A $\Gamma-$incline $M$ with a zero element and unity is said to be field $\Gamma-$incline, if it is commutative and every nonzero element of $M$ is invertible.

**Definition 3.18.** A $\Gamma-$incline $M$ with unity 1 and zero element 0 is called integral $\Gamma-$incline if it has no zero divisors.

**Definition 3.19.** A $\Gamma-$incline $M$ with zero element 0 is said to satisfy the cancellation law if for all $a, b, c \in M, \alpha \in \Gamma$ we have that $a \neq 0, a\alpha b = a\alpha c$ and $b\alpha a = c\alpha a$ implies $b = c$.

**Definition 3.20.** A $\Gamma-$incline $M$ with unity 1 and zero element 0 is called a pre-integral $\Gamma-$incline if $M$ satisfies the cancellation law.

**Example 3.21.** If $M = [0, 1], \Gamma = \{0, 1\}$, binary operation $+$ is maximum, ternary operation $a\alpha b$ is the usual multiplication for all $a, b \in M, \alpha \in \Gamma$, then $M$ is $\Gamma-$incline with unity 1.

**Theorem 3.22.** *Let* $0$ *be a zero element of a* $\Gamma-$ *incline* $M$*. Then* $0 + x = x = x + 0$ *for all* $x \in M$ *and* $0$ *is the least element of* $M$*.*

*Proof.* Let $M$ be a $\Gamma-$ incline with the zero element.

$$\begin{aligned} \text{We have } 0 + x &= 0\alpha x + x = x, \quad \text{for all } \alpha \in \Gamma. \\ x + 0 &= x + x\alpha 0 = x, \quad \text{for all } \alpha \in \Gamma. \\ \text{Therefore } 0 + x &= x + 0 = x. \\ \Rightarrow 0 &\leq x, \quad \text{for all } x \in M. \end{aligned}$$

Hence 0 is the least element of $M$. $\square$

**Theorem 3.23.** *If* $1$ *is an unity of the* $\Gamma-$*incline* $M$*, it is the greatest element of* $M$*.*

*Proof.* Let $M$ be a $\Gamma-$incline with unity 1 and $x \in M$. By definition of unity, there exists $\alpha \in \Gamma$ such that $x = x\alpha 1 \leq 1$. Hence 1 is the greatest element of $M$. $\square$

**Corollary 3.24.** *If a $\Gamma-$incline has an unity and a zero element, they are uniquely determined.*

The following theorem follows from Theorems 3.22 and 3.23

**Theorem 3.25.** *Let $M$ be a $\Gamma-$incline with unity* 1 *and zero element* 0. *If $a \in M$ then $0 \leq a \leq 1$.*

The following theorem is a straightforward verification.

**Theorem 3.26.** *Let $M$ be a $\Gamma-$incline with a* 0 *element and $a, b, c \in M$. Then*

*(i)* *If $a \leq b$ then $a + c \leq b + c$, $a\alpha c \leq b\alpha c$ and $c\alpha a \leq c\alpha b$ for all $\alpha \in \Gamma$.*

*(ii)* $a \leq a + b$, $b \leq a + b$.

*(iii)* $a\alpha b \leq a$, $a\alpha b \leq b$ *for all* $\alpha \in \Gamma$.

*(iv)* $a + b = 0$ *if and only if* $a = 0, b = 0$.

**Theorem 3.27.** *Let $M$ be a $\Gamma-$incline with unity and $a \in M$. $a$ is an idempotent if and only if $a$ is regular and* 1 *is a $g-$inverse of $a$.*

*Proof.* Let $M$ be a $\Gamma-$incline with unity 1. Suppose $a$ is $\alpha$ idempotent and 1 is the unity. There exists $\beta \in \Gamma$ such that $a\beta 1 = 1\beta a = a$.
$a = a\alpha a = a\alpha(1\beta a) = a\alpha 1\beta a$. Hence $a$ is regular and 1 is a $g-$inverse of $a$.

Conversely suppose that $a$ is a regular element of the $\Gamma-$incline $M$. Then there exist $\alpha, \beta \in \Gamma, x \in M$ such that $a = a\alpha x\beta a = a\alpha(x\beta a) \leq a\alpha a \leq a$. Therefore $a = a\alpha a$. Hence $a$ is an idempotent of the $\Gamma-$incline $M$. $\square$

**Theorem 3.28.** *Let $M$ be a $\Gamma-$incline. A regular element $a \in M$ is the smallest $g-$inverse of itself and $a[1, 2] = \{a\}$.*

*Proof.* Let $x \in a[1]$. There exist $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a \leq a\alpha x \leq x$. Since $a$ is idempotent, $a \in a[1]$. Hence $a$ is the smallest $g-$inverse of $a$. Let $x \in a[1, 2]$. There exist $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = a\alpha x\beta a$ and $x = x\gamma a\delta x \Rightarrow a \leq x$ and $x \leq a$. Therefore $a = x$. Hence $a[1, 2] = \{a\}$. $\square$

**Theorem 3.29.** *Let $M$ be a $\Gamma-$incline and $\alpha, \beta \in \Gamma$. If $a, b \in M$ are $\alpha, \beta$ idempotents, respectively, then $a, b, a + b$ are $\alpha + \beta$ idempotents.*

*Proof.* Suppose $a, b \in M$ are $\alpha, \beta$ idempotents, respectively. Then we have $a\alpha a = a, b\beta b = b$.

$$\begin{aligned} a(\alpha+\beta)a &= a\alpha a + a\beta a \\ &= a + a\beta a \\ &= a. \\ \text{Similarly, } b(\alpha+\beta)b &= b. \\ (a+b)(\alpha+\beta)(a+b) &= a\alpha a + a\alpha b + a\beta a + a\beta b + b\alpha a + b\beta a + b\alpha b + b\beta b \\ &= a + a\alpha b + a\beta a + a\beta b + b\alpha a + b\beta a + b\alpha b + b \\ &= a + a\beta a + a\beta b + b\alpha a + b\beta a + b \\ &= a + a\beta b + b\alpha a + b = a + b. \end{aligned}$$

Therefore $a+b$ is $\alpha+\beta$ idempotent of $\Gamma-$incline. Hence the theorem. □

**Theorem 3.30.** *Let $M$ be a $\Gamma-$ incline. An element $a \in M$ is an idempotent if and only if $a = x\alpha a\beta a$ for some $x \in M, \alpha, \beta \in \Gamma$.*

*Proof.* Let $a \in M, a = x\alpha a\beta a$ for some $x \in M, \alpha, \beta \in \Gamma$.
Now $a = x\alpha a\beta a \leq x\alpha a \leq a \Rightarrow a = x\alpha a$.
$a = x\alpha a\beta a = (x\alpha a)\beta a = a\beta a$. Therefore $a$ is a $\beta$ idempotent.

Conversely suppose that $a \in M$ is an $\alpha-$idempotent.
Now $a = a\alpha a = a\alpha(a\alpha a)$. Hence the theorem. □

**Theorem 3.31.** *Let $M$ be a $\Gamma-$ incline. If $a \in M$, and $a\alpha x\beta a = a$ for some $x \in a[1], \alpha, \beta \in \Gamma$ then $a\alpha x, x\beta a$ are idempotents.*

*Proof.* Let $a \in M, a = a\alpha x\beta a$ for some $x \in a[1],\ \alpha, \beta \in \Gamma$.
$a = a\alpha x\beta a \Rightarrow a\alpha x = a\alpha x\beta a\alpha x$. Therefore $a\alpha x$ is a $\beta$ idempotent. Similarly we can prove $x\beta a$ is an $\alpha$ idempotent. □

**Theorem 3.32.** *Let $M$ be a $\Gamma-$ incline. If $a$ is a regular element of $M$ then there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x = x\beta a$.*

*Proof.* Suppose $a$ is a regular element of $M$ then there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a \leq\ a\alpha x \leq a$.
Therefore $a\alpha x = a$.
Now $a = a\alpha x\beta a \leq x\beta a \leq a$. Therefore $a = x\beta a$.
Hence $a = a\alpha x = x\beta a$. □

**Theorem 3.33.** *Let $M$ be a $\Gamma-$incline, $a \in M$ an $\alpha-$idempotent and $a \leq b$ for some $b \in M$. We have $a = a\alpha b = b\alpha a$.*

*Proof.* Let $a, b \in M$ and $a \leq b$.

$$\begin{aligned} a \leq b &\Rightarrow a + b = b \\ &\Rightarrow a\alpha(a+b) = a\alpha b \\ &\Rightarrow a\alpha a + a\alpha b = a\alpha b \\ &\Rightarrow a + a\alpha b = a\alpha b \\ &\Rightarrow a = a\alpha b \\ \text{and } \ a \leq b &\Rightarrow a\alpha a \leq b\alpha a \\ &\Rightarrow a \leq b\alpha a \leq a \\ &\Rightarrow a = b\alpha a \end{aligned}$$

Hence $a = a\alpha b = b\alpha a$. □

**Corollary 3.34.** *Let $M$ be a regular $\Gamma-$incline. If $a, b \in M, a \leq b$ then there exists $\alpha \in \Gamma$ such that $a = a\alpha b = b\alpha a = a\alpha a$.*

The following theorem follows from Corollary 3.34.

**Theorem 3.35.** *Let $M$ be a regular $\Gamma-$incline. Then $a \leq b \Leftrightarrow a + b = b \Leftrightarrow a\alpha b = a$ for some $\alpha \in \Gamma, a, b \in M$*

**Theorem 3.36.** *If $M$ is a linearly ordered regular $\Gamma-$incline, $M$ is also a commutative $\Gamma-$incline.*

*Proof.* Let $a, b \in M, \gamma \in \Gamma$ and $a \leq b$. We have $b\gamma a \leq a$. Let $a$ and $b\gamma a$ be $\delta$ and $\beta$ idempotent, respectively. By Theorem 3.29, $a$ and $b\gamma a$ are $\delta + \beta$ idempotents. Putting $\delta + \beta = \alpha$, by Corollary 3.34, we have

$$\begin{aligned} &(b\gamma a)\alpha a = a\alpha(b\gamma a), a\alpha b = a = a\alpha a \\ &\Rightarrow b\gamma(a\alpha a) = (a\alpha b)\gamma a \\ &\Rightarrow b\gamma a = a\gamma a \\ a \leq b &\Rightarrow a\gamma a \leq a\gamma b. \\ \text{Therefore } \ & b\gamma a \leq a\gamma b. \end{aligned}$$

We have $a\gamma b \leq a$. Let $a$ be an $\alpha-$idempotent and $a\gamma b$ be a $\beta-$idempotent. By Theorem 3.29, $a$ and $a\gamma b$ are $\alpha + \beta$ idempotents. Putting $\alpha + \beta = \delta$ and by Corollary 3.34, we have

$$\begin{aligned} &(a\gamma b)\delta a = a\delta(a\gamma b) \text{ and } a\delta b = b\delta a = a \\ &\Rightarrow a\gamma(b\delta a) = (a\delta a)\gamma b \\ &\Rightarrow a\gamma a = a\gamma b \\ a \leq b &\Rightarrow a\gamma a \leq b\gamma a. \end{aligned}$$

Therefore $a\gamma b \leq b\gamma a$. Hence $a\gamma b = b\gamma a$, $M$ is a commutative $\Gamma-$incline. □

**Theorem 3.37.** *Let $M$ be a $\Gamma-$incline, $a, b \in M$. If there exist $\alpha, \beta, \gamma, \delta \in \Gamma$ and $x, y \in M$ such that $a = a\alpha x\gamma a$, $b = b\beta y\delta b$ and $a\alpha M = b\beta M$ then $a = b$.*

*Proof.* Suppose $a$ and $b$ are regular elements of a $\Gamma-$incline $M$ and there exist $\alpha, \beta, \gamma, \delta \in \Gamma$ and $x, y \in M$ such that

$$\begin{aligned} & a = a\alpha x\gamma a, b = b\beta y\delta b \\ & a = a\alpha x\gamma a \leq a\alpha x \leq a \\ \Rightarrow & a\alpha x = a \Rightarrow a \in a\alpha M \\ \Rightarrow & a \in b\beta M \\ \Rightarrow & a = b\beta z \text{ for some } z \in M \\ \Rightarrow & a \leq b. \end{aligned}$$

Similarly we can prove $b \leq a$. Hence $a = b$. □

**Theorem 3.38.** *Let $M$ be a $\Gamma-$incline. $M$ is a regular $\Gamma-$incline if and only if $M$ is an idempotent $\Gamma-$incline.*

*Proof.* Suppose $M$ is a regular $\Gamma-$incline. Suppose $a \in M$ then $a$ is a regular element. Since $a$ is a regular element, there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. By Theorem 3.32, $a = a\alpha x = x\beta a$.
Now $a = a\alpha x\beta a = a\alpha a$ and $a = a\alpha x\beta a = a\beta a$.
Therefore $a$ is $\alpha-$idempotent and also $\beta-$idempotent. Hence $M$ is an idempotent $\Gamma-$incline.

Conversely suppose that $a \in M$ is an idempotent, $a = a\alpha a = a\alpha a\alpha a$. Hence $M$ is a regular $\Gamma-$incline. □

**Theorem 3.39.** *If $M$ is a regular $\Gamma-$incline with $1$ being its greatest element, then $1$ is the unity.*

*Proof.* Suppose $M$ is a regular $\Gamma-$incline with 1 being the greatest element of $M$. Let $a \in M$. Since $a$ is regular, by Theorem 3.38, there exists $\alpha \in \Gamma$ such that $a\alpha a = a$.

$$\begin{aligned} \text{We have } \quad & a \leq 1 \\ \Rightarrow & a + 1 = 1 \\ \Rightarrow & a\alpha(a+1) = a\alpha 1 \\ \Rightarrow & a\alpha a + a\alpha 1 = a\alpha 1 \\ \Rightarrow & a + a\alpha 1 = a\alpha 1 \\ \Rightarrow & a = a\alpha 1 \\ \text{Therefore } & a = a\alpha 1. \end{aligned}$$

Similarly, we can prove $1\alpha a = a$. Hence 1 is the unity element of $M$. □

**Theorem 3.40.** *Let $M$ be a $\Gamma-$incline and $\alpha, \beta \in \Gamma$. If $b, c \in M$ are $\alpha, \beta$ idempotents, respectively, then $b\alpha c = b\beta c$.*

*Proof.* Suppose $b, c \in M$ are $\alpha, \beta$ idempotents, respectively; then we have $b\alpha b = b$, $c\beta c = c$. $b\alpha c = (b\alpha b)\alpha(c\beta c) = b\alpha(b\alpha c)\beta c \leq b\beta c$. Similarly we can prove $b\beta c \leq b\alpha c$. Hence $b\alpha c = b\beta c$. □

**Theorem 3.41.** *Let $M$ be a $\Gamma-$incline and $\alpha, \beta \in \Gamma$. If $b, c \in M$ are $\alpha, \beta$ idempotents, respectively, $a \in M$, $a + b = a + c$ and $b\alpha a = a\beta c$ then $b = c$.*

*Proof.* Suppose $b, c \in M$ are $\alpha, \beta$ idempotents, respectively; $a \in M, a+b = a+c$ and $b\alpha a = a\beta c$. Then we have $b\alpha b = b, c\beta c = c$. By Theorem 3.40, $b\alpha c = b\beta c$.

$$\begin{aligned} a + b = a + c &\Rightarrow b\alpha(a + b) = b\alpha(a + c) \\ &\Rightarrow b\alpha a + b = b\alpha a + b\alpha c \\ &\Rightarrow b = a\beta c + b\beta c \\ &\Rightarrow b = (a + b)\beta c \\ &\Rightarrow b = (a + c)\beta c \\ &\Rightarrow b = a\beta c + c\beta c \\ &\Rightarrow b = a\beta c + c \\ &\Rightarrow b = c. \end{aligned}$$

Hence the Theorem. □

**Theorem 3.42.** *A pre-integral $\Gamma-$incline is an integral $\Gamma-$incline.*

*Proof.* Let $M$ be a pre-integral $\Gamma-$incline. Suppose $a, b \in M, \alpha \in \Gamma, a\alpha b = 0, b \neq 0 \Rightarrow a\alpha b = 0\alpha b \Rightarrow a = 0$. Hence $M$ is an integral $\Gamma-$incline. □

**Theorem 3.43.** *Let $M$ be a $\Gamma-$incline and $x \in M$ be a regular element. There exist $\alpha, \beta \in \Gamma$ such that for all $y \in M$, $x\alpha y$ is regular if and only if $y\beta x\alpha y = x\alpha y$.*

*Proof.* Let $x \in M$ be a regular element. Then there exist $a \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$. By Theorem 3.38, $x$ is $\alpha$ idempotent and $\beta$ idempotent. Suppose $x\alpha y$ is regular. Then there exists $\delta \in \Gamma$ such that $(x\alpha y)\delta(x\alpha y) = x\alpha y$.

$$\begin{aligned} \text{We have } & x\alpha y\delta x \leq y \\ \Rightarrow & \ y + x\alpha y\delta x = y \\ y\beta(x\alpha y) &= [y + x\alpha y\delta x]\beta(x\alpha y) \\ &= y\beta x\alpha y + x\alpha y\delta x\beta x\alpha y \\ &= y\beta x\alpha y + x\alpha y\delta x\alpha y \\ &= y\beta x\alpha y + x\alpha y \\ &= x\alpha y. \end{aligned}$$

Conversely, suppose that $y\beta x\alpha y = x\alpha y$.
$x\alpha(y\beta x\alpha y) = x\alpha(x\alpha y) = (x\alpha x)\alpha y = x\alpha y$. Hence the Theorem. □

**Theorem 3.44.** *Let $M$ be a field $\Gamma-$incline with unity 1. Then 1 is the only one element which is an invertible.*

*Proof.* Let $M$ be a field $\Gamma-$ incline with unity 1. Let $0 \neq a \in M$. By Definition 3.14 there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a$ and there exist $b \in M, \beta \in \Gamma$ such that $a\beta b = 1 = b\beta a$. We have $a\alpha 1 = a \Rightarrow a \leq 1$.

$$\begin{aligned} \text{We have } a + a\beta b &= a \\ \Rightarrow a + 1 &= a \\ \Rightarrow 1 &\leq a. \end{aligned}$$

Therefore $a = 1$. Hence the Theorem. □

The proof of the corollary follows from Theorem 3.44.

**Corollary 3.45.** *Let $M$ be a field $\Gamma-$incline. Then $M = \{0, 1\}$ or $M$ has only one element (which is both zero and unity).*

## 4. Field $\Gamma-$Semiring

In this section, we introduce the notion of unity element of $\Gamma-$semiring, invertible element of $\Gamma-$semiring, field $\Gamma-$semiring, simple $\Gamma-$semiring, integral $\Gamma-$semiring, pre-integral $\Gamma-$semiring and we study the relations between them.

**Definition 4.1.** Let $M$ be a $\Gamma-$semiring. An element $1 \in M$ is said to be an unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

**Definition 4.2.** In a $\Gamma-$semiring $M$ with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1$ $(a\alpha b = 1)$.

**Definition 4.3.** In a $\Gamma-$semiring $M$ with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

**Definition 4.4.** A $\Gamma-$semiring $M$ is said to be a simple $\Gamma-$semiring if it has no proper ideals other than the zero ideal.

**Definition 4.5.** A nonzero element $a$ in a $\Gamma-$semiring $M$ is said to be a zero divisor if there exist a nonzero element $b \in M$ and $\alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

**Definition 4.6.** A $\Gamma-$semiring $M$ is said to be a field $\Gamma-$semiring if $M$ is a commutative $\Gamma-$semiring with unity 1, zero element 0 and every nonzero element of $M$ is invertible.

**Definition 4.7.** A $\Gamma-$semiring $M$ with unity 1 and zero element 0 is called an integral $\Gamma-$semiring if it has no zero divisors.

**Definition 4.8.** A $\Gamma-$semiring $M$ with zero element 0 is said to satisfy cancellation law if for all $a, b, c \in M, \alpha \in \Gamma$ we have that $a \neq 0, a\alpha b = a\alpha c$ and $b\alpha a = c\alpha a$ implies $b = c$.

**Definition 4.9.** A $\Gamma-$semiring $M$ with unity 1 and zero element 0 is called a pre-integral $\Gamma-$semiring if it satisfies the cancellation law.

**Example 4.10.** Let $M$ and $\Gamma$ be the additive semigroup of all non-negative rational numbers and the additive semigroup of all positive rational numbers, respectively. Define the ternary operation $M \times \Gamma \times M \to M$ by $(a, \alpha, b) \to a\alpha b$, using the usual multiplication. Now $M$ is a field $\Gamma-$semiring.

**Theorem 4.11.** *A commutative $\Gamma-$semiring $M$ with unity and zero is a field $\Gamma-$semiring if and only if for any nonzero elements $a, b \in M$, there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a\alpha x\beta b = b$.*

*Proof.* Let $M$ be a field $\Gamma-$semiring. Let $a, b$ be nonzero elements of $M$. Then there exists $\beta \in \Gamma$ such that $1\beta b = b$ and there exist $\alpha \in \Gamma, x \in M$ such that $a\alpha x = 1$.

$$\begin{aligned} \text{Now} \quad & a\alpha x = 1 \\ \Rightarrow \quad & a\alpha x\beta b = 1\beta b \\ \Rightarrow \quad & a\alpha x\beta b = b \end{aligned}$$

Conversely, suppose that for any nonzero elements $a, b \in M$, there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a\alpha x\beta b = b$. Let $0 \neq x \in M$. Then there exist $\alpha, \beta \in \Gamma$, $y \in M$, such that $x\alpha y\beta 1 = 1 \Rightarrow x\alpha(y\beta 1) = 1$. Hence $M$ is a field $\Gamma-$semiring. $\square$

**Theorem 4.12.** *Let $M$ be a $\Gamma-$semiring in which nonzero elements are invertible, with unity and zero, satisfying the identity $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. Then*

*(i).* $a + 1 = a$, *for all* $a \in M \setminus \{0\}$.

*(ii).* $M$ *is an idempotent* $\Gamma-$*semiring.*

*(iii).* $M$ *is additive idempotent.*

*Proof.* Suppose $M$ is a $\Gamma-$semiring in which nonzero elements are invertible, with unity and zero, satisfying the identity, $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$.

(i). Let $a \in M \setminus \{0\}$. Then there exist $a^{-1} \in M, \alpha \in \Gamma$ such that $a\alpha a^{-1} = 1$. We have $a + a\alpha a^{-1} = a \Rightarrow a + 1 = a$. Hence (i).

(ii). From (i), $a + 1 = a$, $a \in M$. Since $a \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.

$$\begin{aligned} & a + 1 = a \\ \Rightarrow\ & a\gamma a + a\gamma 1 = a\gamma a \\ \Rightarrow\ & a\gamma a + a = a\gamma a \\ \Rightarrow\ & a = a\gamma a. \end{aligned}$$

Hence $M$ is an idempotent $\Gamma-$semiring.

(iii). We have

$$\begin{aligned} & a + a\alpha b = a \quad \text{for all } a, b \in M, \alpha \in \Gamma \\ \Rightarrow\ & a + a\alpha 1 = a, \quad \text{for all } a \in M, \alpha \in \Gamma \\ \Rightarrow\ & a + a = a, \quad \text{for all } a \in M. \end{aligned}$$

Hence $M$ is an additive idempotent $\Gamma-$semiring.

□

**Corollary 4.13.** *A field $\Gamma-$semiring with the identity $a\alpha b + a = a$ for all $a, b \in M, \alpha \in \Gamma$ is a field $\Gamma-$incline*

**Theorem 4.14.** *If $M$ is a $\Gamma-$semiring with unity and $a \in M$ is a left invertible then $a$ is regular.*

*Proof.* Let $M$ be a $\Gamma-$semiring with unity 1. Suppose $a \in M$ is left invertible, there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1$. Since 1 is unity there exists $\delta \in \Gamma$ such that $a\delta 1 = 1\delta a = a$.

$$\begin{aligned} & a\delta 1 = a \\ \Rightarrow & a\delta(b\alpha a) = a. \\ \Rightarrow & a\delta b\alpha a = a. \end{aligned}$$

Hence $a$ is a regular element. □

**Corollary 4.15.** *If $M$ is a $\Gamma-$semiring with unity and $a \in M$ is invertible then $a$ is regular.*

**Theorem 4.16.** *If $M$ is a field $\Gamma-$semiring then $M$ is a regular $\Gamma-$semiring.*

*Proof.* Let $M$ be a field $\Gamma-$semiring. Then each nonzero element is invertible. By Corollary 4.15, each nonzero element is regular; since $0 = 0\alpha a\beta 0$ for all $\alpha, \beta \in \Gamma$ and $a \in M$, the zero element is also regular. Therefore $M$ is a regular $\Gamma-$semiring. □

**Theorem 4.17.** *Let $M$ be a $\Gamma-$semiring with unity 1. If $a, b \in M$, $a$ is left invertible, $a\delta b$ is an idempotent, where $1\delta b = b, \delta \in \Gamma$ then $b$ is a regular element.*

*Proof.* Let $a, b \in M, a$ be left invertible and $a\delta b$ be $\beta-$idempotent. There exist $d \in M, \gamma \in \Gamma$ such that $d\gamma a = 1$.

$$\begin{aligned} d\gamma a = 1 \Rightarrow\ & d\gamma a\delta b = 1\delta b \\ \Rightarrow\ & d\gamma a\delta b = b. \\ \text{Since }\ & a\delta b \text{ is } \ \beta - \text{idempotent, we have} \\ & a\delta b\beta a\delta b = a\delta b \\ \Rightarrow\ & d\gamma a\delta b\beta a\delta b = d\gamma a\delta b \\ \Rightarrow\ & b\beta a\delta b = b. \end{aligned}$$

Hence $b$ is a regular element. □

The proof of the following theorem is similar to Theorem 4.17

**Theorem 4.18.** *Let $M$ be a $\Gamma-$semiring with unity $1$. If $a, b \in M,\ b$ is right invertible, $a\delta b$ is an idempotent, where $a\delta 1 = a, \delta \in \Gamma$ then $a$ is a regular element.*

**Theorem 4.19.** *A pre-integral $\Gamma-$semiring is an integral $\Gamma-$semiring.*

*Proof.* Let $M$ be a pre-integral $\Gamma-$semiring. Suppose $a, b \in M, \alpha \in \Gamma, a\alpha b = 0, b \neq 0 \Rightarrow a\alpha b = 0\alpha b \Rightarrow a = 0$. Hence $M$ is an integral $\Gamma-$semiring. □

**Theorem 4.20.** *If $M$ is a field $\Gamma-$semiring with $1\alpha 1 \neq 0$ for all $\alpha \in \Gamma, M$ is a pre-integral $\Gamma-$semiring.*

*Proof.* Let $M$ be a field $\Gamma-$semiring with $1\alpha 1 \neq 0$ for all $\alpha \in \Gamma$. Suppose $a \neq 0$ and $a\alpha b = a\alpha c$, where $a, b, c \in M, \alpha \in \Gamma$. There exists $\delta, \beta \in \Gamma$ such that $1\delta b = b$ and $1\beta c = c$.

$$\begin{aligned} &\Rightarrow a\alpha(1\delta b) = a\alpha(1\beta c) \\ &\Rightarrow (a\alpha 1)\delta b = (a\alpha 1)\beta c. \end{aligned}$$

If $a \neq 0$ then there exists $\beta \in \Gamma$ such that $a^{-1}\beta a = 1$. Suppose $a\alpha 1 = 0$. Then $a^{-1}\beta a\alpha 1 = 0$ i.e. $1\alpha 1 = 0$. Therefore $a\alpha 1 \neq 0$. Then there exist $\gamma \in \Gamma, d \in M$ such that $d\gamma(a\alpha 1) = 1$

$$\begin{aligned} &a\alpha b = a\alpha c \\ \Rightarrow\ &d\gamma(a\alpha 1)\delta b = d\gamma(a\alpha 1)\beta c \\ \Rightarrow\ &1\delta b = 1\beta c \\ \Rightarrow\ &b = c. \end{aligned}$$

Hence the field $\Gamma-$semiring $M$ is a pre-integral $\Gamma-$semiring . □

**Theorem 4.21.** *Any commutative finite pre-integral $\Gamma-$semiring $M$ is a field $\Gamma-$semiring $M$.*

*Proof.* Let $M = \{a_1, a_2, \cdots, a_n\}$ and $0 \neq a \in M, \alpha \in \Gamma$. We consider the $n$ products $a\alpha a_1, a\alpha a_2, \cdots, a\alpha a_n$. These products are all distinct, since $a\alpha a_i = a\alpha a_j \Rightarrow\ a_i = a_j$. Since $1 \in M$, there exists $a_i \in M$ such that $a\alpha a_i = 1$. Therefore $a$ has multiplicative inverse. Hence any commutative finite pre-integral $\Gamma-$semiring $M$ is a field $\Gamma-$semiring. □

**Theorem 4.22.** *Let $M$ be a $\Gamma-$semiring with unity $1$ and zero element $0$. If $I$ is an ideal of the $\Gamma-$semiring $M$ containing an invertible element then $I = M$.*

*Proof.* Let $I$ be an ideal of $M$ containing an invertible element $u$. Let $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha 1 = x$. We have $x\alpha u \in I$, since $I$ is an ideal. Since $u$ is an invertible element, there exist $\delta \in \Gamma, t \in M$ such that $u\delta t = 1 \Rightarrow x\alpha u\delta t = x\alpha 1 = x \in I$. Hence $I = M$. □

**Theorem 4.23.** *Every field* $\Gamma-$*semiring with* $1\alpha 1 \neq 0$ *for all* $\alpha \in \Gamma$ *is an integral* $\Gamma-$*semiring.*

*Proof.* Let $a, b \in M$ and $a\alpha b = 0, \alpha \in \Gamma$ and $a \neq 0$. Since $a \neq 0$ there exists $\beta \in \Gamma$ such that $a^{-1}\beta a = 1$.

$$\begin{aligned} a\alpha b = 0 &\Rightarrow a^{-1}\beta(a\alpha b) = a^{-1}\beta 0 \\ &\Rightarrow (a^{-1}\beta a)\alpha b = 0 \\ &\Rightarrow 1\alpha b = 0 = 1\alpha 0 \end{aligned}$$

By Theorem 4.20, $b = 0$. Hence $M$ is an integral $\Gamma-$semiring. □

**Theorem 4.24.** $M$ *is a field* $\Gamma-$*semiring with* $1\alpha 1 \neq 0$ *for all* $\alpha \in \Gamma$ *if and only if* $M$ *is an integral, simple and commutative* $\Gamma-$*semiring.*

*Proof.* Let $M$ be a field $\Gamma-$semiring. Let $I$ be a nonzero ideal of field $\Gamma-$semiring $M$. Every nonzero element of $M$ is an invertible. By Theorem 4.22, we have $I = M$. Therefore the field $\Gamma-$semiring $M$ contains no nonzero ideals other than $M$. Hence such a field $\Gamma-$semiring is a simple $\Gamma-$semiring. By Theorem 4.23, $M$ is an integral $\Gamma-$semiring.

Conversely, suppose that $M$ is an integral, simple and commutative $\Gamma-$semiring. Let $0 \neq a \in M, \alpha \in \Gamma$. Consider $a\alpha M, a\alpha M \neq \{0\}$, since $M$ is an integral $\Gamma-$semiring. Clearly $a\alpha M$ is a nonzero ideal of $M \Rightarrow a\alpha M = M$, since $M$ is a simple $\Gamma-$semiring. Therefore, there exists $b \in M$ such that $a\alpha b = 1$. Hence $M$ is a field $\Gamma-$semiring. Since it is integral, we have $1\alpha 1 \neq 0$. □

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