Novi Sad J. Math.
Vol. 45, No. 2, 2015, 143-153

WEAKLY SYMMETRIC AND WEAKLY-RICCI SYMMETRIC LP-SASAKIAN MANIFOLDS ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

Ajit Barman[1]

Abstract

The object of the present paper is to study a necessary condition for the existence of weakly symmetric and weakly Ricci-symmetric LP-Sasakian manifolds admitting a quarter-symmetric metric connection.

AMS Mathematics Subject Classification (2010): 53C15, 53C25

Key words and phrases: Quarter-symmetric metric connection; Levi-Civita connection; LP-Sasakian manifold; weakly symmetric manifold; weakly Ricci-symmetric manifold

1 Introduction

In 1975, Golab [7] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. A liner connection ∇ on an n-dimensional Riemannian manifold (M^n, g) is called a quarter-symmetric connection [7] if its torsion tensor T satisfies $T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y$, where η is a 1-form and ϕ is a (1,1) tensor field.

In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [6]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition $(\bar{\nabla}_X g)(Y,Z) = 0$, for all $X, Y, Z \in \chi(M^n)$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection and the quarter-symmetric metric connection have been studied by De and Kamilya ([3], [2]), De and Mondal [4] and many others.

A relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on (M,g) has been obtained by De, Özgür and Sular [5] which is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{1.1}$$

Further, a relation between the curvature tensor $\bar{R}$ of the quarter-symmetric metric connection $\bar{\nabla}$ and the curvature tensor R of the Levi-Civita connection

[1]Department of Mathematics, Kabi Nazrul Mahavidyalaya, P.O.-Sonamura-799181, P.S.-Sonamura, Dist.- Sepahijala, Tripura, India. e-mail: ajitbarmanaw@yahoo.in

∇ is given by

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z - (\nabla_X \eta)(Y)\phi Z + (\nabla_Y \eta)(X)\phi Z \\ &+ \eta(X)(\nabla_Y \phi)(Z) - \eta(Y)(\nabla_X \phi)(Z). \end{aligned} \tag{1.2}$$

The notions of weakly symmetric and Ricci-symmetric Riemannian manifolds was introduced by Tamássy and Binh ([11], [10]). In [1] De and Gazi introduced the notion of almost pseudo-symmetric Riemannian manifolds.

In the present paper we discuss the existence of weakly symmetric and weakly Ricci-symmetric LP-Sasakian manifolds admitting a quarter-symmetric metric connection. The paper is organized as follows: After introduction in section 2, we give a brief account of the LP-Sasakian manifolds. Section 3 is devoted to obtaining the relation between the curvature tensor of the LP-Sasakian manifolds with respect to the quarter-symmetric metric connection and the Levi-Civita connection. Section 4 deals with the weakly symmetric LP-Sasakian manifold which admits a quarter-symmetric metric connection and prove that there is no weakly symmetric LP-Sasakian manifolds ($n > 3$) admitting a quarter-symmetric metric connection, unless $A + C + D$ vanishes everywhere. In the next section, we investigate the weakly Ricci-symmetric LP-Sasakian manifolds admitting a quarter-symmetric metric connection. Finally, we construct an example for the existence of weakly symmetric and weakly Ricci-symmetric LP-Sasakian manifolds with respect to the quarter-symmetric metric connection.

2 LP-Sasakian Manifold

A differentiable manifold of dimension (n+1) is called Lorentzian Para-Sasakian (briefly, LP-Sasakian) ([9],[8]) if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \quad \phi^2(X) = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\phi X) = 0, \quad \phi(\xi) = 0, \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(X), \tag{2.3}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

$$(\nabla_X \eta)(Y) = g(X, \phi Y) = g(\phi X, Y), \tag{2.6}$$

for all vector fields $X, Y \in \chi(M)$.

Further, on such an LP-Sasakian manifold with the structure (ϕ, ξ, η, g) the following relations hold:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.8}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.9}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.10}$$

where S is the Ricci tensor with respect to Levi-Civita connection.

3 Curvature tensor of a LP-Sasakian manifold with respect to the quarter-symmetric metric connection

Using (2.4) and (2.6) in (1.2), we get

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + \eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. \end{aligned} \tag{3.1}$$

From (3.1) we obtain

$$\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z. \tag{3.2}$$

Putting $X = \xi$ in (3.1) and using (2.1), (2.2) and (2.8), we have

$$\bar{R}(\xi, Y)Z = -2\eta(Z)Y - 2\eta(Y)\eta(Z)\xi. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\bar{R}(Y, \xi)Z = 2\eta(Z)Y + 2\eta(Y)\eta(Z)\xi. \tag{3.4}$$

Again putting $Z=\xi$ (3.1) and using (2.1), (2.2) and (2.7), it follows that

$$\bar{R}(X,Y)\xi = 2\eta(Y)X - 2\eta(X)Y. \tag{3.5}$$

Taking a frame field from (3.1), we get

$$\bar{S}(Y,Z) = S(Y,Z) - g(Y,Z) - n\eta(Y)\eta(Z). \tag{3.6}$$

From (3.6) we get

$$\bar{S}(Y,Z) = \bar{S}(Z,Y).$$

Putting $Z=\xi$ in (3.6) and using (2.1), (2.2) and (2.10), we have

$$\bar{S}(Y,\xi) = 2(n-1)\eta(Y). \tag{3.7}$$

Combining (1.1) and (2.6), it follows that

$$(\bar{\nabla}_X\eta)(Y) = g(X,\phi Y). \tag{3.8}$$

Again combining (1.1) and (2.5), it follows that

$$\bar{\nabla}_X\xi = \phi X. \tag{3.9}$$

From the above discussions we can state the following theorem:

Theorem 3.1. *For a LP-Sasakian manifold M with respect to the quarter-symmetric metric connection $\bar{\nabla}$*
(i) The curvature tensor $\bar{R}$ is given by (3.1),
(ii) The Ricci tensor $\bar{S}$ is given by (3.6),
(iii) $\bar{R}(\xi,Y)Z = -2\eta(Z)Y - 2\eta(Y)\eta(Z)\xi$,
(v) $\bar{R}(X,Y)\xi = 2\eta(Y)X - 2\eta(X)Y$,
(iv) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,
(vi) The Ricci tensor $\bar{S}$ is symmetric,
(vii) $\bar{S}(Y,\xi) = 2(n-1)\eta(Y)$,
(viii) $(\bar{\nabla}_X\eta)(Y) = g(X,\phi Y)$,
(ix) $\bar{\nabla}_X\xi = \phi X$.

4 Weakly symmetric LP-Sasakian manifolds admitting a quarter-symmetric metric connection

A non-flat Riemannian manifold M $(n>3)$ is said to be weakly symmetric [11] if there exist 1-forms A, B, C and D which satisfy the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)V &= A(X)R(Y,Z)V + B(Y)R(X,Z)V + C(Z)R(Y,X)V \\ &\quad + D(V)R(Y,Z)X + g(R(Y,Z)V,X)P,\end{aligned}$$

for all vectors fields $X, Y, Z, V \in \chi(M)$ and where A, B, C, D and P are not simultaneously zero. The 1-forms are called the associated 1-forms of the manifold.

We give the following definition:

A non-flat Riemannian manifold M $(n > 3)$ is said to be weakly symmetric with respect to the quarter-symmetric metric connection if there exist 1-forms A, B, C and D which satisfy the condition

$$\begin{aligned}(\bar{\nabla}_X \bar{R})(Y, Z)V &= A(X)\bar{R}(Y, Z)V + B(Y)\bar{R}(X, Z)V + C(Z)\bar{R}(Y, X)V \\ &\quad + D(V)\bar{R}(Y, Z)X + g(\bar{R}(Y, Z)V, X)P.\end{aligned} \tag{4.1}$$

Let M be a weakly symmetric LP-Sasakian manifold admitting a quarter-symmetric metric connection. So the equation (4.1) holds.

Taking a frame field from (4.1), we get

$$\begin{aligned}(\bar{\nabla}_X \bar{S})(Z, V) &= A(X)\bar{S}(Z, V) + B(\bar{R}(X, Z)V) + C(Z)\bar{S}(X, V) \\ &\quad + D(V)\bar{S}(X, Z) + E(\bar{R}(X, V)Z),\end{aligned} \tag{4.2}$$

where E is defined by $E(X) = g(X, P)$.

Putting $V = \xi$ in (4.2) and using (3.4), (3.5) and (3.7), we have

$$\begin{aligned}&(\bar{\nabla}_X \bar{S})(Z, \xi) \\ &\quad = 2(n-1)A(X)\eta(Z) + 2B(X)\eta(Z) - 2B(Z)\eta(X) \\ &\qquad + 2(n-1)C(Z)\eta(X) + D(\xi)[S(X, Z) - g(X, Z) - n\eta(X)\eta(Z)] \\ &\qquad + 2E(X)\eta(Z) + 2E(\xi)\eta(X)\eta(Z).\end{aligned} \tag{4.3}$$

We know that

$$(\bar{\nabla}_X \bar{S})(Z, V) = \bar{\nabla}_X \bar{S}(Z, V) - \bar{S}(\bar{\nabla}_X Z, V) - \bar{S}(Z, \bar{\nabla}_X V). \tag{4.4}$$

Putting $V = \xi$ in (4.4) and using (3.7), (3.8) and (3.9), we obtain

$$(\bar{\nabla}_X \bar{S})(Z, \xi) = (2n-1)g(\phi X, Z) - S(\phi X, Z). \tag{4.5}$$

Combining (4.3) and (4.5), it follows that

$$\begin{aligned}&(2n-1)g(\phi X, Z) - S(\phi X, Z) \\ &\quad = 2(n-1)A(X)\eta(Z) + 2B(X)\eta(Z) \\ &\qquad - 2B(Z)\eta(X) + 2(n-1)C(Z)\eta(X) + D(\xi)[S(X, Z) \\ &\qquad - g(X, Z) - n\eta(X)\eta(Z)] + 2E(X)\eta(Z) \\ &\qquad + 2E(\xi)\eta(X)\eta(Z).\end{aligned} \tag{4.6}$$

Putting $X = Z = \xi$ in (4.6), and using (2.1), (2.2) and (3.7), we get

$$-(2n-1)[A(\xi) + C(\xi) + D(\xi)] = 0. \tag{4.7}$$

Which implies that (since $n > 3$)

$$[A(\xi) + C(\xi) + D(\xi)] = 0.$$

Putting $Z = \xi$ in (4.2) and using (3.4), (3.5) and (3.7), we have

$$\begin{aligned}(\bar{\nabla}_X \bar{S})(\xi, V) \\ = \; & 2(n-1)A(X)\eta(V) + 2B(X)\eta(V) + 2B(\xi)\eta(X)\eta(V) \\ & + C(\xi)[S(X, V) - g(X, V) - n\eta(X)\eta(V)] \\ & + 2(n-1)D(V)\eta(X) + 2E(X)\eta(V) \\ & - 2E(V)\eta(X). \end{aligned} \tag{4.8}$$

Putting $Z = \xi$ in (4.4) and using (3.7), (3.8) and (3.9), we obtain

$$(\bar{\nabla}_X \bar{S})(\xi, V) = (2n-1)g(\phi X, V) - S(\phi X, V). \tag{4.9}$$

Combining (4.8) and (4.9), it follows that

$$\begin{aligned}(2n-1)g(\phi X, V) - S(\phi X, V) \\ = \; & 2(n-1)A(X)\eta(V) + 2B(X)\eta(V) + 2B(\xi)\eta(X)\eta(V) \\ & + C(\xi)[S(X, V) - g(X, V) - n\eta(X)\eta(V)] \\ & + 2(n-1)D(V)\eta(X) + 2E(X)\eta(V) \\ & - 2E(V)\eta(X). \end{aligned} \tag{4.10}$$

Putting $V = \xi$ in (4.10) and using (2.1), (2.2) and (3.7), we get

$$\begin{aligned} -2(n-1)A(X) - 2B(X) - 2B(\xi)\eta(X) & \\ + C(\xi)[(n-1)\eta(X) - \eta(X) + n\eta(X)] & \\ + 2(n-1)D(\xi)\eta(X) - 2E(X) - 2E(\xi)\eta(X) &= 0. \end{aligned} \tag{4.11}$$

Putting $X = \xi$ in (4.10) and using (2.1), (2.2) and (3.7), we have

$$\begin{aligned} 2(n-1)A(\xi)\eta(V) + 2(n-1)C(\xi)\eta(V) & \\ -2(n-1)D(V) + 2E(\xi)\eta(V) - 2E(V) &= 0. \end{aligned} \tag{4.12}$$

Interchanging V by X in (4.12), it follows that

$$\begin{aligned} 2(n-1)A(\xi)\eta(X) + 2(n-1)C(\xi)\eta(X) & \\ -2(n-1)D(X) + 2E(\xi)\eta(X) - 2E(X) &= 0. \end{aligned} \tag{4.13}$$

Adding (4.11) and (4.13), we obtain

$$
\begin{aligned}
(4.14)\qquad & -2(n-1)A(X) - B(X) - 2(n-1)B(\xi)\eta(X) \\
& +2(n-1)C(\xi)\eta(X) - 2(n-1)D(X) = 0.
\end{aligned}
$$

Putting $Z = \xi$ in (4.6) and using (2.1), (2.2) and (3.7), we get

$$
\begin{aligned}
(4.15)\qquad & 2(n-1)A(\xi)\eta(X) + 2B(\xi)\eta(X) + 2B(X) \\
& -2(n-1)C(X) + 2(n-1)D(\xi)\eta(X) = 0.
\end{aligned}
$$

Adding (4.14) and (4.15) and using (4.7), we have

$$-2(n-1)[A(X) + C(X) + D(X)] = 0.$$

Which implies that (since $n > 3$)

$$[A(X) + C(X) + D(X)] = 0.$$

We can state the following theorem:

Theorem 4.1. *There is no weakly symmetric LP-Sasakian manifold M ($n > 3$) admitting a quarter-symmetric metric connection, unless $A+C+D$ vanishes everywhere.*

5 Weakly Ricci-symmetric LP-Sasakian manifolds admitting a quarter-symmetric metric connection

A non-flat Riemannian manifold M ($n > 3$) is said to be weakly Ricci-symmetric [11] if there exist 1-forms α, β and γ which satisfy the condition

$$(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \gamma(Z)S(Y, X),$$

for all vectors fields $X, Y, Z \in \chi(M)$ and where α, β and γ are not simultaneously zero.

We give the following definition:

A non-flat Riemannian manifold M ($n > 3$) is said to be weakly Ricci-symmetric with respect to the quarter-symmetric metric connection if there exist 1-forms α, β and γ which satisfy the condition

$$(5.1)\qquad (\bar{\nabla}_X \bar{S})(Y, Z) = \alpha(X)\bar{S}(Y, Z) + \beta(Y)\bar{S}(X, Z) + \gamma(Z)\bar{S}(Y, X),$$

where $\bar{S}$ is a Ricci tensor with respect to the quarter-symmetric metric connection.

Let M be a weakly Ricci-symmetric LP-Sasakian manifold admitting a quarter-symmetric metric connection. So the equation (5.1) holds.

Putting $Z = \xi$ in (5.1), we get

$$(\bar{\nabla}_X \bar{S})(Y, \xi) = \alpha(X)\bar{S}(Y, \xi) + \beta(Y)\bar{S}(X, \xi) + \gamma(\xi)\bar{S}(Y, X). \tag{5.2}$$

Interchanging Z by Y in (4.5) and adding with (5.2), we have

$$\begin{aligned} &(2n-1)g(\phi X, Z) - S(\phi X, Z) \\ &= \alpha(X)\bar{S}(Y, \xi) + \beta(Y)\bar{S}(X, \xi) + \gamma(\xi)\bar{S}(Y, X). \end{aligned} \tag{5.3}$$

Putting $X = Y = \xi$ in (5.3) and using (2.1), (2.2) and (3.7), we obtain

$$2(n-1)[\alpha(\xi) + \beta(\xi) + \gamma(\xi)] = 0.$$

Which implies that (since $n > 3$)

$$\alpha(\xi) + \beta(\xi) + \gamma(\xi) = 0. \tag{5.4}$$

Putting $X = \xi$ in (5.3) and using (2.1), (2.2) and (3.7), it follows that

$$\beta(Y) = \beta(\xi)\eta(Y). \tag{5.5}$$

Putting $Y = \xi$ in (5.3) and using (2.1), (2.2) and (3.7), we get

$$\alpha(X) = \alpha(\xi)\eta(X). \tag{5.6}$$

Since $(\bar{\nabla}_\xi \bar{S})(\xi, X) = 0$, then from (5.1), we have

$$\gamma(X) = \gamma(\xi)\eta(X). \tag{5.7}$$

Interchanging Y by X in (5.5) and adding it with (5.6) and (5.7) and using (5.4), we obtain

$$\alpha(X) + \beta(X) + \gamma(X) = 0,$$

for any vector field X on M.

We state the following theorem :

Theorem 5.1. *There is no weakly Ricci-symmetric LP-Sasakian manifold M $(n > 3)$ admitting a quarter-symmetric metric connection, unless $\alpha + \beta + \gamma$ vanishes everywhere.*

6 Example of the weakly symmetric and weakly Ricci-symmetric LP-Sasakian manifold admitting a quarter-symmetric metric connection

Example 6.1. We consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in R^5\}$, where (x, y, z, u, v) are the standard coordinates in R^5.

We choose the vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z},\ \ e_2 = \frac{\partial}{\partial y},\ \ e_3 = \frac{\partial}{\partial z},\ \ e_4 = -2\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial z},\ \ e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_i, e_j) = 0,\ \ i \neq j,\ \ i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_4, e_4) = g(e_5, e_5) = 1, g(e_3, e_3) = -1.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_3),$$

for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi e_1 = e_2,\ \ \phi e_2 = e_1,\ \ \phi e_3 = 0,\ \ \phi e_4 = e_5,\ \ \phi e_5 = e_4.$$

Using the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$

$$\phi^2(Z) = Z + \eta(Z)e_3$$

and

$$g(\phi Z, \phi U) = g(Z, U) + \eta(Z)\eta(U),$$

for any $U, Z \in \chi(M)$.

Then we have

$[e_1, e_2] = -2e_3,\ \ [e_1, e_3] = [e_1, e_4] = [e_1, e_5] = [e_2, e_3] = 0,$

$[e_4, e_5] = -2e_3,\ \ [e_2, e_4] = [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = 0.$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula, which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) \\ & + g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{6.1}$$

Taking $e_3 = \xi$ and using Koszul's formula we get the following

$\nabla_{e_1} e_1 = 0,\ \ \nabla_{e_1} e_2 = -e_3,\ \ \nabla_{e_1} e_3 = e_2,\ \ \nabla_{e_1} e_4 = 0,\ \ \nabla_{e_1} e_5 = 0,$

$\nabla_{e_2} e_1 = e_3,\ \ \nabla_{e_2} e_2 = 0,\ \ \nabla_{e_2} e_3 = e_1,\ \ \nabla_{e_2} e_4 = 0,\ \ \nabla_{e_2} e_5 = 0,$

$\nabla_{e_3} e_1 = e_2,\ \ \nabla_{e_3} e_2 = e_1,\ \ \nabla_{e_3} e_3 = 0,\ \ \nabla_{e_3} e_4 = e_5,\ \ \nabla_{e_3} e_5 = e_4,$

$\nabla_{e_4} e_1 = 0,\ \ \nabla_{e_4} e_2 = 0,\ \ \nabla_{e_4} e_3 = e_5,\ \ \nabla_{e_4} e_4 = 0,\ \ \nabla_{e_4} e_5 = -e_3,$

$\nabla_{e_5} e_1 = 0,\ \ \nabla_{e_5} e_2 = 0,\ \ \nabla_{e_5} e_3 = e_4,\ \ \nabla_{e_5} e_4 = e_3,\ \ \nabla_{e_5} e_5 = 0.$

Using (1.1) in above equation, we obtain

$\bar{\nabla}_{e_1} e_1 = 0,\ \ \bar{\nabla}_{e_1} e_2 = -e_3,\ \ \bar{\nabla}_{e_1} e_3 = e_2,\ \ \bar{\nabla}_{e_1} e_4 = 0,\ \ \bar{\nabla}_{e_1} e_5 = 0,$

$\bar{\nabla}_{e_2} e_1 = e_3,\ \ \bar{\nabla}_{e_2} e_2 = 0,\ \ \bar{\nabla}_{e_2} e_3 = e_1,\ \ \bar{\nabla}_{e_2} e_4 = 0,\ \ \bar{\nabla}_{e_2} e_5 = 0,$

$\bar{\nabla}_{e_3}e_1 = 2e_2, \quad \bar{\nabla}_{e_3}e_2 = 2e_1, \quad \bar{\nabla}_{e_3}e_3 = 0, \quad \bar{\nabla}_{e_3}e_4 = 2e_5, \quad \bar{\nabla}_{e_3}e_5 = 2e_4,$
$\bar{\nabla}_{e_4}e_1 = 0, \quad \bar{\nabla}_{e_4}e_2 = 0, \quad \bar{\nabla}_{e_4}e_3 = e_5, \quad \bar{\nabla}_{e_4}e_4 = 0, \quad \bar{\nabla}_{e_4}e_5 = -e_3,$
$\bar{\nabla}_{e_5}e_1 = 0, \quad \bar{\nabla}_{e_5}e_2 = 0, \quad \bar{\nabla}_{e_5}e_3 = e_4, \quad \bar{\nabla}_{e_5}e_4 = e_3, \quad \bar{\nabla}_{e_5}e_5 = 0.$

With the help of the above results, we can easily calculate the non-vanishing components of the curvature tensor with respect to the Levi-Civita connection and the quarter-symmetric metric connection, respectively, as follows
$R(e_1, e_2)e_4 = 2e_5,\ R(e_1, e_2)e_5 = 2e_4,\ R(e_4, e_5)e_1 = 2e_2,$
$R(e_4, e_5)e_2 = 2e_1,\ R(e_1, e_2)e_2 = 3e_1,\ R(e_1, e_3)e_3 = -e_1,$
$R(e_2, e_1)e_1 = -3e_2,\ R(e_2, e_3)e_3 = -e_2,\ R(e_3, e_1)e_1 = e_3,$
$R(e_3, e_2)e_2 = -e_3,\ R(e_3, e_4)e_4 = e_3,\ R(e_3, e_5)e_5 = -e_3,$
$R(e_4, e_3)e_3 = -e_4,\ R(e_4, e_5)e_5 = 3e_4,\ R(e_5, e_3)e_3 = -e_5,$
$R(e_5, e_4)e_4 = -3e_5,\ R(e_1, e_3)e_2 = -e_1,\ R(e_1, e_4)e_2 = e_5,$
$R(e_1, e_4)e_5 = -e_2,\ R(e_1, e_5)e_2 = e_4,\ R(e_1, e_5)e_4 = e_2,$
$R(e_2, e_4)e_1 = -e_5,\ R(e_2, e_4)e_5 = -e_1,\ R(e_2, e_5)e_1 = -e_4,\ R(e_2, e_5)e_4 = e_1,$
and
$\bar{R}(e_1, e_2)e_4 = 2e_5,\ \bar{R}(e_1, e_2)e_5 = 2e_4,\ \bar{R}(e_4, e_5)e_1 = 2e_2,$
$\bar{R}(e_4, e_5)e_2 = 2e_1,\ \bar{R}(e_1, e_2)e_2 = 3e_1 - e_2,\ \bar{R}(e_1, e_3)e_3 = -2e_1,$
$\bar{R}(e_2, e_1)e_1 = -3e_2 + e_1,\ \bar{R}(e_2, e_3)e_3 = -2e_2,\ \bar{R}(e_3, e_1)e_1 = 2e_3,$
$\bar{R}(e_3, e_2)e_2 = -2e_3,\ \bar{R}(e_3, e_4)e_4 = 2e_3,\ \bar{R}(e_3, e_5)e_5 = -2e_3,$
$\bar{R}(e_4, e_3)e_3 = -2e_4,\ \bar{R}(e_4, e_5)e_5 = 3e_4 - e_5,\ \bar{R}(e_5, e_3)e_3 = -2e_5,$
$\bar{R}(e_5, e_4)e_4 = -3e_5 + e_4,\ \bar{R}(e_1, e_4)e_2 = e_5 - e_4,\ \bar{R}(e_1, e_4)e_5 = -e_2 + e_1,$
$\bar{R}(e_1, e_5)e_2 = e_4 - e_5,\ \bar{R}(e_1, e_5)e_4 = e_2 - e_1,\ \bar{R}(e_2, e_4)e_1 = -e_5 + e_4,$
$\bar{R}(e_2, e_4)e_5 = -e_1 + e_2,\ \bar{R}(e_2, e_5)e_1 = -e_4 + e_5,$
$\bar{R}(e_2, e_5)e_4 = e_1 - e_2,\ \bar{R}(e_3, e_5)e_4 = -e_3.$

From components of the curvature tensor, we can easily calculate components of the Ricci tensor with respect to the Levi-Civita connection and the quarter-symmetric metric connection, respectively, as follows:
$S(e_1, e_1) = S(e_3, e_3) = S(e_4, e_4) = -1, S(e_2, e_2) = 3, S(e_5, e_5) = 1$
and
$\bar{S}(e_1, e_1) = \bar{S}(e_3, e_3) = \bar{S}(e_4, e_4) = -2, \bar{S}(e_2, e_2) = \bar{S}(e_5, e_5) = 2.$

Using the above components of the curvature tensor with respect to the quarter-symmetric metric connection and the equation 4.1, we get

$$A(e_i) + C(e_i) + D(e_i) = 0, \forall i = 1, 2, 3, 4, 5.$$

Again using the above components of the Ricci tensor with respect to the quarter-symmetric metric connection and the equation 5.1, we obtain

$$\alpha(e_i) + \beta(e_i) + \gamma(e_i) = 0, \forall i = 1, 2, 3, 4, 5.$$

Thus, this example is the necessary condition for the existence of weakly symmetric and weakly Ricci-symmetric LP-Sasakian manifolds admitting a quarter-symmetric metric connection, that is, this example supports Theorem 4.1 and Theorem 5.1.

References

[1] De, U.C., Gazi, A.K., On almost pseudo symmetric manifolds, Ann. Univ.Sci.Budapest, Sec.Math., 51(2008), 53-68.

[2] De, U.C., Kamilya, D., Hypersurfaces of a Riemannian manifold with Ricci quarter-symmetric connection. Journal of Scientific Research, B.H.U. 45 (1995), 119-127.

[3] De, U.C., Kamilya, D., Some properties of a Ricci quarter-symmetric metric connection in a Riemannian manifold. Indian J. Pure Appl. Math. 26 (1995), 29-34.

[4] De, U.C., Mondal, A.K., Quarter-symmetric metric connection on 3-dimensional Quasi-Sasakian manifolds. Sut. journal of Mathematics 46 (2010), 35-52.

[5] De, U.C., Özgür, C., Sular, S., Quarter-symmetric metric connection in a Kenmotsu manifold. SUT Journal of Mathematics 44 (2008), 297-306.

[6] Friedmann, A., Schouten, J.A., Über die Geometric der halbsymmetrischen *Ü*bertragung. Math. Zeitschr. 21 (1924), 211-223.

[7] Golab, S., On semi-symmetric and quarter-symmetric liner connections. Tensor (N.S.) 29 (1975), 249-254.

[8] Matsumoto, K., Mihai, I., On a certain transformation in LP-Sasakian manifold. Tensor (N.S.) 47 (1988), 189-197.

[9] Matsumoto, K., On Lorentzian Para contact manifolds. Bull. of Yamagata Uni. Nat. Sci. 12 (1989), 151-156.

[10] Tamássy, L., Binh, T.Q., On weak symmetrics of Einstein and Sasakian manifolds. Tensor (N.S.) 53 (1993), 140-148.

[11] Tamássy, L., Binh, T.Q., On weakly symmetric and weakly projective symmetric Riemannian manifolds. Colloq. Math. Soc. J. Bolyai 50 (1989), 663-670.

Received by the editors October 12, 2014