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CANONICAL CONNECTIONS ON PARA-KENMOTSU MANIFOLDS[1]

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Abstract. In the context of para-Kenmotsu geometry, properties of the φ-conjugate connections of some canonical linear connections (Levi-Civita, Schouten-van Kampen, Golab and Zamkovoy canonical paracontact connections) are established, underlining the relations between them and between their structure and virtual tensors. The case of projectively and dual-projectively equivalent connections is also treated. In particular, it is proved that the structure tensor is invariant under dual-projective transformations.

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1. Introduction

The present paper is dedicated to a brief study of some canonical connections defined on a para-Kenmotsu manifold: Levi-Civita, Schouten-van Kampen, Golab and Zamkovoy canonical paracontact connections with a special view towards φ-conjugation. The structure and the virtual tensors attached to these connections are considered and in the last section, the invariance of the structure tensor under dual-projective transformations is proved. Note that the para-Kenmotsu structure was introduced by J. Wełyczko in [13] for 3-dimensional normal almost paracontact metric structures. A similar notion called P-Kenmotsu structure appears in the paper of B. B. Sinha and K. L. Sai Prasad [12].

Let M be a $(2n+1)$-dimensional smooth manifold, φ a tensor field of $(1,1)$-type, ξ a vector field, η a 1-form and g a pseudo-Riemannian metric on M of signature $(n+1,n)$.

Definition 1.1. [14] We say that (φ, ξ, η, g) defines an *almost paracontact metric structure* on M if

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \quad \varphi^2 = I_{\mathfrak{X}(M)} - \eta \otimes \xi, \quad g(\varphi\cdot, \varphi\cdot) = -g + \eta \otimes \eta$$

and φ induces on the $2n$-dimensional distribution $\mathcal{D} := \ker \eta$ an almost paracomplex structure P; i.e. $P^2 = I_{\mathfrak{X}(M)}$ and the eigensubbundles $\mathcal{D}^+$, $\mathcal{D}^-$, corresponding to the eigenvalues 1, -1 of P, respectively, have equal dimension n; hence $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$.

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In this case, $(M, \varphi, \xi, \eta, g)$ is called an *almost paracontact metric manifold*, φ *the structure endomorphism*, ξ *the characteristic vector field*, η *the paracontact form* and g *compatible metric.*

Examples of almost paracontact metric structures can be found in [8] and [6]. From the definition it follows that η is the g-dual of the unitary vector field ξ:

$$\eta(X) = g(X, \xi) \tag{1.1}$$

and φ is a g-skew-symmetric operator,

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{1.2}$$

Note that the canonical distribution $\mathcal{D}$ is φ-invariant since $\mathcal{D} = Im\varphi$. Moreover, ξ is orthogonal to $\mathcal{D}$ and therefore the tangent bundle splits orthogonally:

$$TM = \mathcal{D} \oplus \langle \xi \rangle. \tag{1.3}$$

An analogue of the Kenmotsu manifold [9] in paracontact geometry will be further considered.

Definition 1.2. [11] We say that the almost paracontact metric structure (φ, ξ, η, g) is para-Kenmotsu if the Levi-Civita connection ∇ of g satisfies $(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$, for any X, $Y \in \mathfrak{X}(M)$.

Example 1.3. Let $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Set

$$\varphi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi := -\frac{\partial}{\partial z}, \quad \eta := -dz,$$

$$g := dx \otimes dx - dy \otimes dy + dz \otimes dz.$$

Then (φ, ξ, η, g) defines a para-Kenmotsu structure on $\mathbb{R}^3$.

Properties of this structure will be given in the next Proposition.

Proposition 1.4. *[1] On a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$, the following relations hold:*

$$\nabla \xi = I_{\mathfrak{X}(M)} - \eta \otimes \xi \tag{1.4}$$

$$\eta(\nabla_X \xi) = 0, \tag{1.5}$$

$$R_\nabla(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{1.6}$$

$$\eta(R_\nabla(X, Y)W) = -\eta(X)g(Y, W) + \eta(Y)g(X, W), \tag{1.7}$$

$$\nabla \eta = g - \eta \otimes \eta, \tag{1.8}$$

$$L_\xi \varphi = 0, \quad L_\xi \eta = 0, \quad L_\xi g = 2(g - \eta \otimes \eta), \tag{1.9}$$

where R_∇ is the Riemann curvature tensor field of the Levi-Civita connection ∇ associated to g. Moreover, η is closed, the distribution $\mathcal{D}$ is involutive and the Nijenhuis tensor field of φ vanishes identically.

2. Canonical connections on $(M, \varphi, \xi, \eta, g)$

Let $(M, \varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. From [1], [11], [7], [14] we get the expressions and the relations with the para-Kenmotsu structure of the canonical connections on M we are interested in.

1. **Levi-Civita connection** ∇ satisfies [1]:

$$\nabla\varphi = g(\varphi\cdot,\cdot)\otimes\xi - \varphi\otimes\eta, \quad \nabla\xi = I_{\mathfrak{X}(M)} - \eta\otimes\xi, \quad \nabla\eta = g - \eta\otimes\eta, \quad \nabla g = 0, \tag{2.1}$$

its torsion and curvature being given by

$$T_{\nabla} = 0 \tag{2.2}$$

$$\eta(R_{\nabla}(X,Y)W) = -\eta(X)g(Y,W) + \eta(Y)g(X,W). \tag{2.3}$$

2. **Schouten-van Kampen connection** $\tilde{\nabla}$ equals to [11]:

$$\tilde{\nabla} := \nabla - I_{\mathfrak{X}(M)} \otimes \eta + g \otimes \xi \tag{2.4}$$

and satisfies [11]:

$$\tilde{\nabla}\varphi = 0, \quad \tilde{\nabla}\xi = 0, \quad \tilde{\nabla}\eta = 0, \quad \tilde{\nabla}g = 0, \tag{2.5}$$

its torsion and curvature being given by

$$T_{\tilde{\nabla}} = \eta \otimes I_{\mathfrak{X}(M)} - I_{\mathfrak{X}(M)} \otimes \eta \tag{2.6}$$

$$R_{\tilde{\nabla}}(X,Y)W = R_{\nabla}(X,Y)W - g(W,X)Y + g(Y,W)X - \eta(W)g(X,Y)\xi. \tag{2.7}$$

3. **Golab connection** ∇^G equals to [7]:

$$\nabla^G := \nabla - \eta \otimes \varphi \tag{2.8}$$

and satisfies [2]:

$$\nabla^G\varphi = \nabla\varphi, \quad \nabla^G\xi = \nabla\xi, \quad \nabla^G\eta = \nabla\eta, \quad \nabla^G g = \nabla g = 0, \tag{2.9}$$

its torsion and curvature being given by

$$T_{\nabla^G} = \varphi \otimes \eta - \eta \otimes \varphi \tag{2.10}$$

(2.11)
$$R_{\nabla^G}(X,Y)W = R_{\nabla}(X,Y)W + g(T,W)\xi - g(\xi,W)T, \ \ \textit{where } T := -T_{\nabla^G}(X,Y).$$

4. **Zamkovoy canonical paracontact connection** ∇^Z equals to [14]:

$$\nabla^Z_X Y := \nabla_X Y + \eta(X)\varphi Y - \eta(Y)\nabla_X\xi + (\nabla_X\eta)Y \cdot \xi \tag{2.12}$$

equivalent to

$$\nabla^Z = \nabla - I_{\mathfrak{X}(M)} \otimes \eta + g \otimes \xi + \eta \otimes \varphi \tag{2.13}$$

and satisfies [14]:

$$\nabla^Z \varphi = 0, \quad \nabla^Z \xi = 0, \quad \nabla^Z \eta = 0, \quad \nabla^Z g = 0, \tag{2.14}$$

its torsion and curvature being given by

$$T_{\nabla^Z} = \eta \otimes (\varphi + I_{\mathfrak{X}(M)}) - (\varphi + I_{\mathfrak{X}(M)}) \otimes \eta (= -T_{(\nabla^G)^*}) \tag{2.15}$$

$$R_{\nabla^Z}(X,Y)W = R_{\nabla}(X,Y)W + g(Y,W)X - g(X,W)Y. \tag{2.16}$$

3. φ-conjugate connections

In this section we shall consider the φ-conjugate connections of the four canonical connections presented above on a para-Kenmotsu manifold.

Recall that for an arbitrary linear connection $\bar{\nabla}$, the φ-conjugate connection of $\bar{\nabla}$ is defined by

$$\bar{\nabla}^{(\varphi)} := \bar{\nabla} + \varphi \circ \bar{\nabla}\varphi, \tag{3.1}$$

that is, $\bar{\nabla}^{(\varphi)}_X Y = \varphi(\bar{\nabla}_X \varphi Y) + \eta(\bar{\nabla}_X Y)\xi$, for any X, $Y \in \mathfrak{X}(M)$. Applying the φ-conjugation by n times, $n \in \mathbb{N}$, we can prove, by mathematical induction, that

$$\bar{\nabla}^{n(\varphi)}_X Y = \varphi^n(\bar{\nabla}_X \varphi^n Y) + \eta(\bar{\nabla}_X Y)\xi. \tag{3.2}$$

Therefore,

$$\bar{\nabla}^{n(\varphi)}_X Y - \bar{\nabla}^{(\varphi)}_X Y = \varphi^n(\bar{\nabla}_X \varphi^n Y) - \varphi(\bar{\nabla}_X \varphi Y), \tag{3.3}$$

for any $n \in \mathbb{N}$ and any X, $Y \in \mathfrak{X}(M)$.

Appearing in the theory of Courant algebroids, the *φ-torsion* of a linear connection can be expressed in terms of torsion of the φ-conjugate connection, namely:

Proposition 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold, $\bar{\nabla}$ a linear connection on M and $\bar{\nabla}^{(\varphi)}$ its φ-conjugate connection. Then:*

$$T_{(\bar{\nabla},\varphi)}(X,Y) = -T_{\bar{\nabla}^{(\varphi)}}(X,Y) + T_{\bar{\nabla}}(X,Y) + \varphi(T_{\bar{\nabla}}(\varphi X, Y)) + \varphi(T_{\bar{\nabla}}(X, \varphi Y)) -$$

$$-\varphi^2(T_{\bar{\nabla}}(X,Y)), \tag{3.4}$$

for any X, $Y \in \mathfrak{X}(M)$.

Proof. For any X, $Y \in \mathfrak{X}(M)$, the φ-torsion of $\bar{\nabla}$ is defined by:

$$T_{(\bar{\nabla},\varphi)}(X,Y) := \varphi(\bar{\nabla}_{\varphi X}Y - \bar{\nabla}_{\varphi Y}X) - [\varphi X, \varphi Y].$$

Observe that

$$T_{\bar{\nabla}}(X,Y) - T_{\bar{\nabla}^{(\varphi)}}(X,Y) = \bar{\nabla}_Y^{(\varphi)}X - \bar{\nabla}_Y X - \bar{\nabla}_X^{(\varphi)}Y + \bar{\nabla}_X Y =$$

$$= (\varphi \circ \bar{\nabla}\varphi)(Y,X) - (\varphi \circ \bar{\nabla}\varphi)(X,Y) =$$

$$= \varphi(\bar{\nabla}_Y \varphi X - \bar{\nabla}_X \varphi Y) - \varphi^2(\bar{\nabla}_Y X - \bar{\nabla}_X Y) =$$

$$= \varphi(T_{\bar{\nabla}}(Y,\varphi X) + \bar{\nabla}_{\varphi X}Y + [Y,\varphi X] - T_{\bar{\nabla}}(X,\varphi Y) - \bar{\nabla}_{\varphi Y}X - [X,\varphi Y]) +$$

$$+\varphi^2(T_{\bar{\nabla}}(X,Y) + [X,Y]) =$$

$$= \varphi(\bar{\nabla}_{\varphi X}Y - \bar{\nabla}_{\varphi Y}X) - \varphi[\varphi X, Y] - \varphi[X,\varphi Y] + \varphi^2[X,Y] +$$

$$+\varphi(T_{\bar{\nabla}}(Y,\varphi X) - T_{\bar{\nabla}}(X,\varphi Y)) + \varphi^2(T_{\bar{\nabla}}(X,Y)) :=$$

$$:= T_{(\bar{\nabla},\varphi)}(X,Y) + N_\varphi(X,Y) + \varphi(T_{\bar{\nabla}}(Y,\varphi X)) - \varphi(T_{\bar{\nabla}}(X,\varphi Y)) + \varphi^2(T_{\bar{\nabla}}(X,Y)) =$$

$$= T_{(\bar{\nabla},\varphi)}(X,Y) - \varphi(T_{\bar{\nabla}}(\varphi X,Y)) - \varphi(T_{\bar{\nabla}}(X,\varphi Y)) + \varphi^2(T_{\bar{\nabla}}(X,Y)).$$

□

Consider now the φ-conjugate connection of the Levi-Civita connection ∇

$$\nabla^{(\varphi)} := \nabla + \varphi \circ \nabla\varphi \tag{3.5}$$

which equals to

$$\nabla_X^{(\varphi)}Y = \nabla_X Y - \eta(Y)X + \eta(X)\eta(Y)\xi. \tag{3.6}$$

Note that $\nabla_X^{2(\varphi)}Y = \varphi(\nabla_X \varphi Y) + \eta(\nabla_X Y)\xi = \nabla_X^{(\varphi)}Y$, for any X, $Y \in \mathfrak{X}(M)$.

Proposition 3.2. *On a para-Kenmotsu manifold (M,φ,ξ,η,g), the φ-conjugate connection of the Levi-Civita connection ∇ satisfies:*

1. $\nabla^{(\varphi)}\varphi = g(\varphi\cdot,\cdot) \otimes \xi$;
2. $\nabla^{(\varphi)}\xi = 0$;
3. $\nabla^{(\varphi)}\eta = \nabla\eta$;
4. $\nabla^{(\varphi)}g = g(\cdot, \eta \otimes I_{\mathfrak{X}(M)} + I_{\mathfrak{X}(M)} \otimes \eta - \eta \otimes \eta \otimes \xi)$;
5. $T_{\nabla^{(\varphi)}} = \eta \otimes I_{\mathfrak{X}(M)} - I_{\mathfrak{X}(M)} \otimes \eta$ *and* $T_{(\nabla,\varphi)} = -T_{\nabla^{(\varphi)}}$;
6. $R_{\nabla^{(\varphi)}}(X,Y)W = R_\nabla(X,Y)W + \eta(W)R_\nabla(X,Y)\xi - g(X,W)(Y - \eta(Y)\xi) + g(Y,W)(X - \eta(X)\xi)$, *for any* X, Y, $W \in \mathfrak{X}(M)$; *in particular,* $R_{\nabla^{(\varphi)}}(X,Y)\xi = R_\nabla(X,Y)\xi$.

Proof. 1.

$$(\nabla^{(\varphi)}_X\varphi)Y := \nabla^{(\varphi)}_X\varphi Y - \varphi(\nabla^{(\varphi)}_X Y) = \nabla_X\varphi Y - \varphi(\nabla_X Y) + \eta(Y)\varphi X :=$$

$$:= (\nabla_X\varphi)Y + \eta(Y)\varphi X = g(\varphi X, Y)\xi;$$

2.

$$\nabla^{(\varphi)}_X\xi = \nabla_X\xi - X + \eta(X)\xi;$$

3.

$$(\nabla^{(\varphi)}_X\eta)Y := X(\eta(Y)) - \eta(\nabla^{(\varphi)}_X Y) = X(\eta(Y)) - \eta(\nabla_X Y) := (\nabla_X\eta)Y;$$

4.

$$(\nabla^{(\varphi)}_X g)(Y,W) := X(g(Y,W)) - g(\nabla^{(\varphi)}_X Y, W) - g(Y, \nabla^{(\varphi)}_X W) =$$
$$= \eta(Y)g(X,W) + \eta(W)g(X,Y) - 2\eta(X)\eta(Y)\eta(W);$$

5.

$$T_{\nabla^{(\varphi)}}(X,Y) := \nabla^{(\varphi)}_X Y - \nabla^{(\varphi)}_Y X - [X,Y] = \eta(X)Y - \eta(Y)X;$$

6.

$$R_{\nabla^{(\varphi)}}(X,Y)W := \nabla^{(\varphi)}_X\nabla^{(\varphi)}_Y W - \nabla^{(\varphi)}_Y\nabla^{(\varphi)}_X W - \nabla^{(\varphi)}_{[X,Y]}W.$$

We obtain

$$\nabla^{(\varphi)}_X\nabla^{(\varphi)}_Y W = \nabla^{(\varphi)}_X(\nabla_Y W) - \eta(W)\nabla^{(\varphi)}_X Y - X(\eta(W))Y + \eta(Y)\eta(W)\nabla^{(\varphi)}_X\xi +$$
$$+ X(\eta(Y))\eta(W)\xi + X(\eta(W))\eta(Y)\xi =$$
$$= \nabla_X\nabla_Y W - \eta(\nabla_Y W)X + \eta(X)\eta(\nabla_Y W)\xi - \eta(W)\nabla_X Y - \eta(Y)\eta(W)X -$$
$$-\eta(X)\eta(Y)\eta(W)\xi - X(\eta(W))Y + X(\eta(Y))\eta(W)\xi + X(\eta(W))\eta(Y)\xi.$$

Similarly,

$$\nabla^{(\varphi)}_Y\nabla^{(\varphi)}_X W = \nabla_Y\nabla_X W - \eta(\nabla_X W)Y + \eta(Y)\eta(\nabla_X W)\xi - \eta(W)\nabla_Y X -$$
$$-\eta(X)\eta(W)Y - \eta(X)\eta(Y)\eta(W)\xi - Y(\eta(W))X +$$
$$+Y(\eta(X))\eta(W)\xi + Y(\eta(W))\eta(X)\xi.$$

Also,

$$\nabla^{(\varphi)}_{[X,Y]}W = \nabla_{[X,Y]}W - \eta(W)[X,Y] + \eta([X,Y])\eta(W)\xi.$$

It follows that

$$R_{\nabla^{(\varphi)}}(X,Y)W = R_\nabla(X,Y)W + (d\eta)(X,Y)\eta(W)\xi + g(W, \nabla_X\xi)\eta(Y)\xi -$$
$$-g(W, \nabla_Y\xi)\eta(X)\xi - \eta(\nabla_Y W)X + \eta(\nabla_X W)Y + Y(\eta(W))X - X(\eta(W))Y -$$
$$-\eta(Y)\eta(W)X + \eta(X)\eta(W)Y =$$

$$= R_\nabla(X,Y)W + g(W,X)\eta(Y)\xi - g(W,Y)\eta(X)\xi - g(W,X)Y + g(W,Y)X +$$
$$+\eta(X)\eta(W)Y - \eta(Y)\eta(W)X =$$
$$= R_\nabla(X,Y)W - g(X,W)(Y - \eta(Y)\xi) + g(Y,W)(X - \eta(X)\xi) +$$
$$+\eta(W)[\eta(X)Y - \eta(Y)X].$$

□

Concerning the Schouten-van Kampen, the Golab and the Zamkovoy canonical paracontact connections, we can state:

Proposition 3.3. *On a para-Kenmotsu manifold (M,φ,ξ,η,g), the φ-conjugate connections of $\tilde{\nabla}$, ∇^G and ∇^Z respectively, are given by:*

$$\tilde{\nabla}^{(\varphi)} = \tilde{\nabla}, \quad (\nabla^G)^{(\varphi)} = \nabla^{(\varphi)} - \eta\otimes\varphi, \quad (\nabla^Z)^{(\varphi)} = \nabla^Z. \tag{3.7}$$

Proof. They follow from relations (2.4), (2.8), (2.12) and (3.1). □

Remark 3.4. For $n\in\mathbb{N}$, applying the φ-conjugation n times, we obtain

$$\nabla^{n(\varphi)} = \nabla^{(\varphi)}, \quad \tilde{\nabla}^{n(\varphi)} = \tilde{\nabla}, \quad (\nabla^G)^{n(\varphi)} = (\nabla^G)^{(\varphi)}, \quad (\nabla^Z)^{n(\varphi)} = \nabla^Z.$$

Indeed, for ∇, $\tilde{\nabla}$ and ∇^Z follows immediately from the previous Proposition. For the Golab connection, notice that $\varphi^{2n+1} = \varphi$, so

$$(\nabla^G)^{n(\varphi)}_X Y = \varphi^n(\nabla^G_X \varphi^n Y) + \eta(\nabla^G_X Y)\xi =$$
$$= \varphi^n(\nabla_X \varphi^n Y) + \eta(\nabla_X Y)\xi - \eta(X)\varphi^{2n+1}Y =$$
$$= \nabla^{n(\varphi)}_X Y - \eta(X)\varphi Y = (\nabla^{(\varphi)} - \eta\otimes\varphi)(X,Y) = (\nabla^G)^{(\varphi)}.$$

4. Relating ∇, $\tilde{\nabla}$, ∇^G and ∇^Z. A view towards the structure and the virtual tensors

Remark that the Golab connection ∇^G is obtained by perturbing the Levi-Civita connection ∇ with $\eta\otimes\varphi$, so the two connections coincide on $\mathcal{D}$. The same thing happens for the Schouten-van Kampen connection $\tilde{\nabla}$ and the Zamkovoy canonical paracontact connection ∇^Z

$$\nabla^G = \nabla - \eta\otimes\varphi, \quad \tilde{\nabla} = \nabla^Z - \eta\otimes\varphi. \tag{4.1}$$

Therefore,

$$\nabla + \tilde{\nabla} = \nabla^G + \nabla^Z. \tag{4.2}$$

Also, from relations (2.4), (2.8) and (2.12) follow that (∇,∇^G) and $(\tilde{\nabla},\nabla^Z)$ behave similarly with respect to (φ,ξ,η,g):

$$\nabla\varphi = \nabla^G\varphi, \quad \nabla\xi = \nabla^G\xi, \quad \nabla\eta = \nabla^G\eta, \quad \nabla g = \nabla^G g = 0; \tag{4.3}$$

$$(4.4)\quad \tilde{\nabla}\varphi = \nabla^Z\varphi = 0, \quad \tilde{\nabla}\xi = \nabla^Z\xi = 0, \quad \tilde{\nabla}\eta = \nabla^Z\eta = 0, \quad \tilde{\nabla}g = \nabla^Z g = 0.$$

Other geometrical structures associated to a pair of a tensor field φ and a linear connection $\bar{\nabla}$ are the *structure* and the *virtual tensors*, respectively defined as follows:

$$C^{\varphi}_{\bar{\nabla}}(X,Y) := \frac{1}{2}[(\bar{\nabla}_{\varphi X}\varphi)Y + (\bar{\nabla}_X\varphi)\varphi Y] \tag{4.5}$$

and

$$B^{\varphi}_{\bar{\nabla}}(X,Y) := \frac{1}{2}[(\bar{\nabla}_{\varphi X}\varphi)Y - (\bar{\nabla}_X\varphi)\varphi Y]. \tag{4.6}$$

These tensors have been introduced in [10] for almost complex structures. They also appear in [3] and, in [4], for almost product structures.

In our context, we have:

Proposition 4.1. *On a para-Kenmotsu manifold (M,φ,ξ,η,g), the structure and the virtual tensors associated with ∇, $\tilde{\nabla}$, ∇^G and ∇^Z satisfy:*

$$C^{\varphi}_{\nabla}(X,Y) = C^{\varphi}_{\nabla^G}(X,Y) = -\frac{1}{2}\eta(Y)\varphi^2 X, \tag{4.7}$$

$$B^{\varphi}_{\nabla}(X,Y) = B^{\varphi}_{\nabla^G}(X,Y) = -\frac{1}{2}\eta(Y)[X + \eta(X)\xi] + g(X,Y)\xi, \tag{4.8}$$

$$C^{\varphi}_{\tilde{\nabla}} = C^{\varphi}_{\nabla^Z} = B^{\varphi}_{\tilde{\nabla}} = B^{\varphi}_{\nabla^Z} = 0. \tag{4.9}$$

Proof. These relations follow if we replace the expressions $(\nabla_X\varphi)Y = (\nabla^G_X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$ and $\tilde{\nabla}\varphi = \nabla^Z\varphi = 0$ in (4.5) and (4.6). □

As a consequence:

Corollary 4.2. *Under the hypotheses above, we have*

$$C^{\varphi}_{\nabla} - B^{\varphi}_{\nabla} = C^{\varphi}_{\nabla^G} - B^{\varphi}_{\nabla^G} = -\frac{1}{2}L_{\xi}g \otimes \xi. \tag{4.10}$$

Concerning their φ-conjugate connections, we can state:

Proposition 4.3. *On a para-Kenmotsu manifold (M,φ,ξ,η,g), the structure tensors of all φ-conjugate connections of ∇, $\tilde{\nabla}$, ∇^G and ∇^Z vanish identically and the virtual tensors satisfy*

$$B^{\varphi}_{\nabla^{(\varphi)}} = B^{\varphi}_{(\nabla^G)^{(\varphi)}} = -g(\varphi\cdot, \varphi\cdot) \otimes \xi, \quad B^{\varphi}_{\tilde{\nabla}^{(\varphi)}} = B^{\varphi}_{(\nabla^Z)^{(\varphi)}} = 0. \tag{4.11}$$

Proof. Note first that:

$$\nabla^{(\varphi)}\varphi = \nabla\varphi + \varphi \otimes \eta, \quad (\nabla^G)^{(\varphi)}\varphi = \nabla^G\varphi + \varphi \otimes \eta,$$

$$\tilde{\nabla}^{(\varphi)}\varphi = \tilde{\nabla}\varphi, \quad (\nabla^Z)^{(\varphi)}\varphi = \nabla^Z\varphi$$

and use the fact that $\nabla\varphi = \nabla^G\varphi = g(\varphi\cdot,\cdot)\otimes\xi - \varphi\otimes\eta$ and $\tilde{\nabla}\varphi = \nabla^Z\varphi = 0$. Then,

$$C^{\varphi}_{\nabla^{(\varphi)}}(X,Y) = C^{\varphi}_{\nabla}(X,Y) + \frac{1}{2}\eta(Y)\varphi^2X = 0,$$

$$C^{\varphi}_{(\nabla^G)^{(\varphi)}}(X,Y) = C^{\varphi}_{\nabla^G}(X,Y) + \frac{1}{2}\eta(Y)\varphi^2X = 0$$

and

$$C^{\varphi}_{\tilde{\nabla}^{(\varphi)}} = C^{\varphi}_{\tilde{\nabla}} = C^{\varphi}_{\nabla^Z} = C^{\varphi}_{(\nabla^Z)^{(\varphi)}}.$$

Also,

$$B^{\varphi}_{\nabla^{(\varphi)}}(X,Y) = B^{\varphi}_{\nabla}(X,Y) + \frac{1}{2}\eta(Y)\varphi^2X = -g(\varphi X,\varphi Y)\xi,$$

$$B^{\varphi}_{(\nabla^G)^{(\varphi)}}(X,Y) = B^{\varphi}_{\nabla^G}(X,Y) + \frac{1}{2}\eta(Y)\varphi^2X = -g(\varphi X,\varphi Y)\xi$$

and $B^{\varphi}_{\tilde{\nabla}^{(\varphi)}} = B^{\varphi}_{\tilde{\nabla}} = B^{\varphi}_{\nabla^Z} = B^{\varphi}_{(\nabla^Z)^{(\varphi)}}$. □

Remark 4.4. In the general case, for $\bar{\nabla}$ arbitrary linear connection, we have:

$$(\bar{\nabla}^{(\varphi)}_X\varphi)Y := \bar{\nabla}^{(\varphi)}_X\varphi Y - \varphi(\bar{\nabla}^{(\varphi)}_X Y) = (\bar{\nabla}_X\varphi)Y + \varphi((\bar{\nabla}_X\varphi)\varphi Y) - \varphi^2((\bar{\nabla}_X\varphi)Y),$$

therefore

$$C^{\varphi}_{\bar{\nabla}^{(\varphi)}}(X,Y) := \frac{1}{2}[(\bar{\nabla}^{(\varphi)}_{\varphi X}\varphi)Y + (\bar{\nabla}^{(\varphi)}_X\varphi)\varphi Y] = \frac{1}{2}[(\bar{\nabla}_{\varphi X}\varphi)Y + (\bar{\nabla}_X\varphi)\varphi Y]+$$

$$+\frac{1}{2}\varphi[(\bar{\nabla}_{\varphi X}\varphi)\varphi Y + (\bar{\nabla}_X\varphi)\varphi^2 Y] - \frac{1}{2}\varphi^2[(\bar{\nabla}_{\varphi X}\varphi)Y + (\bar{\nabla}_X\varphi)\varphi Y] :=$$

$$:= C^{\varphi}_{\bar{\nabla}}(X,Y) + \varphi(C^{\varphi}_{\bar{\nabla}}(X,\varphi Y)) - \varphi^2(C^{\varphi}_{\bar{\nabla}}(X,Y)).$$

Similarly,

$$B^{\varphi}_{\bar{\nabla}^{(\varphi)}}(X,Y) = B^{\varphi}_{\bar{\nabla}}(X,Y) + \varphi(B^{\varphi}_{\bar{\nabla}}(X,\varphi Y)) - \varphi^2(B^{\varphi}_{\bar{\nabla}}(X,Y)).$$

5. Projectively and dual-projectively equivalent connections

In the last section we shall treat the case of projectively and dual-projectively equivalent connections studying their invariance under such transformations. Recall that two linear connections $\bar{\nabla}$ and $\bar{\nabla}'$ are called [5]:

i) *projectively equivalent* if there exists a 1-form η such that:

$$\bar{\nabla}' = \bar{\nabla} + \eta\otimes I_{\mathfrak{X}(M)} + I_{\mathfrak{X}(M)}\otimes\eta; \tag{5.1}$$

ii) *dual-projectively equivalent* if there exists a 1-form η such that:

$$\bar{\nabla}' = \bar{\nabla} - g\otimes\xi, \tag{5.2}$$

where ξ is the g-dual vector field of η and g a pseudo-Riemannian metric.

Consider $\bar{\nabla}$ and $\bar{\nabla}'$, two linear projectively equivalent connections satisfying

$$\bar{\nabla}' = \bar{\nabla} + \eta \otimes I_{\mathfrak{X}(M)} + I_{\mathfrak{X}(M)} \otimes \eta, \tag{5.3}$$

for η the paracontact form, and their φ-conjugate connections, $\bar{\nabla}^{(\varphi)}$ and $(\bar{\nabla}')^{(\varphi)}$

$$(\bar{\nabla}')^{(\varphi)}_X Y := \bar{\nabla}'_X Y + \varphi((\bar{\nabla}'_X \varphi)Y) =$$

$$= \bar{\nabla}_X Y + \varphi((\bar{\nabla}_X \varphi)Y) + \eta(X)Y + \eta(Y)X - \eta(Y)\varphi^2 X :=$$

$$:= \bar{\nabla}^{(\varphi)}_X Y + \eta(X)Y + \eta(X)\eta(Y)\xi. \tag{5.4}$$

From a direct computation follows:

Lemma 5.1. *If $(M, \varphi, \xi, \eta, g)$ is a para-Kenmotsu manifold, then*

1. $\bar{\nabla}'\varphi = \bar{\nabla}\varphi - \varphi \otimes \eta$;
2. $(\bar{\nabla}')^{(\varphi)}\varphi = \bar{\nabla}^{(\varphi)}\varphi$.

Proposition 5.2. *Let $(M, \varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold and $\bar{\nabla}$, $\bar{\nabla}'$ two linear projectively equivalent connections satisfying (5.3). Then the structure and the virtual tensors of them and their φ-conjugate connections satisfy:*

$$C^{\varphi}_{\bar{\nabla}'} = C^{\varphi}_{\bar{\nabla}} - \frac{1}{2}\varphi^2 \otimes \eta, \quad C^{\varphi}_{(\bar{\nabla}')^{(\varphi)}} = C^{\varphi}_{\bar{\nabla}^{(\varphi)}}, \tag{5.5}$$

$$B^{\varphi}_{\bar{\nabla}'} = B^{\varphi}_{\bar{\nabla}} - \frac{1}{2}\varphi^2 \otimes \eta, \quad B^{\varphi}_{(\bar{\nabla}')^{(\varphi)}} = B^{\varphi}_{\bar{\nabla}^{(\varphi)}}. \tag{5.6}$$

Proof. Use the relations from Lemma 5.1 in the expressions of C^{φ} and B^{φ}. □

As a consequence:

Corollary 5.3. *Under the hypotheses above, we have*

$$C^{\varphi}_{\bar{\nabla}'} - C^{\varphi}_{\bar{\nabla}} = B^{\varphi}_{\bar{\nabla}'} - B^{\varphi}_{\bar{\nabla}} = C^{\varphi}_{\bar{\nabla}}. \tag{5.7}$$

Take now $\bar{\nabla}$ and $\bar{\nabla}'$, two linear dual-projectively equivalent connections satisfying

$$\bar{\nabla}' = \bar{\nabla} - g \otimes \xi, \tag{5.8}$$

for ξ the characteristic vector field, and their φ-conjugate connections, $\bar{\nabla}^{(\varphi)}$ and $(\bar{\nabla}')^{(\varphi)}$:

$$(\bar{\nabla}')^{(\varphi)}_X Y := \bar{\nabla}'_X Y + \varphi((\bar{\nabla}'_X \varphi)Y) = \bar{\nabla}_X Y + \varphi((\bar{\nabla}_X \varphi)Y) - g(X,Y)\xi :=$$

$$:= \bar{\nabla}^{(\varphi)}_X Y - g(X,Y)\xi = (\bar{\nabla}^{(\varphi)})'_X Y. \tag{5.9}$$

From a direct computation follows:

Lemma 5.4. *If $(M, \varphi, \xi, \eta, g)$ is a para-Kenmotsu manifold, then*

1. $\bar{\nabla}'\varphi = \bar{\nabla}\varphi + g(\varphi\cdot, \cdot) \otimes \xi$;
2. $(\bar{\nabla}')^{(\varphi)}\varphi = \bar{\nabla}^{(\varphi)}\varphi + g(\varphi\cdot, \cdot) \otimes \xi$.

Proposition 5.5. *Let $(M, \varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold and $\bar{\nabla}$, $\bar{\nabla}'$ two linear dual-projectively equivalent connections satisfying (5.8). Then the structure and the virtual tensors of them and their φ-conjugate connections satisfy:*

$$C^{\varphi}_{\bar{\nabla}'} = C^{\varphi}_{\bar{\nabla}}, \quad C^{\varphi}_{(\bar{\nabla}')^{(\varphi)}} = C^{\varphi}_{\bar{\nabla}^{(\varphi)}}, \tag{5.10}$$

$$B^{\varphi}_{\bar{\nabla}'} = B^{\varphi}_{\bar{\nabla}} - 2g(\varphi\cdot, \varphi\cdot) \otimes \xi, \quad B^{\varphi}_{(\bar{\nabla}')^{(\varphi)}} = B^{\varphi}_{\bar{\nabla}^{(\varphi)}} - 2g(\varphi\cdot, \varphi\cdot) \otimes \xi. \tag{5.11}$$

Proof. Use the relations from Lemma 5.4 in the expressions of C^{φ} and B^{φ}. □

We can conclude:

Theorem 5.6. *On a para-Kenmotsu manifold, the structure tensor is invariant under dual-projective transformations.*

References

[1] Blaga, A.M., η-Ricci solitons on para-Kenmotsu manifolds. Balkan Journal of Geometry and Its Applications 20(1) (2015), 1-13.

[2] Blaga, A.M., Crasmareanu, M., Special connections in almost paracontact metric geometry. Bull. Iranian Math. Soc. 41(6) (2015), in press.

[3] Blaga, A.M., Crasmareanu, M., The geometry of complex conjugate connections. Hacettepe J. Math. and Statistics 41(1) (2012), 119-126.

[4] Blaga, A.M., Crasmareanu, M., The geometry of product conjugate connections. An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Math. 59(1) (2013), 73-84.

[5] Călin, O., Matsuzoe, H., Zhang, J., Generalizations of conjugate connections. Available at: `http://www.lsa.umich.edu/psych/junz/Publication/2009`.

[6] Dacko, P., Olszak, Z., On weakly para-cosymplectic manifolds of dimension 3. J. Geom. Phys. 57 (2007), 561-570.

[7] Golab, S., On semi-symmetric and quarter-symmetric linear connections. Tensor (N.S.) 29 (1975), 293-301.

[8] Ivanov, S., Vassilev, D., Zamkovoy, S., Conformal paracontact curvature and the local flatness theorem. Geom. Dedicata 144 (2010), 79-100.

[9] Kenmotsu, K., A class of almost contact Riemannian manifolds. Tohoku Math. J. 24 (1972), 93-103.

[10] Kirichenko, V.F., Method of generalized Hermitian geometry in the theory of almost contact manifold. Itogi Nauki i Tekhniki, Problems of geometry 18 (1986), 25–71; transl. in J. Soviet. Math. 42(5) (1988), 1885–1919.

[11] Olszak, Z., The Schouten-van Kampen affine connection adapted to an almost (para) contact metric structure. Publ. Inst. Math. nouv. sér. 94 (108) (2013), 31-42.

[12] Sinha, B.B., Sai Prasad, K.L., A class of almost para contact metric manifolds. Bull. Cal. Math. Soc. 87 (1995), 307-312.

[13] Wełyczko, J., Slant curves in 3-dimensional normal almost paracontact metric manifolds. Mediterr. J. Math., DOI 10.1007/s00009-013-0361-2, 2013.

[14] Zamkovoy, S., Canonical connections on paracontact manifolds. Ann. Global Anal. Geom. 36(1) (2008), 37-60.

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