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ON EINSTEIN WARPED PRODUCTS WITH A SEMI-SYMMETRIC CONNECTION

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Abstract. The aim of this paper is to compute the warping functions for a Ricci flat Einstein multiply warped product M having a semi-symmetric connection in the following cases: 1). $\dim M = 2$, 2). $\dim M = 3$, 3). $\dim M \geq 4$ and all the fibres are Ricci flat.

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1. Introduction

Warped products were first introduced in [2] and then Einstein warped products were studied in [1]. Examples of warping functions that make an warped product become an Einstein space were calculated in [3, 4, 5]. Also, in [6] warped products with an affine connection were introduced.

According to [6] we have the following two definitions:

Definition 1.1. A *multiply warped product* (M, g) is a product manifold of the form $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times \dots \times_{f_m} F_m$ with the metric $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus \dots \oplus f_m^2 g_{F_m}$, where $I \subset \mathbb{R}$ is an open interval and $f_i \in \mathcal{C}^\infty(I)$ for every $i \in \{1, \dots, m\}$.

Definition 1.2. A linear connection $\overline{\nabla}$ on M is called a *semi-symmetric connection* if the torsion $T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]$ satisfies $T(X,Y) = \pi(Y)X - \pi(X)Y$, where π is a 1-form associated to a vector field $P \in \Gamma(TM)$ and satisfies $\pi(X) = g(X, P)$. Moreover $\overline{\nabla}$ is called a *semi-symmetric metric connection* if $\overline{\nabla} g = 0$ and $\overline{\nabla}$ is called a *semi-symmetric non-metric connection* if it satisfies $\overline{\nabla} g \neq 0$. Also, if ∇ is the Levi-Civita connection on M, the semi-symmetric non-metric connection $\overline{\nabla}$ is given by $\overline{\nabla}_X Y = \nabla_X Y + \pi(Y)X$.

Regarding the Einstein conditions we have the following result:

Theorem 1.3. ([6]) *Let $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times \dots \times_{f_m} F_m$ be a multiply warped product, $m \geq 1$, $k_i = \dim F_i \geq 1$ for every $i \in \{1, \dots, m\}$ and $P = \frac{\partial}{\partial t}$. Then $(M, \overline{\nabla})$ is Einstein of Einstein constant λ if and only if the following conditions*

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are satisfied:

$$\begin{cases} (F_i, g_{F_i}) \textit{ is Einstein of constant } \lambda_i \text{ for every } i \in \{1, ..., m\}. \\ \sum_{i=1}^{m} k_i \left(1 - \frac{f_i^{''}}{f_i}\right) = \lambda. \\ \lambda_i - f_i f_i^{''} - (k_i - 1)\left(f_i^{'}\right)^2 - f_i f_i^{'} \sum_{j \neq i, j=1}^{m} k_j \frac{f_j^{'}}{f_j} + f_i^2 \sum_{j=1}^{m} k_j \frac{f_j^{'}}{f_j} = \lambda f_i^2, \\ \text{for every } i \in \{1, ..., m\}. \end{cases} \tag{1}$$

2. Main results

The aim this paper is to characterize the warping functions from (1) in the following case: 1). $\dim M = 2$ and M is Ricci flat, 2). $\dim M = 3$ and M is Ricci flat, 3). $\dim M \geq 4$, $M = I \times_{f_1} F_1 \times ... \times_{f_m} F_m$, M is Ricci flat and all the fibres F_i are also Ricci flat for every $i \in \{1, ..., m\}$.

We first consider the cases when $\dim M \in \{2, 3\}$. Hence:

1). M is Ricci flat and $\dim M = 2 \Longrightarrow M = I \times_{f_1} F_1$, $k_1 = \dim F_1 = 1$, $\lambda_1 = 0$.

$$\begin{cases} 1 - \frac{f_1^{''}}{f_1} = 0 \\ -f_1 f_1^{''} + f_1 f_1^{'} = 0. \end{cases}$$

Hence $f_1^{''} = f_1^{'} = f_1$ which implies $f_1(x) = Ce^x$, $C \in \mathbb{R}$.

2). M is Ricci flat and $\dim M = 3$.

i). $M = I \times_{f_1} F_1$, $k_1 = \dim F_1 = 2$.

$$\begin{cases} 2\left(1 - \frac{f_1^{''}}{f_1}\right) = 0 \\ \lambda_1 - f_1 f_1^{''} - \left(f_1^{'}\right)^2 + 2 f_1 f_1^{'} = 0 \end{cases}$$

Hence $\lambda_1 = f_1^2 + \left(f_1^{'}\right)^2 - 2 f_1 f_1^{'} = \left(f_1 - f_1^{'}\right)^2$

a) If $\lambda_1 < 0$, then we have no solution.

b) If $\lambda_1 \geq 0$, then $f_1^{'} = f_1 + \sqrt{\lambda_1}$ or $f_1^{'} = f_1 - \sqrt{\lambda_1}$. In both cases $f_1^{''} = f_1^{'} = f_1$ and $\lambda_1 = 0$. Thus we obtain $f_1(x) = Ce^x$, $C \in \mathbb{R}$.

ii). $M = I \times_{f_1} F_1 \times_{f_2} F_2$, $k_i = \dim F_i = 1$, $\lambda_i = 0$ for every $i \in \{1, 2\}$.

$$\begin{cases} 2 = \frac{f_1^{''}}{f_1} + \frac{f_2^{''}}{f_2} \\ -\frac{f_1^{''}}{f_1} - \frac{f_1^{'}}{f_1} \cdot \frac{f_2^{'}}{f_2} + \frac{f_1^{'}}{f_1} + \frac{f_2^{'}}{f_2} = 0 \\ -\frac{f_2^{''}}{f_2} - \frac{f_1^{'}}{f_1} \cdot \frac{f_2^{'}}{f_2} + \frac{f_1^{'}}{f_1} + \frac{f_2^{'}}{f_2} = 0. \end{cases}$$

Hence $\frac{f_1''}{f_1} = \frac{f_2''}{f_2} = 1$ and $-\frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} + \frac{f_1'}{f_1} + \frac{f_2'}{f_2} = 1 \Longrightarrow \left(1 - \frac{f_1'}{f_1}\right)\left(1 - \frac{f_2'}{f_2}\right) = 0.$

a). $f_1' = f_1 \Longrightarrow f_1(x) = Ce^x$ and $\frac{f_2''}{f_2} = 1 \Longrightarrow f_2(x) = C_1e^x + C_2e^{-x}$, $C, C_1, C_2 \in \mathbb{R}$.

b). $f_2' = f_2 \Longrightarrow f_2(x) = Ce^x$ and $\frac{f_1''}{f_1} = 1 \Longrightarrow f_1(x) = C_1e^x + C_2e^{-x}$, $C, C_1, C_2 \in \mathbb{R}$.

3). Consider now $M = I \times_{f_1} F_1 \times ... \times_{f_m} F_m$, M and F_i Ricci flat, $k_i = \dim F_i \geq 1$ for every $i \in \{1, ..., m\}$ and $\dim M = 1 + \sum_{i=1}^{m} k_i \geq 4$. The main theorem of this paragraph is the following:

Theorem 2.1. *Let $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times ... \times_{f_m} F_m$ be a multiply warped product, $m \geq 1$, $k_i = \dim F_i \geq 1$ for every $i \in \{1, ..., m\}$, $\dim M \geq 4$ and $P = \frac{\partial}{\partial t}$. Then, $(M, \overline{\nabla})$ and all the fibres (F_i, g_{F_i}) are Ricci flat for every $i \in \{1, ..., m\}$ if and only if the warping functions satisfy $f_i(x) = \eta_i e^x$, $\eta_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$.*

Proof.
" $\Longrightarrow$ " In the above hypothesis the system (1) becomes:

$$\begin{cases} \sum_{i=1}^{m} k_i \left(1 - \frac{f_i''}{f_i}\right) = 0. \\ -f_i f_i'' - (k_i - 1)\left(f_i'\right)^2 - f_i f_i' \sum_{j \neq i, j=1}^{m} k_j \frac{f_j'}{f_j} + f_i^2 \sum_{j=1}^{m} k_j \frac{f_j'}{f_j} = 0 \\ \text{for every } i \in \{1, ..., m\}. \end{cases} \tag{2}$$

Dividing by f_i^2 the second equation of (2), one obtains the equivalent form:

$$\begin{cases} \sum_{i=1}^{m} k_i \left(1 - \frac{f_i''}{f_i}\right) = 0. \\ -\frac{f_i''}{f_i} - (k_i - 1)\left(\frac{f_i'}{f_i}\right)^2 - \frac{f_i'}{f_i} \sum_{j \neq i,\ j=1}^{m} k_j \frac{f_j'}{f_j} + \sum_{j=1}^{m} k_j \frac{f_j'}{f_j} = 0 \\ \text{for every } i \in \{1, ..., m\}. \end{cases} \tag{3}$$

We make now the following notations: $h_i = \frac{f_i'}{f_i}$ for every $i \in \{1, ..., m\}$ and

$$H = \sum_{i=1}^{m} k_i \frac{f_i'}{f_i} = \sum_{i=1}^{m} k_i h_i.$$

We remark that $\frac{f_i''}{f_i} = h_i' + h_i^2$ for every $i \in \{1, ..., m\}$ and $H' = \sum_{i=1}^{m} k_i h_i'$.

Thus the system (3) becomes:

$$\begin{cases} \sum_{i=1}^{m} k_i\left(1-h_i^{'}-h_i^2\right)=0. \\ -h_i^{'}-h_i^2-(k_i-1)\,h_i^2-h_i\,(H-k_ih_i)+H=0, \\ \text{ for every } i\in\{1,\ldots,m\}. \end{cases} \tag{4}$$

The second equation is equivalent to:

$$-h_i^{'}-h_i^2-(k_i-1)\,h_i^2-h_i\,(H-k_ih_i)+H=0 \Longleftrightarrow$$
$$-h_i^{'}-h_i^2-k_ih_i^{2}+h_i^2-h_iH+k_ih_i^2+H=0 \Longleftrightarrow$$

$$-h_i^{'}-h_iH+H=0 \Longleftrightarrow -k_ih_i^{'}-k_ih_iH+k_iH=0, \tag{4'}$$

for every $i\in\{1,...,m\}$.

Summing over i the above m equations of $(4^{'})$ we obtain:

$$\begin{gathered} -\sum_{i=1}^{m} k_ih_i^{'}-H\sum_{i=1}^{m} k_ih_i+H\sum_{i=1}^{m} k_i=0 \Longleftrightarrow \\ -H^{'}-H^2+AH=0 \Longleftrightarrow H^{'}=-H^2+AH, \end{gathered} \tag{5}$$

where $A=\sum_{i=1}^{m} k_i\geq 3>0$.

The reduced equation of (5), $\overline{H}^{'}=-\overline{H}^2$, has the solution $\overline{H}(x)=\frac{1}{x+C}$, $C\in\mathbb{R}$. We search now solutions of the form $H(x)=\frac{1}{x+C(x)}$. Putting this in (5) we obtain:

$$C^{'}(x)+AC(x)+Ax=0. \tag{6}$$

The reduced equation of (6), $\overline{C}^{'}(x)=-A\overline{C}(x)$, has the solution $\overline{C}(x)=De^{-Ax}$, $D\in\mathbb{R}$. We search now solutions of the form $C(x)=D(x)e^{-Ax}$. Thus equation (6) implies:

$$D'(x)=-Axe^{Ax} \Longrightarrow D(x)=-xe^{Ax}+\frac{e^{Ax}}{A}-E\,,\ \ E\in\mathbb{R}.$$

Thus, $C(x)=-x+A-Ee^{-Ax}$ and hence

$$H(x)=\frac{1}{x-x+\frac{1}{A}-Ee^{-Ax}}=\frac{Ae^{Ax}}{e^{Ax}+\beta},\ \beta\in\mathbb{R}.$$

1). The case when $\beta\neq 0:$

Returning to relation $(4^{'})$ we have:

$$-h_i^{'}-Hh_i+H=0 \Longrightarrow h_i^{'}=-Hh_i+H, \tag{7}$$

for every $i\in\{1,...,m\}$.

The reduced equation of (7), $\overline{h_i}' = -H\overline{h_i}$, has the solution $\overline{h_i}(x) = \frac{E_i}{e^{Ax}+\beta}$, $E_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$. We search for solutions of the form $h_i(x) = \frac{E_i(x)}{e^{Ax}+\beta}$. Putting this in (7) we obtain $E_i'(x) = Ae^{Ax} \Longrightarrow E_i(x) = e^{Ax} + \delta_i$ where $\delta_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$. Thus $h_i(x) = \frac{e^{Ax}+\delta_i}{e^{Ax}+\beta}$, $\beta, \delta_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$.

We remark that since $H = \sum_{i=1}^{m} k_i h_i$ we obtain $\sum_{i=1}^{m} k_i\delta_i = 0$.

Hence, $h_i = \frac{f_i'}{f_i} = \frac{e^{Ax}+\delta_i}{e^{Ax}+\beta} \Longrightarrow f_i(x) = \sigma_i e^{\int \frac{e^{Ax}+\delta_i}{e^{Ax}+\beta}dx}$, $\sigma_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$. But, if $\beta \neq 0$, then $\int \frac{e^{Ax}+\delta_i}{e^{Ax}+\beta}dx = \frac{\delta_i}{\beta}x - \frac{\delta_i-\beta}{A\beta}\ln\left(e^{Ax}+\beta\right) + D_i$ and thus $f_i(x) = \theta_i \frac{e^{\frac{\delta_i}{\beta}x}}{(e^{Ax}+\beta)^{\frac{\delta_i-\beta}{A\beta}}}$, $\theta_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$.

We remark that since $\sum_{i=1}^{m} k_i\left(1 - h_i' - h_i^2\right) = 0$ we have equivalently

$$\sum_{i=1}^{m} k_i\left(\beta^2 - \delta_i^2\right) = (A-2)\sum_{i=1}^{m} k_i\left(\beta - \delta_i\right) = 0$$

which implies

$$\sum_{i=1}^{m} k_i\left(\beta^2 - \delta_i^2\right) = \sum_{i=1}^{m} k_i\left(\beta - \delta_i\right) = 0$$

since $A \geq 3$. Hence $\sum_{i=1}^{m} k_i\beta = \sum_{i=1}^{m} k_i\delta_i = 0$ and thus $\beta = 0$ which is a contradiction.

2). The case when $\beta = 0$:

In this case $H(x) = A$ and $h_i(x) = \frac{e^{Ax}+\delta_i}{e^{Ax}}$ in a similar way as case 1), where $\delta_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$. Hence, since

$$\int \frac{e^{Ax}+\delta_i}{e^{Ax}}dx = x - \frac{\delta_i}{Ae^{Ax}} + D_i$$

we obtain $f_i(x) = \eta_i e^{x - \frac{\delta_i}{Ae^{Ax}}}$, $\eta_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$.

But, verifying the relation

$$\sum_{i=1}^{m} k_i\left(1 - h_i' - h_i^2\right) = 0$$

one can obtain

$$\sum_{i=1}^{m} k_i\delta_i^2 = \sum_{i=1}^{m} k_i\delta_i = 0$$

which implies $\delta_i = 0$ for every $i \in \{1,...,m\}$. Thus, in this way we have obtained $h_i(x) = 1$ and $f_i(x) = \eta_i e^x$, $\eta_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$.

So, the warping functions are only of the form $f_i(x) = \eta_i e^x$, $\eta_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$.

" $\Longleftarrow$ " This implication follows by straightforward computation. Suppose the warping functions are $f_i(x) = \eta_i e^x$, $\eta_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$ and satisfy the system (1). Then $f_i^{''} = f_i^{'} = f_i$ for every $i \in \{1, ..., m\}$.

Thus $\sum_{i=1}^{m} k_i \left(1 - \frac{f_i^{''}}{f_i}\right) = 0 = \lambda$ and so M is Ricci flat.

Also:

$$\lambda_i - f_i f_i^{''} - (k_i - 1)\left(f_i^{'}\right)^2 - f_i f_i^{'} \sum_{j \neq i, j=1}^{m} k_j \frac{f_j^{'}}{f_j} + f_i^2 \sum_{j=1}^{m} k_j \frac{f_j^{'}}{f_j} = \lambda f_i^2 \Longleftrightarrow$$

$$\lambda_i - f_i^2 - (k_i - 1) f_i^2 - f_i^2 \sum_{j=1,\ j \neq i}^{m} k_j + f_i^2 \sum_{j=1}^{m} k_j = 0 \Longleftrightarrow$$

$$\lambda_i - f_i^2 - (k_i - 1) f_i^2 + k_i f_i^2 = 0 \Longleftrightarrow \lambda_i = 0,$$

for every $i \in \{1, ..., m\}$. Hence F_i is Ricci flat for every $i \in \{1, ..., m\}$. □

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