Novi Sad J. Math.
Vol. 45, No. 2, 2015, 113-123

D-HOMOTHETIC DEFORMATION OF LP-SASAKIAN MANIFOLDS

Krishnendu De[1]

Abstract. The object of the present paper is to study a transformation called D-homothetic deformation of LP-Sasakian manifolds. Among others it is shown that in an LP-Sasakian manifold, the Ricci operator Q commutes with the structure tensor ϕ. We also discuss about the invariance of η-Einstein manifolds, ϕ-sectional curvature, the locally ϕ-Ricci symmetry and η-parallelity of the Ricci tensor under the D-homothetic deformation. Finally, we give an example of such a manifold .

AMS Mathematics Subject Classification (2010): 53C15, 53C25

Key words and phrases: D-homothetic deformation; η-Einstein manifold; ϕ- sectional curvature; locally ϕ-Ricci symmetry

1. Introduction

The notion of Lorentzian para-Sasakian manifold was introduced by Matsumoto [5] in 1989. Then Mihai and Rosca [7] defined the same notion independently and they obtained several results on this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [6], De and Shaikh [3], Ozgur [8] and others.

An LP-Sasakian manifold is said to be η-Einstein if its Ricci tensor S is of the form

$$S = \lambda g + \mu \eta \otimes \eta \tag{1.1}$$

where λ and μ are smooth functions on the manifold.

The notion of local ϕ-symmetry was first introduced by Takahashi [10] on a Sasakian manifold. Again in a recent paper [2] De and Sarkar introduced the notion of locally ϕ-Ricci symmetric Sasakian manifolds. Also ϕ-Ricci symmetric Kenmotsu manifolds have been studied by Shukla and Shukla [9].

An LP-Sasakian manifold is said to be locally ϕ-Ricci symmetric if

$$\phi^2(\nabla_X Q)(Y) = 0, \tag{1.2}$$

where Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and X, Y are orthogonal to ξ.

The Ricci tensor S of an LP-Sasakian manifold is said to be η-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{1.3}$$

[1]Konnagar High School(H.S.), 68 G.T. Road (West),Konnagar,Hooghly, Pin.712235, West Bengal, India, e-mail: krishnendu_de@yahoo.com

for all vector fields X, Y and Z. The notion of η- parallelity in a Sasakian manifold was introduced by Kon [4].

Let $M\ (\phi,\xi,\eta,g)$ be an almost contact metric manifold with dim $M = m = 2n+1$. The equation $\eta = 0$ defines an $(m-1)$-dimensional distribution D on M [11]. By an $(m-1)$-homothetic deformation or D-homothetic deformation [12] we mean a change of structure tensors of the form

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta\otimes\eta,$$

where a is a positive constant. If $M(\phi,\xi,\eta,g)$ is an almost contact metric structure with contact form η, then $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ is also an almost contact metric structure [12]. Denoting by W^i_{jk} the difference $\bar{\Gamma}^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols we have in an almost contact metric manifold [12]

$$\begin{aligned} W(X,Y) &= (1-a)[\eta(Y)\phi X + \eta(X)\phi Y] \\ &+\frac{1}{2}(1-\frac{1}{a})[(\nabla_X\eta)(Y) + (\nabla_Y\eta)(X)]\xi \end{aligned} \tag{1.4}$$

for all $X, Y \in \chi(M)$. If R and $\bar{R}$ denote respectively the curvature tensor of the manifold $M(\phi,\xi,\eta,g)$ and $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$, then we have [12]

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + (\nabla_X W)(Z,Y) - (\nabla_Y W)(Z,X) \\ &+W(W(Z,Y),X) - W(W(Z,X),Y) \end{aligned} \tag{1.5}$$

for all $X, Y, Z \in \chi(M)$.

A plane section in the tangent space $T_p(M)$ is called a ϕ-section if there exists a unit vector X in $T_p(M)$ orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature

$$K(X,\phi X) = g(R(X,\phi X)X, \phi X)$$

is called a ϕ-sectional curvature. A para contact metric manifold $M(\phi,\xi,\eta,g)$ is said to be of constant ϕ-sectional curvature if at any point $p \in M$, the sectional curvature $K(X,\phi X)$ is independent of the choice of non-zero $X \in D_p$, where D denotes the contact distribution of the para contact metric manifold defined by $\eta = 0$.

The present paper is organized as follows:

After preliminaries in section 2, we prove some important lemmas. Section 4 deals with the study of $(2n+1)$-dimensional $\eta-$Einstein LP-Sasakian manifolds and prove that these manifolds are invariant under a D-homothetic deformation. Also we study ϕ-sectional curvature, locally ϕ-Ricci symmetry and η-parallelity of the Ricci tensor in a $(2n+1)$-dimensional LP-Sasakian manifold under a D-homothetic deformation. Finally in section 5, we cited an example of LP-Sasakian manifold which validates a theorem of section 4.

2. Preliminaries

Let M^{2n+1} be an $2n+1$-dimensional differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor g_p: $T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,.....,+)$, where T_pM denotes the tangent space of M at p and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2.1}$$

$$g(X,\xi) = \eta(X), g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold M^{2n+1} with the structure (ϕ, ξ, η, g) is called Lorentzian almost paracontact manifold [5]. In the Lorentzian almost paracontact manifold M^{2n+1}, the following relations hold [5] :

$$\phi\xi = 0, \eta(\phi X) = 0, \tag{2.3}$$

$$\Omega(X,Y) = \Omega(Y,X), \tag{2.4}$$

where $\Omega(X,Y) = g(X, \phi Y)$.

Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor S and the scalar curvature r are defined by

$$S(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1$, $\epsilon_2 = \cdots = \epsilon_n = 1$.

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called Lorentzian paracontact manifold if

$$\Omega(X,Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}.$$

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called an LP-Sasakian manifold [5] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In an LP-Sasakian manifold the 1-form η is closed. Also in [5], it is proved that if an n- dimensional Lorentzian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then M^n admits an LP-Sasakian structure.

Further, on such an LP-Sasakian manifold M^n (ϕ, ξ, η, g), the following relations hold [5]:

$$\eta(R(X,Y)Z) = [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.5}$$

$$S(X,\xi) = 2n\eta(X), \tag{2.6}$$

$$R(X,Y)\xi = [\eta(Y)X - \eta(X)Y], \tag{2.7}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$(\nabla_X \phi)(Y) = [g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.9}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold. Also since the vector field η is closed in an LP-Sasakian manifold, we have ([6],[5])

$$(\nabla_X \eta)Y = \Omega(X,Y), \tag{2.10}$$

$$\Omega(X,\xi) = 0, \tag{2.11}$$

$$\nabla_X \xi = \phi X, \tag{2.12}$$

for any vector field X and Y.

3. Some Lemmas

In this section we shall state and prove some Lemmas which will be needed to prove the main results.

Lemma 3.1. *[1] In an LP-Sasakian manifold, the following relation holds*

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) &= g(R(X,Y)Z,W) + g(X,W)\eta(Y)\eta(Z) \\ &\quad - g(X,Z)\eta(W)\eta(Y) + g(Y,Z)\eta(X)\eta(W) \\ &\quad - g(Y,W)\eta(X)\eta(Z). \end{aligned} \tag{3.1}$$

Lemma 3.2. *Let (M^{2n+1}, g) be an LP-Sasakian manifold. Then the Ricci operator Q commutes with ϕ.*

Proof. From (3.1), it follows that

$$\begin{aligned}\phi R(\phi X, \phi Y)\phi Z &= R(X,Y)Z - [\eta(Z)Y - g(Y,Z)\xi]\eta(X) \\ &+[X\eta(Z) - g(X,Z)\xi]\eta(Y).\end{aligned} \tag{3.2}$$

Let $\{e_i, \phi e_i, \xi\}$, $i = 1, 2, \ldots.., n$ be an orthonormal frame at any point of the manifold. Then putting $Y = Z = e_i$ in (3.2) and taking summation over i and using $\eta(e_i) = 0$, we get

$$\sum_{i=1}^{n} \epsilon_i \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{n} \epsilon_i R(X, e_i)e_i - n\eta(X)\xi, \tag{3.3}$$

where $\epsilon_i = g(e_i, e_i)$.

Again setting $Y = Z = \phi e_i$ in (3.2), taking summation over i and using $\eta.\phi = 0$, we get

$$\sum_{i=1}^{n} \epsilon_i \phi R(\phi X, e_i)e_i = \sum_{i=1}^{n} \epsilon_i R(X, \phi e_i)\phi e_i - n\eta(X)\xi. \tag{3.4}$$

Adding (3.3) and (3.4) and using the definition of the Ricci tensor, we obtain

$$\phi(Q\phi X - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi - 2n\eta(X)\xi.$$

Using (2.7) and $\phi\xi = 0$ in the above relation, we have

$$\phi(Q\phi X) = QX - 2n\eta(X)\xi.$$

Operating both sides by ϕ and using (2.1), symmetry of Q and $\phi\xi = 0$, we get $\phi Q = Q\phi$. This proves the lemma.

□

Proposition 3.1. *In an $2n+1$-dimensional η-Einstein LP-Sasakian manifold, the Ricci tensor S is expressed as*

$$\begin{aligned}S(X,Y) &= [\frac{r}{2n} - 1]g(X,Y) \\ &-[\frac{r}{2n} - 2n - 1]\eta(X)\eta(Y).\end{aligned} \tag{3.5}$$

4. Main results

In this section we study η-Einstein LP-Sasakian manifolds, ϕ-sectional curvature, locally ϕ-Ricci symmetry and η-parallelity of the Ricci tensor of an odd dimensional LP-Sasakian manifold under a D-homothetic deformation.

In virtue of (2.10), the relation (1.4) reduces to

$$W(X,Y) = (1-a)[\eta(Y)\phi X + \eta(X)\phi Y] + (1-\frac{1}{a})g(\phi X, Y)\xi. \tag{4.1}$$

In view of (2.9), (2.10) and (2.12), the relation (4.1) yields

$$\begin{aligned}(\nabla_Z W)(X,Y) &= (1-a)[\{g(\phi Z, Y)\phi X \\ &+g(X,Z)\eta(Y)\xi + 2\eta(X)\eta(Y)Z + 4\eta(X)\eta(Y)\eta(Z)\xi \\ &+g(\phi Z, X)\phi Y + \eta(X)g(Y,Z)\xi] \\ &+\frac{a-1}{a}g(\phi X, Y)\phi Z.\end{aligned} \tag{4.2}$$

Using (4.1) and (4.2) into (1.5), we obtain by virtue of (2.7) and (2.10) that

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + (1-a)[g(X,Z)\eta(Y)\xi \\ &-g(Y,Z)\eta(X)\xi + 2\eta(Y)\eta(Z)X \\ &-2\eta(X)\eta(Z)Y \\ &+g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X] \\ &+\frac{a-1}{a}[g(\phi Z, Y)\phi X - g(\phi Z, X)\phi Y] \\ &+(1-a)^2[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\ &-\frac{(1-a)^2}{a}[g(\phi Z, X)\phi Y - g(\phi Z, Y)\phi X]\end{aligned} \tag{4.3}$$

Putting $Y = Z = \xi$ in (4.3) and using (2.1) we obtain

$$\bar{R}(X,\xi)\xi = R(X,\xi)\xi + 2(1-a)[-X + \eta(X)\xi] - (1-a)^2\phi^2 X. \tag{4.4}$$

Let $\{e_i, \phi e_i, \xi\}$, $i = 1, 2, \ldots.., n$ be an orthonormal frame at any point of the manifold. Then putting $Y = Z = e_i$ in (4.3) and taking summation over i and using $\eta(e_i) = 0$, we get

$$\sum_{i=1}^{n} \epsilon_i \bar{R}(X, e_i)e_i = \sum_{i=1}^{n} \epsilon_i R(X, e_i)e_i - (1-a)n\eta(X)\xi, \tag{4.5}$$

where $\epsilon_i = g(e_i, e_i)$.

Again setting $Y = Z = \phi e_i$ in (4.3) and taking summation over i and using $\eta.\phi = 0$, we get

$$\sum_{i=1}^{n} \epsilon_i \bar{R}(X, \phi e_i)\phi e_i = \sum_{i=1}^{n} \epsilon_i R(X, \phi e_i)\phi e_i - (1-a)n\eta(X)\xi. \tag{4.6}$$

Adding (4.5) and (4.6) and using the definition of Ricci operator we have

$$\bar{Q}X - \bar{R}(X,\xi)\xi = QX - R(X,\xi)\xi - 2(1-a)n\eta(X)\xi. \tag{4.7}$$

In view of (4.4) we get from (4.7)

$$\begin{aligned} \bar{S}(X,Y) &= S(X,Y) - [2(1-a)+(1-a)^2]g(X,Y) \\ &\quad -[2(1-a)(n-1)+(1-a)^2]\eta(X)\eta(Y), \end{aligned} \tag{4.8}$$

which implies that

$$\begin{aligned} \bar{Q}X &= QX - [2(1-a)+(1-a)^2]X \\ &\quad -[2(1-a)(n-1)+(1-a)^2]\eta(X)\xi. \end{aligned} \tag{4.9}$$

Operating $\bar{\phi} = \phi$ on both sides of (4.9) from the left we have

$$\bar{\phi}\bar{Q}X = \phi QX - [2(1-a)+(1-a)^2]\phi X. \tag{4.10}$$

Again, putting $\bar{\phi}X = \phi X$ in (4.9) from the right we have

$$\bar{Q}\bar{\phi}X = Q\phi X - [2(1-a)+(1-a)^2]\phi X. \tag{4.11}$$

Subtracting (4.10) and (4.11) we get

$$(\bar{\phi}\bar{Q} - \bar{Q}\bar{\phi})X = (\phi Q - Q\phi)X. \tag{4.12}$$

Therefore using Lemma 3.2 we can state the following:

Theorem 4.1. *Under a D-homothetic deformation, the expression* $\bar{Q}\bar{\phi} = \bar{\phi}\bar{Q}$ *holds in an* $(2n+1)$*-dimensional LP-Sasakian manifold.*

4.1. η-Einstein LP-Sasakian manifolds

Let $M(\phi,\xi,\eta,g)$ be a $(2n+1)$-dimensional η-Einstein LP-Sasakian manifold which reduces to $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ under a D-homothetic deformation. Then from (4.8) it follows by virtue of (3.5)that

$$\bar{S}(X,Y) = \bar{\lambda}\bar{g}(X,Y) + \bar{\mu}\bar{\eta}(X)\bar{\eta}(Y), \tag{4.13}$$

where $\bar{\lambda}, \bar{\mu}$ are smooth functions given by

$$\bar{\lambda} = [\frac{r}{2n} - (a-2)^2] \tag{4.14}$$

and

$$\bar{\mu} = [\frac{r}{2n} - 4n + 2an - a^2]. \tag{4.15}$$

In view of the relation (4.13) we state the following:

Theorem 4.2. *Under a D-homothetic deformation, a* $(2n+1)$*-dimensional* η*-Einstein LP-Sasakian manifold is invariant.*

4.2. ϕ-sectional curvature of LP-Sasakian manifolds

In this section we consider the ϕ-sectional curvature on a $(2n+1)$-dimensional LP-Sasakian manifold.

From (4.3) it can be easily seen that

$$\bar{K}(X,\phi X) - K(X,\phi X) = -2(a-1) \tag{4.16}$$

and hence we state the following theorem.

Theorem 4.3. *The ϕ-sectional curvature of $(2n+1)$-dimensional LP-Sasakian manifolds is not an invariant property under D-homothetic deformations.*

If a $(2n+1)$-dimensional LP-Sasakian manifold $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ satisfies $R(X,Y)\xi = 0$ for all X, Y, then it can be easily seen that $K(X,\phi X) = 0$ and hence from (4.16) it follows that

$$\bar{K}(X,\phi X) = -2(a-1) \neq 0,$$

where X is a unit vector field orthogonal to ξ and $K(X,\phi X)$ is the ϕ-sectional curvature. This implies that the ϕ-sectional curvature $\bar{K}(X,\phi X)$ is non-vanishing. Therefore we state the following:

Theorem 4.4. *There exists $(2n+1)$-dimensional LP-Sasakian manifold with non-zero ϕ-sectional curvature.*

4.3. Locally ϕ-Ricci symmetric LP-Sasakian manifolds

In this section we study locally ϕ-Ricci symmetry on an LP-Sasakian manifold.

Differentiating (4.9) covariantly with respect to W we obtain

$$\begin{aligned}(\nabla_W\bar{Q})(X) &= (\nabla_W Q)(X)\\ &\quad -[2(1-a)(n-1)+(1-a)^2](\nabla_W\eta)(X)\xi\\ &\quad -[2(1-a)(n-1)+(1-a)^2]\eta(X)\nabla_W\xi.\end{aligned} \tag{4.17}$$

Operating ϕ^2 on both sides of (4.17) and taking X as an orthonormal vector to ξ we obtain

$$\bar{\phi}^2(\nabla_W\bar{Q})(X) = \phi^2(\nabla_W Q)(X). \tag{4.18}$$

In view of the relation (4.18) we state the following:

Theorem 4.5. *The local ϕ-Ricci symmetry on LP-Sasakian manifolds is an invariant property under D-homothetic deformations.*

4.4. $\eta-$ parallel Ricci tensor of an LP-Sasakian manifolds

Let us consider the η-parallelity of the Ricci tensor on an LP-Sasakian manifold.

Differentiating (4.8) covariantly with respect to W and using (2.10) we obtain

$$\begin{aligned}(\nabla_W \bar{S})(X,Y) &= (\nabla_W S)(X,Y)\\ &\quad -[2(1-a)(n-1)+(1-a)^2]\\ &\quad [g(\phi W,X)\eta(Y)+g(\phi W,Y)\eta(X)].\end{aligned} \tag{4.19}$$

In (4.19) replacing X by $\phi X, Y$ by ϕ Y and using (2.3) we get

$$(\nabla_W \bar{S})(\phi X,\phi Y) = (\nabla_W S)(\phi X,\phi Y). \tag{4.20}$$

Hence we can state the following:

Theorem 4.6. *The η-parallelity of the Ricci tensor on LP-Sasakian manifolds is an invariant property under D-homothetic deformations.*

5. Example

We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) are standard coordinates of $\mathbb{R}^3$.

The vector fields

$$e_1 = e^z\frac{\partial}{\partial y}, \quad e_2 = e^z\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right), \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_1,e_3) = g(e_1,e_2) = g(e_2,e_3) = 0,$$

$$g(e_1,e_1) = g(e_2,e_2) = 1,$$

$$g(e_3,e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_3,$$

$$g(\phi Z,\phi W) = g(Z,W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and let R be the curvature tensor of g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1 \quad and \quad [e_2, e_3] = -e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3,$$
$$\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0,$$
$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0. \tag{5.1}$$

From the above it can be easily seen that $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. With the help of the above results it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,$$
$$R(e_1, e_2)e_2 = e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0,$$
$$R(e_1, e_2)e_1 = -e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3.$$

From the above expressions of the curvature tensor we obtain

$$\begin{aligned} S(e_1, e_1) &= g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_3)e_3, e_1) \\ &= 2. \end{aligned}$$

Similarly we have

$$S(e_2, e_2) = 2$$

and

$$S(e_3, e_3) = -2.$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = 6.$$

From [3] we know that in a 3- dimensional LP-Sasakian manifold

$$\begin{aligned} R(X, Y)Z &= (\frac{r-4}{2})[g(Y, Z)X - g(X, Z)Y] + (\frac{r-6}{2})[g(Y, Z)\eta(X)\xi \\ &\quad -g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{5.2}$$

Now using (5.2) we get

$$\begin{aligned} g(R(X, Y)Z, W) &= (\frac{r-4}{2})[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ &\quad +(\frac{r-6}{2})[g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W) \\ &\quad +\eta(Y)\eta(Z)g(X, W) - \eta(X)\eta(Z)g(Y, W)]. \end{aligned} \tag{5.3}$$

From (5.3), it follows that the ϕ- sectional curvature of the manifold is given by

$$K(X, \phi X) = \frac{r-4}{2}$$

for any vector field X orthogonal to ξ.

In view of the above relation we get

$$K(e_1, \phi e_1) = K(e_2, \phi e_2) = \frac{r-4}{2}$$

Again it can be easily shown from (4.3) that

$$\bar{K}(e_1, \phi e_1) - K(e_1, \phi e_1) = -2(a-1)$$

and

$$\bar{K}(e_2, \phi e_2) - K(e_2, \phi e_2) = -2(a-1)$$

Therefore Theorem 4.3 is verified.

References

[1] Al-Aqeel, A., De, U.C., Ghosh, G.C., On Lorentzian para- Sasakian manifolds. Kuwait J. Sci. Eng. 31 (2004), 1-13.

[2] De, U.C., Sarkar, A., On $\phi-$Ricci symmetric Sasakian manifolds. Proceedings of the Jangjeon Mathematical Society 11(1) (2008), 47-52.

[3] De, U.C., Shaikh, A.A., On 3- dimensional LP-Sasakian manifolds. Soochow J. Math. 26 (2000), 359-368.

[4] Kon, M., Invariant submanifolds in Sasakian manifolds. Math. Ann. 219 (1976), 277-290.

[5] Matsumoto, K., On Lorentzian paracontact manifolds. Bull. Yamagata Univ. Natur. Sci. 12 (1989), 151-156.

[6] Matsumoto, K., Mihai, I., On a certain transformation in a Lorentzian Para-Sasakian Manifold. Tensor (N.S.) 47 (1988), 189-197.

[7] Mihai, I., Rosca, R., On Lorentzian P-Sasakian Manifolds. Classical Analysis. World Scientific, Singapore (1992), 155-169.

[8] Ozgur, C., ϕ-conformally flat Lorentzian Para-Sasakian Manifolds. Radovi Matematički 12 (2003), 99-106.

[9] Shukla, S.S., Shukla, M.K., On ϕ-Ricci symmetric Kenmotsu manifolds. Novi Sad J. Math. 39(2) (2009), 89-95.

[10] Takahashi, T., Sasakian $\phi-$symmetric spaces. Tohoku Math. J. 29 (1977), 91-113.

[11] Tanno, S., Partially conformal transformations with respect to $(m-1)$-dimensional distribution of m-dimensional Riemannian manifolds. J. Tohoku Math. 2(17) (1965), 358-409.

[12] Tanno, S., The topology of contact Riemannian manifolds. J. Tohoku Math. 12 (1968), 700-717.

Received by the editors July 1, 2014