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# NOTES ON PSEUDO-SEQUENCE-COVERING MAPS

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**Abstract.** In this paper, we prove that each sequentially-quotient and boundary-compact map on $g$-metrizable spaces is pseudo-sequence-covering, and each finite subsequence-covering (or 1-sequentially-quotient) map on $snf$-countable spaces is pseudo-sequence-covering.

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## 1. Introduction

Pseudo-sequence-covering maps and sequentially-quotient maps play an important role in the study of images of metric spaces. It is well known that every pseudo-sequence-covering map on metric spaces is sequentially-quotient. But this implication can not be reversed. In 2005, S. Lin proved that each sequentially-quotient and compact map on metric spaces is pseudo-sequence-covering, and that there exists a sequentially-quotient $\pi$-map on metric spaces which is not pseudo-sequence-covering ([12]). After that, F. C. Lin and S. Lin proved that each sequentially-quotient and boundary-compact map on metric spaces is pseudo-sequence-covering ([9]). Recently, the same authors proved that if $X$ is an open image of metric spaces, then each sequentially-quotient and boundary-compact map on $X$ is pseudo-sequence-covering ([10]).

In this paper, we prove that each sequentially-quotient and boundary-compact map on $g$-metrizable spaces is pseudo-sequence-covering, and each finite subsequence-covering (or 1-sequentially-quotient) map on $snf$-countable spaces is pseudo-sequence-covering.

Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ be a collection of subsets of $X$, we denote $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$.

**Definition 1.1.** Let $X$ be a space, $\{x_n\} \subset X$ and $P \subset X$.

1. $\{x_n\}$ is called *eventually* in $P$, if $\{x_n\}$ converges to $x$, and there exists $m \in \mathbb{N}$ such that $\{x\} \bigcup \{x_n : n \geq m\} \subset P$.

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2. $\{x_n\}$ is called *frequently* in $P$, if some subsequence of $\{x_n\}$ is eventually in $P$.

3. $P$ is called a *sequential neighborhood* of $x$ in $X$ [4], if whenever $\{x_n\}$ is a sequence converging to $x$ in $X$, then $\{x_n\}$ is eventually in $P$.

**Definition 1.2.** Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space $X$. Assume that $\mathcal{P}$ satisfies the following (a) and (b) for every $x \in X$.

(a) $\mathcal{P}_x$ is a network at $x$.
(b) If $P_1$, $P_2 \in \mathcal{P}_x$, then there exists $P \in \mathcal{P}_x$ such that $P \subset P_1 \cap P_2$.

1. $\mathcal{P}$ is a *weak base* of $X$ [1], if for $G \subset X$, $G$ is open in $X$ if and only if for every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$; $\mathcal{P}_x$ is said to be a *weak neighborhood base at* $x$ in $X$.

2. $\mathcal{P}$ is an *sn-network* for $X$ [11], if each element of $\mathcal{P}_x$ is a sequential neighborhood of $x$ for all $x \in X$; $\mathcal{P}_x$ is said to be an *sn-network at* $x$ in $X$.

**Definition 1.3.** Let $X$ be a space. Then,

1. $X$ is *$gf$-countable* [1] (resp., *$snf$-countable* [5]), if $X$ has a weak base (resp., *sn*-network) $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ such that each $\mathcal{P}_x$ is countable.

2. $X$ is *$g$-metrizable* [14], if $X$ is regular and has a $\sigma$-locally finite weak base.

3. $X$ is *sequential* [4], if whenever $A$ is a non closed subset of $X$, then there is a sequence in $A$ converging to a point not in $A$.

4. $X$ is *strongly $g$-developable* [15], if $X$ is sequential has a $\sigma$-locally finite strong network consisting of *cs*-covers.

*Remark* 1.4.

1. Each strongly $g$-developable space is $g$-metrizable.

2. A space $X$ is $gf$-countable if and only if it is sequential and $snf$-countable.

**Definition 1.5.** Let $f : X \longrightarrow Y$ be a map.

1. $f$ is a *compact* map [3], if each $f^{-1}(y)$ is compact in $X$.

2. $f$ is a *boundary-compact* map [3], if each $\partial f^{-1}(y)$ is compact in $X$.

3. $f$ is a *pseudo-sequence-covering* map [7], if for each convergent sequence $L$ in $Y$, there is a compact subset $K$ in $X$ such that $f(K) = \overline{L}$.

4. $f$ is a *sequentially-quotient* map [2], if whenever $\{y_n\}$ is a convergent sequence in $Y$, there is a convergent sequence $\{x_k\}$ in $X$ with each $x_k \in f^{-1}(y_{n_k})$.

5. $f$ is a *finite subsequence-covering* map [13], if for each $y \in Y$, there is a finite subset $F$ of $f^{-1}(y)$ such that for any sequence $S$ converging to $y$ in $Y$, there is a sequence $L$ converging to some $x \in F$ in $X$ and $f(L)$ is a subsequence of $S$.

6. $f$ is a *1-sequentially-quotient* map [6], if for each $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to $y$ in $Y$, there is a sequence $\{x_{n_k}\}$ converging to $x_y$ in $X$ with each $x_{n_k} \in f^{-1}(y_{n_k})$.

*Remark* 1.6.

1. Each compact map is a compact-boundary map.
2. Each 1-sequentially-quotient map is a finite subsequence-covering map.

**Definition 1.7** ([8])**.** A function $g : \mathbb{N} \times X \longrightarrow \mathcal{P}(X)$ is a CWC-*map*, if it satisfies the following conditions.

1. $x \in g(n, x)$ for all $x \in X$ and $n \in \mathbb{N}$.
2. $g(n + 1, x) \subset g(n, x)$ for all $n \in \mathbb{N}$.
3. $\{g(n, x) : n \in \mathbb{N}\}$ is a weak neighborhood base at $x$ for all $x \in X$.

## 2. Main results

**Theorem 2.1.** *Let $f : X \longrightarrow Y$ be a boundary-compact map. If $X$ is a g-metrizable space, then $f$ is a sequentially-quotient map if and only if it is a pseudo-sequence-covering map.*

*Proof.* Necessity. Let $f$ be a sequentially-quotient map and $\{y_n\}$ be a non-trivial sequence converging to $y$ in $Y$. Since $X$ is $g$-metrizable, it follows from Theorem 2.5 in [16] that there exists a CWC-map $g$ on $X$ satisfying, for any sequences $\{x_n\}$ and $\{y_n\}$ of $X$, if $x_n \to x$ and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $y_n \to x$. For $n \in \mathbb{N}$, let

$$U_{y,n} = \bigcup\{g(n, x) : x \in \partial f^{-1}(y)\}, \text{ and } P_{y,n} = f(U_{y,n}).$$

It is obvious that $\{P_{y,n} : n \in \mathbb{N}\}$ is a decreasing sequence in $X$. Furthermore, $P_{n,y}$ is a sequential neighborhood of $y$ in $Y$ for all $n \in \mathbb{N}$. If not, there exists $n \in \mathbb{N}$ such that $P_{y,n}$ is not a sequential neighborhood of $y$ in $Y$. Thus, there exists a sequence $L$ converging to $y$ in $Y$ such that $L \cap P_{y,n} = \emptyset$. Since $f$ is sequentially-quotient, there exists a sequence $S$ converging to $x \in \partial f^{-1}(y)$ such that $f(S)$ is a subsequence of $L$. On the other hand, because $g(n, x)$ is a sequential neighborhood of $x$ in $X$, $S$ is eventually in $g(n, x)$. Thus, $S$ is eventually in $U_{y,n}$. Therefore, $L$ is frequently in $P_{y,n}$. This contradicts $L \cap P_{y,n} = \emptyset$.

Then for each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $y_i \in P_{y,n}$ for all $i \geq i_n$. So $f^{-1}(y_i) \cap U_{y,n} \neq \emptyset$. We can suppose that $1 < i_n < i_{n+1}$. For each $j \in \mathbb{N}$, we take

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1, \\ f^{-1}(y_j) \cap U_{y,n}, & \text{if } i_n \leq j < i_{n+1}. \end{cases}$$

Let $K = \partial f^{-1}(y) \cup \{x_j : j \in \mathbb{N}\}$. Clearly, $f(K) = \{y\} \cup \{y_n : n \in \mathbb{N}\}$. Furthermore, $K$ is a compact subset in $X$. In fact, let $\mathcal{U}$ be an open cover for $K$ in $X$. Since $\partial f^{-1}(y)$ is a compact subset in $X$, there exists a finite subfamily $\mathcal{H} \subset \mathcal{U}$ such that $\partial f^{-1}(y) \subset \bigcup \mathcal{H}$. Then there exists $m \in \mathbb{N}$ such that $U_{n,y} \subset$

$\bigcup\mathcal{H}$ for all $n \geq m$. If not, for each $n \in \mathbb{N}$, there exists $v_n \in U_{y,n} - \bigcup\mathcal{H}$. It implies that $v_n \in g(n, u_n) - \bigcup\mathcal{H}$ for some $u_n \in \partial f^{-1}(y)$. Since $\{u_n\} \subset \partial f^{-1}(y)$ and each compact subset of $X$ is metrizable, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to x \in \partial f^{-1}(y)$. Now, for each $i \in \mathbb{N}$, we put

$$a_i = \begin{cases} u_{n_1} & \text{if} \quad i \leq n_1 \\ u_{n_{k+1}} & \text{if} \quad n_k < i \leq n_{k+1}; \end{cases}$$

$$b_i = \begin{cases} v_{n_1} & \text{if} \quad i \leq n_1 \\ v_{n_{k+1}} & \text{if} \quad n_k < i \leq n_{k+1}. \end{cases}$$

Then $a_i \to x$. Because $g(n+1, x) \subset g(n, x)$ for all $x \in X$ and $n \in \mathbb{N}$, it implies that $b_i \in g(i, a_i)$ for all $i \in \mathbb{N}$. By property of $g$, it implies that $b_i \to x$. Thus, $v_{n_k} \to x$. This contradicts to $\bigcup\mathcal{H}$ is a neighborhood of $x$ and $v_{n_k} \notin \bigcup\mathcal{H}$ for all $k \in \mathbb{N}$.

Because $P_{y,i+1} \subset P_{y,i}$ for all $i \in \mathbb{N}$, it implies that $\partial f^{-1}(y) \cup \{x_i : i \geq m\} \subset \bigcup\mathcal{H}$. For each $i < m$, take $V_i \in \mathcal{U}$ such that $x_i \in V_i$. Put $\mathcal{V} = \mathcal{U} \cup \{V_i : i < m\}$. Then $\mathcal{V} \subset \mathcal{U}$ and $K \subset \bigcup\mathcal{V}$. Therefore, $K$ is compact in $X$, and $f$ is pseudo-sequence-covering.

Sufficiency. Suppose that $f$ is a pseudo-sequence-covering map. If $\{y_n\}$ is a convergent sequence in $Y$, then there is a compact subset $K$ in $X$ such that $f(K) = \overline{\{y_n\}}$. For each $n \in \mathbb{N}$, take a point $x_n \in f^{-1}(y_n) \cap K$. Since $K$ is compact and metrizable, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, and $\{f(x_{n_k})\}$ is a subsequence of $\{y_n\}$. Therefore, $f$ is sequentially-quotient. □

By Remark 1.4, Remark 1.6 and Theorem 2.1, we obtain the following corollaries.

**Corollary 2.2.** *Let $f : X \longrightarrow Y$ be a boundary-compact map. If $X$ is strongly $g$-developable, then $f$ is sequentially-quotient if and only if it is pseudo-sequence-covering.*

**Corollary 2.3.** *Let $f : X \longrightarrow Y$ be a compact map. If $X$ is $g$-metrizable, then $f$ is sequentially-quotient if and only if it is pseudo-sequence-covering.*

**Corollary 2.4.** *Let $f : X \longrightarrow Y$ be a compact map. If $X$ is strongly $g$-developable, then $f$ is sequentially-quotient if and only if it is pseudo-sequence-covering.*

**Theorem 2.5.** *Let $f : X \longrightarrow Y$ be a finite subsequence-covering map. If $X$ is an $snf$-countable space, then $f$ is a pseudo-sequence-covering map.*

*Proof.* Let $\{y_n\}$ be a non-trivial sequence converging to $y$ in $Y$ and $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ be an $sn$-network for $X$ such that each $\mathcal{B}_x$ is countable. Since $f$ is finite subsequence-covering, there exists a finite subset $F_y \subset f^{-1}(y)$ such that for each sequence $S$ converging to $y$ in $Y$, there is a sequence $L$ in $X$ such that $L$ converging to some $x \in F_y$ and $f(L)$ is a subsequence of $S$. Since each $\mathcal{B}_x$ is a countable $sn$-network at $x$, for each $x \in X$, we can choose a decreasing countable network $\{B_{x,n} : n \in \mathbb{N}\} \subset \mathcal{B}_x$. Put

$$U_{y,n} = \bigcup\{B_{x,n} : x \in F_y\} \text{ and } P_{y,n} = f(U_{y,n}).$$

Then $P_{y,n+1} \subset P_{y,n}$ for all $n \in \mathbb{N}$. Furthermore, each $P_{y,n}$ is a sequential neighborhood of $y$ in $Y$. If not, there exists $n \in \mathbb{N}$ such that $P_{y,n}$ is not a sequential neighborhood of $y$ in $Y$. Thus, there exists a sequence $L$ converging to $y$ in $Y$ such that $L \cap P_{y,n} = \emptyset$. Since $f$ is finite subsequence-covering, there exists a sequence $S$ in $X$ such that $S$ converges to some $x \in F_y$ and $f(S)$ is a subsequence of $L$. On the other hand, because $B_{x,n}$ is a sequential neighborhood of $x$ in $X$, $S$ is eventually in $B_{x,n}$. It implies that $S$ is eventually in $U_{y,n}$. Therefore, $L$ is frequently in $P_{y,n}$. This contradicts to $L \cap P_{y,n} = \emptyset$.

Thus, for each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $y_i \in f(U_{y,n})$ for all $i \geq i_n$. So $g^{-1}(y_i) \cap U_{y,n} \neq \emptyset$. We can suppose that $1 < i_n < i_{n+1}$. For each $j \in \mathbb{N}$, we take

$$x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1, \\ f^{-1}(y_j) \cap U_{y,n}, & \text{if } i_n \leq j < i_{n+1}. \end{cases}$$

Let $K = F_y \cup \{x_j : j \in \mathbb{N}\}$. Clearly, $f(K) = \{y\} \cup \{y_n : n \in \mathbb{N}\}$. Furthermore, $K$ is a compact subset in $X$. In fact, let $\mathcal{U}$ be an open cover for $K$ in $X$. Since $F_y$ is finite, there exists a finite subfamily $\mathcal{H} \subset \mathcal{U}$ such that $F_y \subset \bigcup\mathcal{H}$. For each $x \in F_y$, since $\{B_{x,n} : n \in \mathbb{N}\}$ is a decreasing network at $x$ and $\bigcup\mathcal{H}$ is a neighborhood of $x$ in $X$, $B_{x,n_x} \subset \bigcup\mathcal{H}$ for some $n_x \in \mathbb{N}$. If we put $k = \max\{n_x : x \in F_y\}$, then $U_{y,k} \subset \bigcup\mathcal{H}$. Furthermore, because $U_{y,i+1} \subset U_{y,i}$ for all $i \in \mathbb{N}$, $F_y \cup \{x_i : i \geq k\} \subset \bigcup\mathcal{H}$. For each $i < k$, take $V_i \in \mathcal{U}$ such that $x_i \in V_i$, and put $\mathcal{V} = \mathcal{H} \cup \{V_i : i < k\}$. Then $\mathcal{V} \subset \mathcal{U}$, $K \subset \bigcup\mathcal{V}$ and $K$ is compact in $X$. Therefore, $f$ is a pseudo-sequence-covering map. □

By Remark 1.4, Remark 1.6 and Theorem 2.5, we have

**Corollary 2.6.** *Let $f : X \longrightarrow Y$ be a 1-sequentially-quotient map. If $X$ is an $snf$-countable space, then $f$ is a pseudo-sequence-covering map.*

**Corollary 2.7.** *Let $f : X \longrightarrow Y$ be a finite subsequence-covering map. If $X$ is a $gf$-countable space, then $f$ is a pseudo-sequence-covering map.*

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