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PARTIAL REDUCTION FOR LINEAR SYSTEMS OF OPERATOR EQUATIONS WITH SYSTEM MATRIX IN COMPANION FORM[1]

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Abstract. In this paper we will consider a partial reduction for nonhomogeneous linear systems of the operator equations with the system matrix in the companion form and with different operators. As a result of this method we will get an equivalent system consisting of the linear operator equations having only one or two variables. Homogeneous part of the equation in one unknown is obtained using generalized characteristic polynomial of the system matrix. We will also look more closely at some properties of the doubly companion matrix.

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1. Introduction

A common method for solving linear systems of the operator equations with constant coefficients is to decompose it into several subsystems using Jordan canonical form, and then to solve each of these subsystems separately. In paper [6] the idea was to use the rational instead of the Jordan canonical form to reduce a linear system of the first order operator equations to an equivalent partially reduced system, i.e. to the system which consists of a higher order linear operator equations having only one variable and the first order linear operator equations in two variables. By the order of a linear operator equation we mean the highest power of the operator in the equation. The reduction process was divided into two steps. The first step was to reduce a linear system, by using some basic properties of the rational canonical form, into a proper system for a further study. More precisely, let K be a field and V be a vector space over K. For a positive integer n, let $K^{n\times n}$ be the set of all $n\times n$ matrices over K and let $V^{n\times 1}$ be the set of all $n\times 1$ matrices over V. Let $\varphi_1\,\varphi_2\,\ldots\,\varphi_n$ be given vectors in V and let $A : V \to V$ be a linear operator on V. We write $x_1\,x_2\,\ldots\,x_n$ for an unknown vectors in V. If we assume that the system matrix $B = [b_{ij}] \in K^{n\times n}$ is similar to the matrix C in the companion form, then the

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linear system of the first order operator equations in unknowns $x_1, x_2, \ldots, x_n$ of the form

$$\begin{aligned} A(x_1) &= b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n + \varphi_1 \\ A(x_2) &= b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n + \varphi_2 \\ &\vdots \\ A(x_n) &= b_{n1}x_1 + b_{n2}x_2 + \ldots + b_{nn}x_n + \varphi_n, \end{aligned}$$

can be transformed to an equivalent system in unknowns $y_1, y_2, \ldots, y_n$ of the form

$$\begin{aligned} A(y_1) &= y_2 + \psi_1 \\ A(y_2) &= y_3 + \psi_2 \\ &\vdots \\ A(y_{n-1}) &= y_n + \psi_{n-1} \\ A(y_n) &= -d_n y_1 - d_{n-1} y_2 - \ldots - d_1 y_n + \psi_n, \end{aligned}$$

where the columns $\vec{y} = [y_1 \ \ldots \ y_n]^T \in V^{n\times 1}$ and $\vec{\psi} = [\psi_1 \ \ldots \ \psi_n]^T \in V^{n\times 1}$ are determined by $\vec{y} = P^{-1}\vec{x}$ i $\vec{\psi} = P^{-1}\vec{\varphi}$ for regular matrix $P \in K^{n\times n}$ such that $C = P^{-1}BP$ and where $d_1, d_2, \ldots, d_n$ are coefficients of the characteristic polynomial of the matrices B and C. The second step of reduction is to obtain the partially reduced system from the system in companion form. If we write $\delta_k^1(C; A^{n-k}(\psi_1), \ldots, A^{n-k}(\psi_n))$ for the sum of principal minors of order k containing elements of the first column of the matrix obtained by substituting column $[A^{n-k}(\psi_1) \ldots A^{n-k}(\psi_n)]^T \in V^{n\times 1}$ in place of the first column of C and if we set $\Delta_C(A) = A^n + d_1A^{n-1} + \ldots + d_{n-1}A + d_nI$, then the partially reduced system is of the form

$$\begin{aligned} \Delta_C(A)(y_1) &= \sum_{k=1}^{n} (-1)^{k+1} \delta_k^1(C; A^{n-k}(\psi_1), \ldots, A^{n-k}(\psi_n)) \\ y_2 &= A(y_1) - \psi_1 \\ y_3 &= A(y_2) - \psi_2 \\ &\vdots \\ y_n &= A(y_{n-1}) - \psi_{n-1}. \end{aligned}$$

In paper [5] we introduced a method for total reduction for linear systems of the operator equations with the system matrix in the companion form, not by a change of basis, but by finding the adjugate matrix of the characteristic matrix of the system matrix. We also indicated how this technique may be used to connect differential transcendence of the solution with the coefficients of the system. In this paper we will replace single operator A with a sequence of linear operators $A_1, A_2, \ldots, A_n$, and concern more closely the second step of the reduction process.

Let C be a $n \times n$ matrix with coefficients in the field K in companion form, i.e.

$$C = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -d_n & -d_{n-1} & \dots & -d_2 & -d_1 \end{bmatrix} \in K^{n\times n}.$$

Characteristic polynomial of the matrix C is

$$\Delta_C(\lambda) = \lambda^n + d_1\lambda^{n-1} + \ldots + d_{n-1}\lambda + d_n,$$

please refer to [3, p. 488]. Throughout the paper columns $\vec{x} = [x_1\, x_2\, \ldots\, x_n]^T \in V^{n\times 1}$ and $\vec{\varphi} = [\varphi_1\, \varphi_2\, \ldots\, \varphi_n]^T \in V^{n\times 1}$ will be called column of unknowns and nonhomogeneous term, respectively. Vector operator $\vec{A} : V^{n\times 1} \to V^{n\times 1}$ is defined componentwise by $\vec{A}(\vec{x}) = [A_1(x_1)\, A_2(x_2)\, \ldots\, A_n(x_n)]^T$. Matrix form of the linear system of the operator equations with the system matrix in the companion form and with different operators is

$$\vec{A}(\vec{x}) = C\vec{x} + \vec{\varphi}. \tag{1}$$

The matrix form can be rewritten in the following system companion-like form

$$\begin{array}{lcc} A_1(x_1) & = & x_2 + \varphi_1 \\ A_2(x_2) & = & x_3 + \varphi_2 \\ \vdots & & \\ A_{n-1}(x_{n-1}) & = & x_n + \varphi_{n-1} \\ A_n(x_n) & = & -d_n x_1 - d_{n-1}x_2 - \ldots - d_1 x_n + \varphi_n. \end{array} \tag{2}$$

As we have already mentioned, the main topic of the paper is reduction of the linear system (2) to the partially reduced system. In fulfilling this task we will get some auxiliary results on doubly companion matrices.

2. Properties of Doubly Companion Matrix

Butcher and Chartier in [1, p. 274] introduced the notion of the doubly companion matrix of polynomials $\alpha(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$ and $\beta(\lambda) = \lambda^n + b_1\lambda^{n-1} + \ldots + b_{n-1}\lambda + b_n$ as an $n \times n$ matrix over the field K of the form

$$C = (\alpha, \beta) \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n - b_n \\ 1 & 0 & \dots & 0 & -b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -b_2 \\ 0 & 0 & \dots & 1 & -b_1 \end{bmatrix}.$$

If $b_1 = b_2 = \ldots = b_n = 0$, we obtain the companion matrix of the polynomial $\alpha(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$ of the form

$$C(\alpha) = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix},$$

and if $a_1 = a_2 = \ldots = a_n = 0$, we get matrix

$$C(\beta) = \begin{bmatrix} 0 & 0 & \ldots & 0 & -b_n \\ 1 & 0 & \ldots & 0 & -b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -b_2 \\ 0 & 0 & \ldots & 1 & -b_1 \end{bmatrix}.$$

with the characteristic polynomial $\beta(\lambda) = \lambda^n + b_1\lambda^{n-1} + \ldots + b_{n-1}\lambda + b_n$.
Wanicharpichat in paper [10, p. 262], inspired by paper [1], defined the notion of the lower doubly companion matrix as a matrix

$$C(\alpha, \beta) = \begin{bmatrix} -b_1 & 1 & 0 & \ldots & 0 \\ -b_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{n-1} & 0 & 0 & \ldots & 1 \\ -b_n - a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix},$$

which is for us more convenient to follow.
Butcher and Wright in [2, pp. 363-364], and Wright in [7] used the doubly companion matrix as a tool for analyzing various extension of classical methods with inherent Runge–Kutta stability. Wanicharpichat in [8] proved that the doubly companion matrix is nonderogatory and calculated its minimal polynomial. In Wanicharpichat's paper [10] we can find eigenvectors formulas for the doubly companion matrix and in paper [9] explicit formula for a determinant and an inverse formula of the doubly companion matrix were proved.
Butcher and Chartier in paper [1, Lemma 1.] asserted that the characteristic polynomial of $C(\alpha, \beta)$ is given by omitting the negative powers of λ in $\lambda^{-n}\alpha(\lambda)\beta(\lambda)$. Wanicharpichat in paper [8, pp. 367–368] gave a direct calculation for finding the characteristic polynomial $\Delta_{C(\alpha,\beta)} = det(\lambda I - C(\alpha, \beta))$ by performing an elementary row and column operation on the matrix $\lambda I - C(\alpha, \beta)$.

Lemma 2.1. *The characteristic polynomial of the doubly companion matrix*

$$M = \begin{bmatrix} b_1 & 1 & 0 & \ldots & 0 \\ b_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \ldots & 1 \\ b_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{bmatrix}$$

is polynomial

$$\begin{aligned}\Delta_M(\lambda) &= \lambda^n - (b_1 + a_1)\lambda^{n-1} + (b_1 a_1 - b_2 - a_2)\lambda^{n-2} + \ldots \\ &\ldots + b_1 a_{n-1} + b_2 a_{n-2} + \ldots b_{n-1} a_1 - b_n,\end{aligned}$$

i.e. for the coefficients d_k of λ^k in the characteristic polynomial $\Delta_M(\lambda)$ of the matrix M following equality holds

$$d_k = \sum_{j=1}^{k-1} b_j a_{k-j} - b_k - a_k,$$

where $1 \le k \le n$ and $a_n = 0$.

Let M be an arbitrary $n \times n$ matrix over the field K. Let us denote by $\delta_k(M)$ the sum of principal minors of the matrix M of the order k and by $\delta_k^1(M)$ the sum of the principal minors of the matrix M of the order k containing the first column, $1 \le k \le n$. Coefficients of the characteristic polynomial $\Delta_M(\lambda) = \lambda^n + d_1\lambda^{n-1} + \ldots + d_n$ of the matrix M can be expressed in the terms of sums of its principal minors. More precisely we have $d_k = (-1)^k \delta_k(M)$, for $1 \le k \le n$ (please see [4, p. 203]).

The following result can be also found in paper [6]. In this paper we give an elegant proof using previous lemma.

Lemma 2.2. *If matrix M has the form*

$$M = \begin{bmatrix} b_1 & 1 & 0 & \ldots & 0 \\ b_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \ldots & 1 \\ b_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{bmatrix}$$

then it follows

$$\delta_k^1(M) = (-1)^k \Big(\sum_{j=1}^{k-1} b_j a_{k-j} - b_k \Big) \qquad (1 \le k \le n).$$

Proof. For the coefficients of the characteristic polynomial

$$\Delta_M(\lambda) = \lambda^n + d_1\lambda^{n-1} + \ldots + d_{n-1}\lambda + d_n$$

of the matrix M the equality $d_k = (-1)^k \delta_k(M)$ holds .
By deleting the first row and column of the matrix M we obtain $(n-1) \times (n-1)$ matrix

$$\widetilde{M} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ a_{n-1} & a_{n-2} & \ldots & a_1 \end{bmatrix}.$$

The matrix $\widetilde{M}$ is the companion matrix of the polynomial

$$\alpha(\lambda) = \lambda^{n-1} - a_1\lambda^{n-2} - \ldots - a_{n-2}\lambda - a_{n-1},$$

and hence its characteristic polynomial is $\alpha(\lambda)$. Consequently it follows that $-a_k = (-1)^k \delta_k(\widetilde{M})$, for $1 \leq k \leq n-1$.
The sum of all principal minors of the matrix M of the order k can be expressed as the sum of its principal minors of the order k containing the first row, and consequently the first column, and the sum of those which do not. Therefore, we get connection between sums of principal minors of the order k of matrices M and $\widetilde{M}$ which states $\delta_k(M) = \delta_k^1(M) + \delta_k(\widetilde{M})$. Lemma 2.1 provides that $d_k = \sum_{j=1}^{k-1} b_j a_{k-j} - b_k - a_k$ and by assuming that $a_n = 0$, we conclude

$$\delta_k^1(M) = \delta_k(M) - \delta_k(\widetilde{M}) = (-1)^k d_k + (-1)^k a_k = (-1)^k \left(\sum_{j=1}^{k-1} b_j a_{k-j} - b_k \right),$$

for $1 \leq k \leq n$. □

3. Main Result

In this section we are returning on the reduction process of the system (2). We write $M^1(v_1, \ldots, v_n)$ for the matrix obtained by substituting column $v = [v_1 \ldots v_n]^T \in V^{n\times 1}$ in place of the first column of M. As we mentioned above, it is convenient to use $\delta_k^1(M; v_1, \ldots, v_n) = \delta_k^1(M^1(v_1, \ldots, v_n))$ for the sum of principal minors of the order k containing the first column of the matrix $M^1(v_1, \ldots, v_n)$.

Theorem 3.1. *The linear system of the operator equations*

$$\begin{array}{rcl} A_1(x_1) & = & x_2 + \varphi_1 \\ A_2(x_2) & = & x_3 + \varphi_2 \\ \vdots & & \\ A_{n-1}(x_{n-1}) & = & x_n + \varphi_{n-1} \\ A_n(x_n) & = & -d_n x_1 - d_{n-1} x_2 - \ldots - d_1 x_n + \varphi_n, \end{array}$$

can be transformed into partially reduced system

$$\begin{array}{l} L(\vec{A})(x_1) \\ \quad = \displaystyle\sum_{k=1}^{n} (-1)^{k+1} \delta_k^1(C; \underbrace{A_{n-k+1} \circ \ldots \circ A_2}_{n-k}(\varphi_1), \ldots, \underbrace{A_{n-k} \circ \ldots \circ A_1}_{n-k}(\varphi_n)) \end{array}$$

$$\begin{array}{rcl} x_2 & = & A_1(x_1) - \varphi_1 \\ x_3 & = & A_2(x_2) - \varphi_2 \\ \vdots & & \\ x_{n-1} & = & A_{n-2}(x_{n-2}) - \varphi_{n-2} \\ x_n & = & A_{n-1}(x_{n-1}) - \varphi_{n-1}, \end{array}$$

where

$$L(\vec{A})(x_1) \;=\; A_n \circ A_{n-1} \circ \ldots \circ A_1(x_1) + d_1 A_{n-1} \circ A_{n-2} \circ \ldots \ldots \circ A_1(x_1) + \ldots + d_{n-1}A_1(x_1) + d_n x_1$$

and where

$$\delta_k^1(C; A_{n-k+1}\circ\ldots\circ A_2(\varphi_1), \ldots, A_{n-k}\circ\ldots\circ A_1(\varphi_n))$$

is the sum of principal minors of the order k containing the first column of matrix

$$\begin{bmatrix}
A_{n-k+1}\circ A_{n-k} \circ\ldots\circ A_2(\varphi_1) & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0\\
A_{n-k+2}\circ A_{n-k+1}\circ\ldots\circ A_3(\varphi_2) & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
A_n\circ A_{n-1}\circ\ldots\circ A_{k+1}(\varphi_k) & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0\\
A_1\circ A_n \circ\ldots\circ A_{k+2}(\varphi_{k+1}) & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0\\
A_2\circ A_1 \circ\ldots\circ A_{k+3}(\varphi_{k+2}) & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0\\
A_{n-k-1}\circ A_{n-k-2}\circ\ldots\circ A_n(\varphi_{n-1}) & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1\\
A_{n-k} \circ A_{n-k-1}\circ\ldots\circ A_1(\varphi_n) & -d_{n-1} & -d_{n-2} & \ldots & -d_{n-k} & -d_{n-k-1} & -d_{n-k-2} & \ldots & -d_1
\end{bmatrix}.$$

Proof. From the first equation of the system we have $x_2 = A_1(x_1) - \varphi_1$. Substituting this expression into the second equation we get $x_3 = A_2(x_2) - \varphi_2 = A_2 \circ A_1(x_1) - A_2(\varphi_1) - \varphi_2$. So each x_k, $2 \le k \le n$, can be expressed as a function of x_1 in the following way

$$x_k = A_{k-1} \circ A_{k-2} \circ \ldots \circ A_1(x_1) - \sum_{j=1}^{k-1} \underbrace{A_{k-1} \circ A_{k-2} \circ \ldots \circ A_{j+1}}_{k-1-j}(\varphi_j).$$

Hence substituting these expressions into the last equation yields

$$A_n \circ A_{n-1} \circ \ldots \circ A_1(x_1) - \textstyle\sum_{j=1}^{n-1} A_n \circ A_{n-1} \circ \ldots \circ A_{j+1}(\varphi_j) =$$

$$-d_n x_1 - d_{n-1}(A_1(x_1) - \varphi_1) - d_{n-2}(A_2 \circ A_1(x_1) - A_2(\varphi_1) - \varphi_2) - \ldots -$$

$$-d_1(A_{n-1} \circ A_{n-2} \circ \ldots \circ A_1(x_1) - \textstyle\sum_{j=1}^{n-1} A_{n-1} \circ A_{n-2} \circ \ldots \circ A_{j+1}(\varphi_j)) + \varphi_n$$

i.e,

$$\begin{aligned}
L(\vec{A})(x_1) = {} & \Big(A_n \circ A_{n-1} \circ \ldots \circ A_2(\varphi_1)\Big)\\
& + \Big(A_n \circ A_{n-1} \circ \ldots \circ A_3(\varphi_2) + d_1 A_{n-1} \circ A_{n-2} \circ \ldots \circ A_2(\varphi_1)\Big)\\
& \vdots\\
& + \Big(A_n(\varphi_{n-1}) + d_1 A_{n-1}(\varphi_{n-2}) + d_2 A_{n-2}(\varphi_{n-3}) + \ldots + d_{n-2} A_2(\varphi_1)\Big)\\
& + \Big(\varphi_n + d_1\varphi_{n-1} + d_2\varphi_{n-2} + \ldots + d_{n-2}\varphi_2 + d_{n-1}\varphi_1\Big).
\end{aligned}$$

We have

$$L(\vec{A})(x_1) = \sum_{k=1}^{n} \sum_{j=1}^{k} d_{k-j} \underbrace{A_{n-k+j} \circ A_{n-k+j-1} \circ \ldots \circ A_{j+1}}_{n-k}(\varphi_j),$$

where $d_0 = 1$.
Lemma 2.2 leads to $\delta_k^1(C; A_{n-k+1} \circ \ldots \circ A_2(\varphi_1), \ldots, A_{n-k} \circ \ldots \circ A_1(\varphi_n)) = (-1)^{k+1} \sum_{j=1}^{k} d_{k-j} A_{n-k+j} \circ A_{n-k+j-1} \circ \ldots \circ A_{j+1}(\varphi_j)$, and in consequence $L(\vec{A})(x_1) = \sum_{k=1}^{n} (-1)^{k+1} \delta_k^1(C; A_{n-k+1} \circ \ldots \circ A_2(\varphi_1), \ldots, A_{n-k} \circ \ldots \circ A_1(\varphi_n))$, which completes the proof. $\square$

The following construction was motivated by [6]. Operator L can also be obtained by using generalized characteristic polynomial of the system matrix C. *Generalized characteristic polynomial* of matrix C is defined by

$$\Delta_C(\vec{\lambda}) = \Delta_C(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{vmatrix} \lambda_1 & -1 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_{n-1} & -1 \\ d_n & d_{n-1} & \ldots & d_2 & \lambda_n + d_1 \end{vmatrix}.$$

Let us denote by $L(\vec{\lambda}) = L(\lambda_1, \lambda_2, \ldots, \lambda_n)$ polynomial

$$\lambda_n \lambda_{n-1} \ldots \lambda_1 + d_1 \lambda_{n-1} \lambda_{n-2} \ldots \lambda_1 + \ldots + d_{n-1} \lambda_1 + d_n.$$

Multiplying the last column of determinant $\Delta_C(\vec{\lambda})$ by λ_{n-1} and adding it to the penultimate, then multiplying obtained column with λ_{n-2} and adding to the previous one, and continuing in this fashion, we obtain

$$\Delta_C(\vec{\lambda}) = \begin{vmatrix} 0 & -1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -1 \\ L(\vec{\lambda}) & \frac{L(\vec{\lambda}) - d_n}{\lambda_1} & \ldots & \lambda_n \lambda_{n-1} + d_1 \lambda_{n-1} + d_2 & \lambda_n + d_1 \end{vmatrix}.$$

The Laplace expansion along the first column of the previous determinant yields $\Delta_C(\vec{\lambda}) = L(\vec{\lambda})$.

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