

ELLIPTICITY IN THE SG CALCULUS

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Dedicated to Professor Bogoljub Stanković on the occasion of his 90th birthday

Abstract. In this paper we report on recent results and applications of the so-called SG calculus, focusing, in particular, on its elliptic elements. The topics we discuss include propagation of singularities of (global versions of) wave-front sets, global regularity of solutions to linear and nonlinear partial differential equations on \mathbb{R}^d in appropriate function and distribution spaces, and spectral asymptotics of elliptic, selfadjoint, positive operators on \mathbb{R}^d .

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1. Introduction

A general calculus and corresponding definition of global ellipticity for partial differential operators and pseudo-differential operators on noncompact manifolds represent a relevant issue of the modern Mathematical Analysis. In the case of the Euclidean space \mathbb{R}^d , we may refer to the Weyl-Hörmander calculus cf. [50, Vol. 3]. In this framework, we mention two basic examples. One is given by the Γ -operators of M.A. Shubin [75], used in semi-classical analysis (see also D. Robert [70]), and many other contexts. The other is given by the so-called **SG** classes, having a related, but somewhat more ductile, structure. The basic properties of these classes can be found, for example, in the book [64] by F. Nicola and L. Rodino.

The present paper is a survey on recent results concerning **SG** classes. In fact, during the last few years, many new results appeared, concerning, in particular, global versions of wave-front sets, spectral theory and semilinear equations in this setting (the latter coming from problems in Mathematical Physics). Our attention will be mainly focused on **SG**-elliptic equations, by limiting to a short information on **SG**-hyperbolic problems and related Fourier integral operators. Similarly, extensions to a wide class of non-compact manifolds, namely, the so-called manifolds with ends, and other relevant topics will be omitted here. Nevertheless, we will try to give the flavor of the new ideas in the field.

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Let us start with a few historical notes and the basic definitions. Originally, the **SG** classes have been independently introduced by H.O. Cordes [22] (see also [23]) and C. Parenti [66]. Explicitly, denoting by \widehat{f} the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$, defined as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int e^{-ix \cdot \xi} f(x) dx,$$

SG-pseudodifferential operators $A = a(x, D) = \text{Op}(a)$ can be defined via the usual left-quantization

$$Au(x) = \frac{1}{(2\pi)^{d/2}} \int e^{ix \cdot \xi} a(x, \xi) \widehat{f}(\xi) d\xi,$$

starting from symbols $a(x, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with the property that, for arbitrary multiindices α, β , there exist constants $C_{\alpha\beta} \geq 0$ such that the estimates

$$(1.1) \quad |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{m-|\alpha|} \langle \xi \rangle^{\mu-|\beta|}$$

hold for fixed $m, \mu \in \mathbb{R}$ and all $x, \xi \in \mathbb{R}^d$. In (1.1) we used the notation $\langle y \rangle = \sqrt{1 + |y|^2}$, $y \in \mathbb{R}^d$. Symbols of this type belong to the class denoted by $\mathbf{SG}^{m, \mu}(\mathbb{R}^d)$, and the corresponding operators constitute the class $\mathbf{L}^{m, \mu}(\mathbb{R}^d) = \text{Op}(\mathbf{SG}^{m, \mu}(\mathbb{R}^d))$. In the sequel we will often simply write $\mathbf{SG}^{m, \mu}$ and $\mathbf{L}^{m, \mu}$, respectively.

These classes of operators form a graded algebra, i.e., $\mathbf{L}^{s, \sigma} \circ \mathbf{L}^{t, \tau} \subseteq \mathbf{L}^{s+t, \sigma+\tau}$, whose residual elements are operators with symbols in

$$\mathbf{SG}^{-\infty, -\infty}(\mathbb{R}^d) = \bigcap_{(m, \mu) \in \mathbb{R}^2} \mathbf{SG}^{m, \mu}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d}),$$

that is, those having kernel in $\mathcal{S}(\mathbb{R}^{2d})$, continuously mapping $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. An operator $A = \text{Op}(a) \in \mathbf{L}^{m, \mu}$ is called **SG**-elliptic if there exists $R \geq 0$ such that $a(x, \xi)$ is invertible for $|x| + |\xi| \geq R$ and

$$a(x, \xi)^{-1} = O(\langle x \rangle^{-m} \langle \xi \rangle^{-\mu}), \quad |x| + |\xi| \geq R.$$

Operators in $\mathbf{L}^{m, \mu}$ act continuously from $\mathcal{S}(\mathbb{R}^d)$ to itself, and extend as continuous operators from $\mathcal{S}'(\mathbb{R}^d)$ to itself and from $H_{s, \sigma}(\mathbb{R}^d)$ to $H_{s-m, \sigma-\mu}(\mathbb{R}^d)$, where $H_{t, \tau}(\mathbb{R}^d)$, $t, \tau \in \mathbb{R}$, denotes the weighted Sobolev space (or Sobolev-Kato space)

$$H_{t, \tau}(\mathbb{R}^d) = H_{t, \tau}^2(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{t, \tau} = \|\text{Op}(\vartheta_{t, \tau})u\|_{L^2} < \infty\},$$

$$\vartheta_{t, \tau}(x, \xi) = \langle x \rangle^t \langle \xi \rangle^\tau.$$

Incidentally, notice that $H_{s, \sigma}(\mathbb{R}^d) \hookrightarrow H_{t, \tau}(\mathbb{R}^d)$ when $s \geq t$ and $\sigma \geq \tau$, with compact embedding when both inequalities are strict, and $H_{0, 0}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, while

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{(t, \tau) \in \mathbb{R}^2} H_{t, \tau}(\mathbb{R}^d) \text{ and } \mathcal{S}'(\mathbb{R}^d) = \bigcup_{(t, \tau) \in \mathbb{R}^2} H_{t, \tau}(\mathbb{R}^d).$$

An elliptic **SG**-operator $A \in \mathbf{L}^{m,\mu}$ admits a parametrix $P \in \mathbf{L}^{-m,-\mu}$ such that

$$PA = I + K_1, \quad AP = I + K_2,$$

for suitable $K_1, K_2 \in \mathbf{L}^{-\infty,-\infty} = \text{Op}(\mathbf{SG}^{-\infty,-\infty})$, and it turns out to be a Fredholm operator. In 1987, E. Schrohe [74] introduced a class of non-compact manifolds, the so-called **SG**-manifolds, on which it is possible to transfer from \mathbb{R}^d the whole **SG**-calculus. In short, these are manifolds which admit a finite atlas whose changes of coordinates behave like symbols of order $(0,1)$ (see [74] for details and additional technical hypotheses). The manifolds with cylindrical ends are a special case of **SG**-manifolds, on which also the concept of **SG**-classical operator makes sense. Moreover, the principal symbol of a **SG**-classical operator A on a manifold with cylindrical ends M , in this case a triple $\sigma(A) = (\sigma_\psi(A), \sigma_e(A), \sigma_{\psi e}(A))$, has an invariant meaning on M , see, e.g., Y. Egorov, B.-W. Schulze [38], L. Maniccia, P. Panarese [58], and Section 5 below.

For a more geometric approach, leading to the so-called *scattering calculus*, on \mathbb{R}^d and on suitable classes of noncompact manifolds (including the manifolds with ends), see the theory developed by R. Melrose [60] and coauthors.

The paper is organized as follows. In Section 2 we fix the notation and recall the definition of the generalized **SG** symbols introduced by S. Coriasco, K. Johansson, J. Toft in [27]. In Section 3 we describe the regularity of solutions of **SG**-elliptic equations in the framework of general, weighted modulation spaces, in terms of an appropriate notion of (global) wave-front set. In Section 4 we describe the regularity of solutions of **SG**-elliptic equations in the framework of the Gel'fand-Shilov classes. In Section 5 we give the Weyl formula for an **SG**-elliptic, selfadjoint, positive operator on \mathbb{R}^d . To conclude, in Section 6 we give a short indication of the existing results on the manifolds with ends and in the **SG**-hyperbolic setting.

2. Preliminaries

We begin by fixing the notation and recalling some basic concepts which will be needed below. In Subsections 2.1-2.3 we summarize part of the contents of Sections 2 in [27, 28]. Here and in what follows, $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$, where $A \lesssim B$ means that $A \leq c \cdot B$, for a suitable constant $c > 0$.

2.1. Weight functions

Let ω and v be positive measurable functions on \mathbb{R}^d . Then ω is called v -moderate if

$$(2.1) \quad \omega(x+y) \lesssim \omega(x)v(y)$$

If v in (2.1) can be chosen as a polynomial, then ω is called a function or weight of *polynomial type*. We let $\mathcal{P}(\mathbb{R}^d)$ be the set of all polynomial type functions on \mathbb{R}^d . If $\omega(x, \xi) \in \mathcal{P}(\mathbb{R}^{2d})$ is constant with respect to the x -variable or the ξ -variable, then we write $\omega(\xi)$, respectively $\omega(x)$, instead of $\omega(x, \xi)$, and consider

ω as an element in $\mathcal{P}(\mathbb{R}^{2d})$ or in $\mathcal{P}(\mathbb{R}^d)$ depending on the situation. We say that v is submultiplicative if (2.1) holds for $\omega = v$. For convenience we assume that all submultiplicative weights are even, and v always stands in the sequel for a submultiplicative weight, if nothing else is stated.

Without loss of generality we may assume that every $\omega \in \mathcal{P}(\mathbb{R}^d)$ is smooth and satisfies the ellipticity condition $\partial^\alpha \omega / \omega \in L^\infty$. In fact, by Lemma 1.2 in [76] it follows that for each $\omega \in \mathcal{P}(\mathbb{R}^d)$, there is a smooth and elliptic $\omega_0 \in \mathcal{P}(\mathbb{R}^d)$ which is equivalent to ω in the sense

$$(2.2) \quad \omega \asymp \omega_0.$$

The weights involved in the sequel have to satisfy additional conditions. More precisely let $r, \rho \geq 0$. Then $\mathcal{P}_{r,\rho}(\mathbb{R}^{2d})$ is the set of all $\omega(x, \xi)$ in $\mathcal{P}(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d})$ such that

$$(2.3) \quad \langle x \rangle^{r|\alpha|} \langle \xi \rangle^{\rho|\beta|} \frac{\partial_x^\alpha \partial_\xi^\beta \omega(x, \xi)}{\omega(x, \xi)} \in L^\infty(\mathbb{R}^{2d}),$$

for every multi-indices α and β . Any weight $\omega \in \mathcal{P}_{r,\rho}(\mathbb{R}^{2d})$ is then called **SG**-moderate on \mathbb{R}^{2d} , of order r and ρ . Notice that $\mathcal{P}_{r,\rho}$ is different here compared to [26], $\mathcal{P}_{0,0}(\mathbb{R}^{2d}) = \mathcal{P}(\mathbb{R}^{2d})$, and for $r > 0, \rho > 0$ there are elements in $\mathcal{P}(\mathbb{R}^{2d})$ which have no equivalent elements in $\mathcal{P}_{r,\rho}(\mathbb{R}^{2d})$. On the other hand, if $m, \mu \in \mathbf{R}$ and $r, \rho \in [0, 1]$, then $\mathcal{P}_{r,\rho}(\mathbb{R}^{2d})$ contains all weights of the form

$$(2.4) \quad \vartheta_{m,\mu}(x, \xi) \equiv \langle x \rangle^m \langle \xi \rangle^\mu,$$

which are one of the most common type of weights.

2.2. Modulation spaces

Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then the *short-time Fourier transform* of $f \in \mathcal{S}(\mathbb{R}^d)$ with respect to (the window function) ϕ is defined by

$$(2.5) \quad V_\phi f(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \overline{\phi(y-x)} e^{-i\langle y, \xi \rangle} dy.$$

More generally, the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to $\phi \in \mathcal{S}'(\mathbb{R}^d)$ is defined by

$$(2.5)' \quad (V_\phi f) = \mathcal{F}_2 F, \quad \text{where} \quad F(x, y) = (f \otimes \overline{\phi})(y, y-x).$$

Here $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathcal{S}'(\mathbb{R}^{2d})$ with respect to the y -variable. We refer to [42, 44] for more facts about the short-time Fourier transform. To introduce the modulation spaces, we first recall that a Banach function space \mathcal{B} , continuously embedded in $L^1_{\text{loc}}(\mathbb{R}^d)$, is called a (*translation*) *invariant BF-space on \mathbb{R}^d* , with respect to a submultiplicative weight $v \in \mathcal{P}(\mathbb{R}^d)$, if there is a constant C such that the following conditions are fulfilled:

1. $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbb{R}^d)$ (continuous embeddings);

2. if $x \in \mathbb{R}^d$ and $f \in \mathcal{B}$, then $f(\cdot - x) \in \mathcal{B}$, and

$$\|f(\cdot - x)\|_{\mathcal{B}} \leq C v(x) \|f\|_{\mathcal{B}};$$

3. if $f, g \in L^1_{\text{loc}}(\mathbb{R}^d)$ satisfy $g \in \mathcal{B}$ and $|f| \leq |g|$ almost everywhere, then $f \in \mathcal{B}$ and

$$\|f\|_{\mathcal{B}} \leq C \|g\|_{\mathcal{B}};$$

4. if $f \in \mathcal{B}$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $f * \varphi \in \mathcal{B}$, and

$$\|f * \varphi\|_{\mathcal{B}} \leq \|\varphi\|_{L^1_{(v)}} \|f\|_{\mathcal{B}}.$$

The following definition of modulation spaces is due to Feichtinger [41]. Let \mathcal{B} be a translation invariant BF-space on \mathbb{R}^{2d} with respect to $v \in \mathcal{P}(\mathbb{R}^{2d})$, $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ and let $\omega \in \mathcal{P}(\mathbb{R}^{2d})$ be such that ω is v -moderate. The *modulation space* $M(\omega, \mathcal{B})$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_\phi f \cdot \omega \in \mathcal{B}$. We notice that $M(\omega, \mathcal{B})$ is a Banach space with the norm

$$\|f\|_{M(\omega, \mathcal{B})} \equiv \|V_\phi f \cdot \omega\|_{\mathcal{B}}$$

(cf. [40]).

Remark 2.1. Assume that $p, q \in [1, \infty]$, and let $L_1^{p,q}(\mathbb{R}^{2d})$ and $L_2^{p,q}(\mathbb{R}^{2d})$ be the sets of all $F \in L^1_{\text{loc}}(\mathbb{R}^{2d})$ such that

$$\|F\|_{L_1^{p,q}} \equiv \left(\int \left(\int |F(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty$$

and

$$\|F\|_{L_2^{p,q}} \equiv \left(\int \left(\int |F(x, \xi)|^q d\xi \right)^{p/q} dx \right)^{1/p} < \infty.$$

Then $M(\omega, L_1^{p,q}(\mathbb{R}^{2d}))$ is equal to the *classical* modulation space $M_{(\omega)}^{p,q}(\mathbb{R}^d)$, and $M(\omega, L_2^{p,q}(\mathbb{R}^{2d}))$ is equal to the space $W_{(\omega)}^{p,q}(\mathbb{R}^d)$, related to Wiener-amalgam spaces (cf. [39, 40, 41, 44]).

Remark 2.2. Several important spaces agree with certain modulation spaces. In fact, let $s, \sigma \in \mathbf{R}$. If $\omega = \vartheta_{s, \sigma}$ (cf. (2.4)), then $M_{(\omega)}^2(\mathbb{R}^d)$ is equal to $H_{\sigma, s}^2(\mathbb{R}^d)$ in [30, 60], the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\langle x \rangle^s \langle D \rangle^\sigma f \in L^2(\mathbb{R}^d)$. In particular, if $s = 0$ ($\sigma = 0$), then $M_{(\omega)}^2(\mathbb{R}^d)$ equals to $H_\sigma(\mathbb{R}^d) = H_\sigma^2(\mathbb{R}^d)$ ($L_s^2(\mathbb{R}^d)$). Furthermore, if instead $\omega(x, \xi) = \langle x, \xi \rangle^s$, then $M_{(\omega)}^2(\mathbb{R}^d)$ is equal to the Sobolev-Shubin space of order s . (cf. e.g. [7, 57]).

2.3. Pseudo-differential operators and SG symbol classes

Let $a \in \mathcal{S}(\mathbb{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\text{Op}_t(a)$ is the linear and continuous operator on $\mathcal{S}(\mathbb{R}^d)$ defined by the formula

$$(\text{Op}_t(a)f)(x) = (2\pi)^{-d} \iint e^{i\langle x-y, \xi \rangle} a((1-t)x + ty, \xi) f(y) dy d\xi$$

(cf. Chapter XVIII in [50]). For general $a \in \mathcal{S}'(\mathbb{R}^{2d})$, the pseudo-differential operator $\text{Op}_t(a)$ is defined as the continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ with distribution kernel

$$K_{t,a}(x, y) = (2\pi)^{-d/2} (\mathcal{F}_2^{-1}a)((1-t)x + ty, x - y).$$

If $t = 0$, then $\text{Op}_t(a)$ is the Kohn-Nirenberg representation $\text{Op}(a) = a(x, D)$, and if $t = 1/2$, then $\text{Op}_t(a)$ is the Weyl quantization.

In the sequel, a belongs to a generalized **SG**-symbol class, which we shall consider now. Let $m, \mu, r, \rho \in \mathbf{R}$ be fixed. Then the **SG**-class $\mathbf{SG}_{r,\rho}^{m,\mu}(\mathbb{R}^{2d})$ is the set of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \lesssim \langle x \rangle^{m-r|\alpha|} \langle \xi \rangle^{\mu-\rho|\beta|},$$

for all multi-indices α and β . Usually we assume that $r, \rho \geq 0$ and $\rho + r > 0$. When $r = \rho = 1$, we write $\mathbf{SG}^{m,\mu}$ in place of $\mathbf{SG}_{1,1}^{m,\mu}$, cf. (1.1) in the Introduction.

More generally, assume that $\omega \in \mathcal{P}_{r,\rho}(\mathbb{R}^{2d})$. Then $\mathbf{SG}_{r,\rho}^{(\omega)}(\mathbb{R}^{2d})$ consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \lesssim \omega(x, \xi) \langle x \rangle^{-r|\alpha|} \langle \xi \rangle^{-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^d,$$

for all multi-indices α and β . We notice that $\mathbf{SG}_{r,\rho}^{(\omega)} = \mathbf{SG}_{r,\rho}^{m,\mu}$ when $\omega = \vartheta_{m,\mu}$ (see (2.4)). For conveniency, we set

$$\mathbf{SG}_\rho^{(\omega\vartheta_{-\infty,0})}(\mathbb{R}^{2d}) = \mathbf{SG}_{r,\rho}^{(\omega\vartheta_{-\infty,0})}(\mathbb{R}^{2d}) \equiv \bigcap_{N \geq 0} \mathbf{SG}_{r,\rho}^{(\omega\vartheta_{-N,0})}(\mathbb{R}^{2d}),$$

$$\mathbf{SG}_r^{(\omega\vartheta_{0,-\infty})}(\mathbb{R}^{2d}) = \mathbf{SG}_{r,\rho}^{(\omega\vartheta_{0,-\infty})}(\mathbb{R}^{2d}) \equiv \bigcap_{N \geq 0} \mathbf{SG}_{r,\rho}^{(\omega\vartheta_{0,-N})}(\mathbb{R}^{2d}),$$

and

$$\mathbf{SG}^{(\omega\vartheta_{-\infty,-\infty})}(\mathbb{R}^{2d}) = \mathbf{SG}_{r,\rho}^{(\omega\vartheta_{-\infty,-\infty})}(\mathbb{R}^{2d}) \equiv \bigcap_{N \geq 0} \mathbf{SG}_{r,\rho}^{(\omega\vartheta_{-N,-N})}(\mathbb{R}^{2d}).$$

We observe that $\mathbf{SG}_{r,\rho}^{(\omega\vartheta_{-\infty,0})}(\mathbb{R}^{2d})$ is independent of r , $\mathbf{SG}_{r,\rho}^{(\omega\vartheta_{0,-\infty})}(\mathbb{R}^{2d})$ is independent of ρ , and that $\mathbf{SG}_{r,\rho}^{(\omega\vartheta_{-\infty,-\infty})}(\mathbb{R}^{2d})$ is independent of both r and ρ . Furthermore, for any $x_0, \xi_0 \in \mathbb{R}^d$ we have

$$\mathbf{SG}_\rho^{(\omega\vartheta_{-\infty,0})}(\mathbb{R}^{2d}) = \mathbf{SG}_\rho^{(\omega_0\vartheta_{-\infty,0})}(\mathbb{R}^{2d}), \quad \text{when } \omega_0(\xi) = \omega(x_0, \xi),$$

$$\mathbf{SG}_r^{(\omega\vartheta_{0,-\infty})}(\mathbb{R}^{2d}) = \mathbf{SG}_r^{(\omega_0\vartheta_{0,-\infty})}(\mathbb{R}^{2d}), \quad \text{when } \omega_0(x) = \omega(x, \xi_0),$$

and

$$\mathbf{SG}^{(\omega^\vartheta - \infty, -\infty)}(\mathbb{R}^{2d}) = \mathcal{S}(\mathbb{R}^{2d}).$$

The following result shows that the concept of asymptotic expansion extends to the classes $\mathbf{SG}_{r,\rho}^{(\omega)}(\mathbb{R}^{2d})$. We refer to [37, Theorem 8] for the proof.

Proposition 2.3. *Let $r, \rho \geq 0$ satisfy $r + \rho > 0$, and let $\{s_j\}_{j \geq 0}$ and $\{\sigma_j\}_{j \geq 0}$ be sequences of non-positive numbers such that $\lim_{j \rightarrow \infty} s_j = -\infty$ when $r > 0$ and $s_j = 0$ otherwise, and $\lim_{j \rightarrow \infty} \sigma_j = -\infty$ when $\rho > 0$ and $\sigma_j = 0$ otherwise. Also let $a_j \in \mathbf{SG}_{r,\rho}^{(\omega_j)}(\mathbb{R}^{2d})$, $j = 0, 1, \dots$, where $\omega_j = \omega \cdot \vartheta_{s_j, \sigma_j}$. Then there is a symbol $a \in \mathbf{SG}_{r,\rho}^{(\omega)}(\mathbb{R}^{2d})$ such that*

$$a - \sum_{j=0}^N a_j \in \mathbf{SG}_{r,\rho}^{(\omega_{N+1})}(\mathbb{R}^{2d}).$$

The symbol a is uniquely determined modulo a remainder h , where

$$h \in \mathbf{SG}_{\rho}^{(\omega^{\vartheta - \infty, 0})}(\mathbb{R}^{2d}) \quad \text{when } r > 0,$$

$$h \in \mathbf{SG}_r^{(\omega^{\vartheta 0, -\infty})}(\mathbb{R}^{2d}) \quad \text{when } \rho > 0,$$

$$h \in \mathcal{S}(\mathbb{R}^{2d}) \quad \text{when } r > 0, \rho > 0.$$

Definition 2.4. The notation $a \sim \sum a_j$ is used when a and a_j fulfill the hypothesis in Proposition 2.3. Furthermore, the formal sum

$$\sum_{j \geq 0} a_j$$

is called an *asymptotic expansion*.

It is a well-known fact that \mathbf{SG} -operators give rise to linear continuous mappings from $\mathcal{S}(\mathbb{R}^d)$ to itself, extendable as linear continuous mappings from $\mathcal{S}'(\mathbb{R}^d)$ to itself. They also act continuously between modulation spaces. Indeed, if $a \in \mathbf{SG}_{r,\rho}^{(\omega_0)}(\mathbb{R}^{2d})$, then $\text{Op}_t(a)$ is continuous from $M(\omega, \mathcal{B})$ to $M(\omega/\omega_0, \mathcal{B})$ (cf. [27]). Moreover, for every fixed $\omega_0 \in \mathcal{P}_{r,\rho}(\mathbb{R}^{2d})$, $r, \rho \geq 0$, there exist $a \in \mathbf{SG}_{r,\rho}^{(\omega_0)}(\mathbb{R}^{2d})$ and $b \in \mathbf{SG}_{r,\rho}^{(1/\omega_0)}(\mathbb{R}^{2d})$ such that, for every choice of $\omega \in \mathcal{P}(\mathbb{R}^{2d})$ and every translation invariant BF-space \mathcal{B} on \mathbb{R}^{2d} , the mappings

$$\text{Op}_t(a) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d), \quad \text{Op}_t(a) : \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$$

$$\text{and } \text{Op}_t(a) : M(\omega, \mathcal{B}) \rightarrow M(\omega/\omega_0, \mathcal{B}).$$

are continuous bijections with inverses $\text{Op}_t(b)$.

3. Regularity of solutions in modulation spaces

We express the regularity of solutions to **SG**-elliptic equations within the environment of the general, weighted modulation spaces, in terms of appropriate global wave-front sets. We first recall the corresponding definition, given by S. Coriasco, K. Johansson, J. Toft in [27], which generalizes the analysis performed in S. Coriasco, K. Johansson, J. Toft [26] and in S. Coriasco, L. Maniccia [30], see also S. Coriasco, R. Schulz [35]. The contents of Subsections 3.1 and 3.2 again come from [27, 28].

3.1. Global Wave-front Sets

Here we recall the definition given in [27] of global wave-front sets for temperate distributions with respect to Banach or Fréchet spaces and state some of their properties (see also [28]). First of all, we recall the definitions of the set of characteristic points which we use in this framework. Remember that if $a \in \mathbf{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$, then

$$|a(x, \xi)| \lesssim \omega_0(x, \xi).$$

On the other hand, a is invertible, in the sense that $1/a$ is a symbol in $\mathbf{SG}_{r,\rho}^{(1/\omega_0)}(\mathbf{R}^{2d})$, if and only if

$$(3.1) \quad \omega_0(x, \xi) \lesssim |a(x, \xi)|.$$

We need to deal with the situations where (3.1) holds only in certain (conic-shaped) subset of $\mathbf{R}^d \times \mathbf{R}^d$. Here we let Ω_m , $m = 1, 2, 3$, be the sets

$$(3.2) \quad \begin{aligned} \Omega_1 &= \mathbf{R}^d \times (\mathbf{R}^d \setminus 0), & \Omega_2 &= (\mathbf{R}^d \setminus 0) \times \mathbf{R}^d, \\ \Omega_3 &= (\mathbf{R}^d \setminus 0) \times (\mathbf{R}^d \setminus 0), \end{aligned}$$

Definition 3.1. Let $r, \rho \geq 0$, $\omega_0 \in \mathcal{P}_{r,\rho}(\mathbf{R}^{2d})$, Ω_m , $m = 1, 2, 3$ be as in (3.2), and let $a \in \mathbf{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$.

1. a is called *locally* or *type-1 invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \Omega_1$, if there exist a neighbourhood X of x_0 , an open conical neighbourhood Γ of ξ_0 and a positive constant R such that (3.1) holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \geq R$.
2. a is called *Fourier-locally* or *type-2 invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \Omega_2$, if there exist an open conical neighbourhood Γ of x_0 , a neighbourhood X of ξ_0 and a positive constant R such that (3.1) holds for $x \in \Gamma$, $|x| \geq R$ and $\xi \in X$.
3. a is called *oscillating* or *type-3 invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \Omega_3$, if there exist open conical neighbourhoods Γ_1 of x_0 and Γ_2 of ξ_0 , and a positive constant R such that (3.1) holds for $x \in \Gamma_1$, $|x| \geq R$, $\xi \in \Gamma_2$ and $|\xi| \geq R$.

If $m \in \{1, 2, 3\}$ and a is *not* type- m invertible with respect to ω_0 at $(x_0, \xi_0) \in \Omega_m$, then (x_0, ξ_0) is called *type- m characteristic* for a with respect to ω_0 . The set of type- m characteristic points for a with respect to ω_0 is denoted by $\text{Char}_{(\omega_0)}^m(a)$.

The (*global*) *set of characteristic points* (the characteristic set), for a symbol $a \in \mathbf{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$ with respect to ω_0 is defined as

$$\text{Char}(a) = \text{Char}_{(\omega_0)}(a) = \text{Char}_{(\omega_0)}^1(a) \bigcup \text{Char}_{(\omega_0)}^2(a) \bigcup \text{Char}_{(\omega_0)}^3(a).$$

Remark 3.2. Let $X \subseteq \mathbf{R}^d$ be open and $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbf{R}^d \setminus 0$ be open cones. Then the following is true.

1. if $x_0 \in X$, $\xi_0 \in \Gamma$, $\varphi \in \mathcal{C}_{x_0}(X)$ and $\psi \in \mathcal{C}_{\xi_0}^{\text{dir}}(\Gamma)$, then $c_1 = \varphi \otimes \psi$ belongs to $\mathbf{SG}_{1,1}^{0,0}(\mathbf{R}^{2d})$, and is type-1 invertible at (x_0, ξ_0) ;
2. if $x_0 \in \Gamma$, $\xi_0 \in X$, $\psi \in \mathcal{C}_{x_0}^{\text{dir}}(\Gamma)$ and $\varphi \in \mathcal{C}_{\xi_0}(X)$, then $c_2 = \varphi \otimes \psi$ belongs to $\mathbf{SG}_{1,1}^{0,0}(\mathbf{R}^{2d})$, and is type-2 invertible at (x_0, ξ_0) ;
3. if $x_0 \in \Gamma_1$, $\xi_0 \in \Gamma_2$, $\psi_1 \in \mathcal{C}_{x_0}^{\text{dir}}(\Gamma_1)$ and $\psi_2 \in \mathcal{C}_{\xi_0}^{\text{dir}}(\Gamma_2)$, then $c_3 = \psi_1 \otimes \psi_2$ belongs to $\mathbf{SG}_{1,1}^{0,0}(\mathbf{R}^{2d})$, and is type-3 invertible at (x_0, ξ_0) .

Remark 3.3. In the case $\omega_0 = 1$ we exclude the phrase “with respect to ω_0 ” in Definition 3.1. For example, $a \in \mathbf{SG}_{r,\rho}^{0,0}(\mathbf{R}^{2d})$ is *type-1 invertible* at $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ if $(x_0, \xi_0) \notin \text{Char}_{(\omega_0)}^1(a)$ with $\omega_0 = 1$. This means that there exist a neighbourhood X of x_0 , an open conical neighbourhood Γ of ξ_0 and $R > 0$ such that (3.1) holds for $\omega_0 = 1$, $x \in X$ and $\xi \in \Gamma$ satisfies $|\xi| \geq R$.

In the next definition we introduce different classes of cutoff functions (see also Definition 1.9 in [26]).

Definition 3.4. Let $X \subseteq \mathbf{R}^d$ be open, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.

1. A smooth function φ on \mathbf{R}^d is called a *cutoff (function)* with respect to x_0 and X , if $0 \leq \varphi \leq 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighbourhood of x_0 . The set of cutoffs with respect to x_0 and X is denoted by $\mathcal{C}_{x_0}(X)$ or \mathcal{C}_{x_0} .
2. A smooth function ψ on \mathbf{R}^d is called a *directional cutoff (function)* with respect to ξ_0 and Γ , if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1 \subseteq \Gamma$ of ξ_0 such that the following is true:
 - $0 \leq \psi \leq 1$ and $\text{supp } \psi \subseteq \Gamma$;
 - $\psi(t\xi) = \psi(\xi)$ when $t \geq 1$ and $|\xi| \geq R$;
 - $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \geq R$.

The set of directional cutoffs with respect to ξ_0 and Γ is denoted by $\mathcal{C}_{\xi_0}^{\text{dir}}(\Gamma)$ or $\mathcal{C}_{\xi_0}^{\text{dir}}$.

The next proposition shows that $\text{Op}_t(a)$ for $t \in \mathbf{R}$ satisfies convenient invertibility properties of the form

$$(3.3) \quad \text{Op}_t(a) \text{Op}_t(b) = \text{Op}_t(c) + \text{Op}_t(h),$$

outside the set of characteristic points for a symbol a . Here $\text{Op}_t(b)$, $\text{Op}_t(c)$ and $\text{Op}_t(h)$ have the roles of “local inverse”, “local identity” and smoothing operators respectively. From these statements it also follows that our set of characteristic points in Definition 3.1 are related to those in [30, 50]. We let \mathbb{I}_m , $m = 1, 2, 3$, be the sets

$$(3.4) \quad \mathbb{I}_1 \equiv [0, 1] \times (0, 1], \mathbb{I}_2 \equiv (0, 1] \times [0, 1], \mathbb{I}_3 \equiv (0, 1] \times (0, 1] = \mathbb{I}_1 \cap \mathbb{I}_2.$$

which will be useful in the sequel.

Proposition 3.5. *Let $m \in \{1, 2, 3\}$, $(r, \rho) \in \mathbb{I}_m$, $\omega_0 \in \mathcal{P}_{r, \rho}(\mathbf{R}^{2d})$ and let $a \in \mathbf{SG}_{r, \rho}^{(\omega_0)}(\mathbf{R}^{2d})$. Also let Ω_m be as in (3.2), $(x_0, \xi_0) \in \Omega_m$, and let (r_0, ρ_0) be equal to $(r, 0)$, $(0, \rho)$ and (r, ρ) when m is equal to 1, 2 and 3, respectively. Then the following conditions are equivalent:*

1. $(x_0, \xi_0) \notin \text{Char}_{(\omega_0)}^m(a)$;
2. there is an element $c \in \mathbf{SG}_{r, \rho}^{0,0}$ which is type- m invertible at (x_0, ξ_0) , and an element $b \in \mathbf{SG}_{r, \rho}^{(1/\omega_0)}$ such that $ab = c$;
3. (3.3) holds for some $c \in \mathbf{SG}_{r, \rho}^{0,0}$ which is type- m invertible at (x_0, ξ_0) , and some elements $h \in \mathbf{SG}_{r, \rho}^{-r_0, -\rho_0}$ and $b \in \mathbf{SG}_{r, \rho}^{(1/\omega_0)}$;
4. (3.3) holds for some $c_m \in \mathbf{SG}_{r, \rho}^{0,0}$ in Remark 3.2 which is type- m invertible at (x_0, ξ_0) , and some elements h and $b \in \mathbf{SG}_{r, \rho}^{(1/\omega_0)}$, where $h \in \mathcal{S}$ when $m \in \{1, 3\}$ and $h \in \mathbf{SG}^{-\infty, 0}$ when $m = 2$.

Furthermore, if $t = 0$, then the supports of b and h can be chosen to be contained in $X \times \mathbf{R}^d$ when $m = 1$, in $\Gamma \times \mathbf{R}^d$ when $m = 2$, and in $\Gamma_1 \times \mathbf{R}^d$ when $m = 3$.

We can now introduce the complements of the wave-front sets. More precisely, let Ω_m , $m \in \{1, 2, 3\}$, be given by (3.2), \mathcal{B} be a Banach or Fréchet space such that $\mathcal{S}(\mathbf{R}^d) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbf{R}^d)$, and let $f \in \mathcal{S}'(\mathbf{R}^d)$. Then the point $(x_0, \xi_0) \in \Omega_m$ is called *type- m regular* for f with respect to \mathcal{B} , if

$$(3.5) \quad \text{Op}(c_m)f \in \mathcal{B},$$

for some c_m in Remark 3.2. The set of all type- m regular points for f with respect to \mathcal{B} , is denoted by $\Theta_{\mathcal{B}}^m(f)$.

Definition 3.6. Let $m \in \{1, 2, 3\}$, Ω_m be as in (3.2), and let \mathcal{B} be a Banach or Fréchet space such that $\mathcal{S}(\mathbf{R}^d) \subseteq \mathcal{B} \subset \mathcal{S}'(\mathbf{R}^d)$.

1. the *type- m wave-front set* of $f \in \mathcal{S}'(\mathbf{R}^d)$ with respect to \mathcal{B} is the complement of $\Theta_{\mathcal{B}}^m(f)$ in Ω_m , and is denoted by $\text{WF}_{\mathcal{B}}^m(f)$;
2. the *global wave-front set* $\text{WF}_{\mathcal{B}}(f) \subseteq (\mathbf{R}^d \times \mathbf{R}^d) \setminus \{0\}$ is the set

$$\text{WF}_{\mathcal{B}}(f) \equiv \text{WF}_{\mathcal{B}}^1(f) \cup \text{WF}_{\mathcal{B}}^2(f) \cup \text{WF}_{\mathcal{B}}^3(f).$$

The sets $\text{WF}_{\mathcal{B}}^1(f)$, $\text{WF}_{\mathcal{B}}^2(f)$ and $\text{WF}_{\mathcal{B}}^3(f)$ in Definition 3.6, are also called the *local*, *Fourier-local* and *oscillating* wave-front set of f with respect to \mathcal{B} .

Remark 3.7. Let Ω_m , $m = 1, 2, 3$ be the same as in (3.2).

1. If $\Omega \subseteq \Omega_1$, and $(x_0, \xi_0) \in \Omega \iff (x_0, \sigma\xi_0) \in \Omega$ for $\sigma \geq 1$, then Ω is called *1-conical*;
2. If $\Omega \subseteq \Omega_2$, and $(x_0, \xi_0) \in \Omega \iff (sx_0, \xi_0) \in \Theta_{\mathcal{B}}^2(f)$ for $s \geq 1$, then Ω is called *2-conical*;
3. If $\Omega \subseteq \Omega_3$, and $(x_0, \xi_0) \in \Omega \iff (sx_0, \sigma\xi_0) \in \Omega$ for $s, \sigma \geq 1$, then Ω is called *3-conical*.

By (3.5) and the paragraph before Definition 3.6, it follows that if $m = 1, 2, 3$, then $\Theta_{\mathcal{B}}^m(f)$ is m -conical. The same holds for $\text{WF}_{\mathcal{B}}^m(f)$, $m = 1, 2, 3$, by Definition 3.6, noticing that, for any $x_0 \in \mathbf{R}^r \setminus \{0\}$, any open cone $\Gamma \ni x_0$, and any $s > 0$, $\mathcal{C}_{x_0}^{\text{dir}}(\Gamma) = \mathcal{C}_{sx_0}^{\text{dir}}(\Gamma)$. For any $R > 0$ and $m \in \{1, 2, 3\}$, we set

$$\Omega_{1,R} \equiv \{(x, \xi) \in \Omega_1; |\xi| \geq R\}, \quad \Omega_{2,R} \equiv \{(x, \xi) \in \Omega_2; |x| \geq R\},$$

$$\Omega_{3,R} \equiv \{(x, \xi) \in \Omega_3; |x|, |\xi| \geq R\}$$

Evidently, Ω_m^R is m -conical for every $m \in \{1, 2, 3\}$.

We now specify the subspaces of $\mathcal{S}'(\mathbf{R}^d)$ which are “admissible” in the present context.

Definition 3.8. Let $r, \rho \in [0, 1]$, $t \in \mathbf{R}$, \mathcal{B} be a topological vector space of distributions on \mathbf{R}^d such that

$$\mathcal{S}(\mathbf{R}^d) \subseteq \mathcal{B} \subseteq \mathcal{S}'(\mathbf{R}^d)$$

with continuous embeddings. Then \mathcal{B} is called **SG-admissible** (with respect to r , ρ and d) when $\text{Op}_t(a)$ maps \mathcal{B} continuously into itself, for every $a \in \mathbf{SG}_{r,\rho}^{0,0}(\mathbf{R}^d)$. If \mathcal{B} and \mathcal{C} are **SG-admissible** with respect to r , ρ and d , and $\omega_0 \in \mathcal{P}_{r,\rho}(\mathbf{R}^{2d})$, then the pair $(\mathcal{B}, \mathcal{C})$ is called **SG-ordered** (with respect to ω_0), when the mappings

$$\text{Op}_t(a) : \mathcal{B} \rightarrow \mathcal{C} \quad \text{and} \quad \text{Op}_t(b) : \mathcal{C} \rightarrow \mathcal{B}$$

are continuous for every $a \in \mathbf{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$ and $b \in \mathbf{SG}_{r,\rho}^{(1/\omega_0)}(\mathbf{R}^{2d})$.

From now on we assume that \mathcal{B} in Definition 3.6 is **SG-admissible**, and recall that Sobolev-Kato spaces and, more generally, modulation spaces, and $\mathcal{S}(\mathbf{R}^d)$ are **SG-admissible**, see [27, 28].

3.2. Global wave-front sets of solutions of SG-elliptic equations

The next result describes the relation between “regularity with respect to \mathcal{B} ” of temperate distributions and global wave-front sets.

Proposition 3.9. *Let \mathcal{B} be SG-admissible, and let $f \in \mathcal{S}'(\mathbf{R}^d)$. Then*

$$f \in \mathcal{B} \iff \text{WF}_{\mathcal{B}}(f) = \emptyset.$$

The main results in this theory are microlocality and microellipticity for these global wave-front sets and pseudo-differential operators in $\text{Op}(\mathbf{SG}_{r,\rho}^{(\omega_0)})$, see [27].

Theorem 3.10. *Let $r, \rho \in [0, 1]$, $t \in \mathbf{R}$, $\omega_0 \in \mathcal{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathbf{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$ and let $f \in \mathcal{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B}, \mathcal{C})$ be a SG-ordered pair with respect to ω_0 . Then, the following properties hold true.*

1. *If in addition $\rho > 0$, then*

$$(3.6) \quad \text{WF}_{\mathcal{C}}^{\psi}(\text{Op}_t(a)f) \subseteq \text{WF}_{\mathcal{B}}^{\psi}(f) \subseteq \text{WF}_{\mathcal{C}}^{\psi}(\text{Op}_t(a)f) \bigcup \text{Char}_{(\omega_0)}^{\psi}(a).$$

2. *If in addition $r > 0$, then*

$$(3.7) \quad \text{WF}_{\mathcal{C}}^e(\text{Op}_t(a)f) \subseteq \text{WF}_{\mathcal{B}}^e(f) \subseteq \text{WF}_{\mathcal{C}}^e(\text{Op}_t(a)f) \bigcup \text{Char}_{(\omega_0)}^e(a).$$

3. *If in addition $r, \rho > 0$, then*

$$(3.8) \quad \text{WF}_{\mathcal{C}}^{\psi e}(\text{Op}_t(a)f) \subseteq \text{WF}_{\mathcal{B}}^{\psi e}(f) \subseteq \text{WF}_{\mathcal{C}}^{\psi e}(\text{Op}_t(a)f) \bigcup \text{Char}_{(\omega_0)}^{\psi e}(a).$$

The above results imply that operators which are elliptic with respect to $\omega_0 \in \mathcal{P}_{\rho,\delta}(\mathbf{R}^{2d})$ when $0 < r, \rho \leq 1$ preserve the global wave-front set of temperate distributions. The next proposition is an immediate corollary of microlocality and microellipticity for operators in $\text{Op}(\mathbf{SG}_{r,\rho}^{(\omega_0)})$:

Proposition 3.11. *Let $m \in \{1, 2, 3\}$, $(r, \rho) \in \mathbb{I}_m$, $t \in \mathbf{R}$, $\omega_0 \in \mathcal{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathbf{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$ be SG-elliptic with respect to ω_0 and let $f \in \mathcal{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B}, \mathcal{C})$ be a SG-ordered pair with respect to ω_0 . Then,*

$$\text{WF}_{\mathcal{C}}^m(\text{Op}_t(a)f) = \text{WF}_{\mathcal{B}}^m(f).$$

Finally, the next result expresses SG-elliptic regularity in the general context of SG-admissible spaces. In particular, as already explained above, it holds for general, weighted modulation spaces $(\mathcal{B}, \mathcal{C})$ which are a SG-ordered pair with respect to ω_0 .

Theorem 3.12. *Assume that the hypothesis in Theorem 3.11 is fulfilled, let $f \in \mathcal{C}$ and let $u \in \mathcal{S}'(\mathbb{R}^d)$ be a solution to the equation*

$$(3.9) \quad \text{Op}_t(a)u = f.$$

Then, $f \in \mathcal{B}$.

Remark 3.13. In view of Theorems 3.9 and 3.12, and of the theory developed in [30] and in [27, 28], it follows immediately that the kernel of **SG**-elliptic operators consists of smooth functions belonging to $\mathcal{S}(\mathbb{R}^d)$.

The results above can be applied, in particular, to families of modified heat operators, which turn out to be elliptic elements in the generalized **SG** classes, with a suitable choice of the involved weight, see the examples in [27].

Other types of wave-front sets, capturing different notions of regularity, and somehow related to the versions of **SG**-wave-front set described above exist. To mention a few, we recall the works by M. Borsero and R. Schulz [9], L. Robbiano and C. Zuily [71], L. Rodino and P. Wahlberg [72].

4. Regularity of solutions in Gel'fand-Shilov classes

In the previous sections, see in particular Remark 3.13, we observed that, under the assumption of **SG**-ellipticity, the solutions in $\mathcal{S}'(\mathbb{R}^d)$ of the homogeneous equations belong to $\mathcal{S}(\mathbb{R}^d)$. In particular, in the self-adjoint case, the eigenfunctions are in $\mathcal{S}(\mathbb{R}^d)$. Under additional assumptions of regularity of the symbol, this information can be strongly improved, namely, there are precise results of exponential decay and holomorphic extension of the solutions. A precise language to describe the above mentioned properties is given by the classes of I.M. Gel'fand and G.E. Shilov [43]. Let us start with the following definition, which reflects the symmetrical role of the variables x and ξ in the definition of **SG** classes.

Definition 4.1. The function $f(x)$ is in $S_s^\sigma(\mathbb{R}^d)$, $\sigma > 0$, $s > 0$, if $f \in \mathcal{S}(\mathbb{R}^d)$ and there exists a constant $\epsilon > 0$ such that

$$(4.1) \quad |f(x)| \lesssim e^{-\epsilon|x|^{\frac{1}{s}}}, \quad x \in \mathbb{R}^d,$$

$$(4.2) \quad |\widehat{f}(\xi)| \lesssim e^{-\epsilon|\xi|^{\frac{1}{\sigma}}}, \quad \xi \in \mathbb{R}^d.$$

Classes $S_s^\sigma(\mathbb{R}^d)$ and related generalizations were widely studied, and used in the applications to partial differential equations, see, for example, H.A. Biagioni, T. Gramchev [5], B.S. Mitjagin [61], S. Pilipović, N. Teofanov [67], E. Cordero, S. Pilipović, L. Rodino, N. Teofanov [20], J. Chung, S.Y. Chung, D. Kim [19], K. Gröchenig, G. Zimmermann [45]. Concerning the tempered ultradistributions, dual spaces of the Gel'fand-Shilov functions, see, for example, S. Pilipović [68].

We recall the following result, giving some equivalent definitions of the class $S_s^\sigma(\mathbb{R}^d)$.

Theorem 4.2. *Assume $\sigma > 0$, $s > 0$, $\sigma + s \geq 1$. For $f \in \mathcal{S}(\mathbb{R}^d)$, the following conditions are equivalent:*

1. $f \in S_s^\sigma(\mathbb{R}^d)$;

2. there exists $C > 0$ such that

$$\begin{aligned} |x^\alpha f(x)| &\lesssim C^{|\alpha|} (\alpha!)^s, & x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d, \\ |\xi^\beta \widehat{f}(\xi)| &\lesssim C^{|\beta|} (\beta!)^\sigma, & \xi \in \mathbb{R}^d, \beta \in \mathbb{N}^d; \end{aligned}$$

3. there exists $C > 0$ such that

$$\begin{aligned} \|x^\alpha f(x)\|_{L^2} &\lesssim C^{|\alpha|} (\alpha!)^s, & \alpha \in \mathbb{N}^d, \\ \|\xi^\beta \widehat{f}(\xi)\|_{L^2} &\lesssim C^{|\beta|} (\beta!)^\sigma, & \beta \in \mathbb{N}^d; \end{aligned}$$

4. there exists $C > 0$ such that

$$\begin{aligned} \|x^\alpha f(x)\|_{L^2} &\lesssim C^{|\alpha|} (\alpha!)^s, & \alpha \in \mathbb{N}^d, \\ \|\partial^\beta f(x)\|_{L^2} &\lesssim C^{|\beta|} (\beta!)^\sigma, & \beta \in \mathbb{N}^d; \end{aligned}$$

5. there exists $C > 0$ such that

$$\|x^\alpha \partial^\beta f(x)\|_{L^2} \lesssim C^{|\alpha|+|\beta|} (\alpha!)^s (\beta!)^\sigma, \quad \alpha, \beta \in \mathbb{N}^d;$$

6. there exists $C > 0$ such that

$$|x^\alpha \partial^\beta f(x)| \lesssim C^{|\alpha|+|\beta|} (\alpha!)^s (\beta!)^\sigma, \quad x \in \mathbb{R}^d, \alpha, \beta \in \mathbb{N}^d;$$

The previous Theorem 4.2 does not include the case $\sigma + s < 1$. In fact, in this case the space $S_s^\sigma(\mathbb{R}^d)$ turns out to be trivial, i.e., the estimates (4.1) and (4.2) imply $f \equiv 0$. From condition 6 in Theorem 4.2 we have that $\sigma < 1$ implies the possibility of extending f to an entire function, whereas for $\sigma = 1$ the extension is limited to a strip in the complex domain. Let us introduce the subclass $\mathbf{ASG}_s^{m,\mu,\sigma}(\mathbb{R}^d) \subset \mathbf{SG}^{m,\mu}(\mathbb{R}^d)$.

Definition 4.3. For s, σ fixed, $s \geq 1$, $\sigma \geq 1$, we define $\mathbf{ASG}_s^{m,\mu,\sigma}(\mathbb{R}^d)$ as the set of all functions $a(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ such that, for suitable positive constants R and C ,

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq R C^{|\alpha|+|\beta|} (\alpha!)^s (\beta!)^\sigma \langle x \rangle^{m-|\alpha|} \langle \xi \rangle^{\mu-|\beta|}, \quad x, \xi \in \mathbb{R}^d.$$

The corresponding pseudo-differential operators act on the Gel'fand-Shilov classes, and the solutions of the \mathbf{SG} -elliptic equations have Gel'fand-Shilov regularity, as detailed by the following theorems. For the proofs, see M. Capiello [10, 11], M. Capiello, L. Rodino [16], M. Capiello, T. Gramchev, L. Rodino [12], M. Capiello, F. Nicola [15].

Theorem 4.4. *Let $a \in \mathbf{ASG}_s^{m,\mu,\sigma}(\mathbb{R}^d)$ and let θ be a real number such that $\theta \geq \max\{s, \sigma\}$. Then, the operator $\text{Op}(a) = a(\cdot, D)$ with symbol a is a linear continuous operator from $S_\theta^\theta(\mathbb{R}^d)$ to $S_\theta^\theta(\mathbb{R}^d)$, and, when $\theta > 1$, it extends to a linear continuous map on the tempered distributions $S_\theta^{\theta'}(\mathbb{R}^d)$*

Theorem 4.5. *Let $a \in \mathbf{ASG}_s^{m,\mu,\sigma}(\mathbb{R}^d)$ be \mathbf{SG} -elliptic and let $f \in S_\theta^\theta(\mathbb{R}^d)$ for some $\theta \geq s + \sigma - 1$. If $u \in \mathcal{S}'(\mathbb{R}^d)$, or $u \in S_\theta^{\theta'}(\mathbb{R}^d)$ if $\theta > 1$, is a solution of the equation $\text{Op}(a)u = f$, then $u \in S_\theta^\theta(\mathbb{R}^d)$.*

Therefore, when $a \in \mathbf{ASG}_1^{m,\mu,1}(\mathbb{R}^d)$, in particular when $\text{Op}(a)$ is a partial differential operator with polynomial coefficients, we have that all the solutions $u \in \mathcal{S}'(\mathbb{R}^d)$ of the \mathbf{SG} -elliptic equation

$$\text{Op}(a)u = 0$$

belong to $S_1^1(\mathbb{R}^d)$. Hence, for some $\epsilon > 0$ we have

$$|u(x)| \lesssim e^{-\epsilon|x|}, \quad x \in \mathbb{R}^d,$$

and $u(x)$ extends as holomorphic function to a strip-neighborhood of \mathbb{R}^d in the complex domain. This result is optimal in the frame of the classes $S_s^\sigma(\mathbb{R}^d)$, $\sigma + s \geq 1$. In fact, consider the \mathbf{SG} -elliptic differential operator

$$\text{Op}(a)u(x) = -(1+x^2)u''(x) - 2xu'(x) + x^2u(x), \quad x \in \mathbb{R}.$$

The operator $\text{Op}(a)$ above is self-adjoint with compact resolvent, see the results of the next Section 5, and there exists a sequence $\lambda_j \in \mathbb{R}$, $j = 1, 2, \dots$, of eigenvalues with eigenfunctions $u_j \in \mathcal{S}(\mathbb{R})$. From the classical theory of the asymptotic integration, we have

$$u_j(x) = Cx^{-1}e^{-|x|} + O(x^{-2}e^{-|x|}), \quad |x| \rightarrow +\infty,$$

and from Fuchs theory we may indeed expect singularities at $x_{1,2} = \pm i$.

Nevertheless, Theorem 4.5 can be improved in other directions. Let us mention the following result by M. Capiello and F. Nicola [15], where operators with \mathbf{SG} -elliptic symbol $a \in \mathbf{ASG}_1^{m,\mu,1}(\mathbb{R}^d)$ are considered. Nonlinear perturbations $F[u]$ are admitted, which are linear combinations of terms of the form

$$x^h \prod_{k=1}^L (\partial_x^{l_k} u)(x), \quad |h| \leq \max\{m-1, 0\}, |l_k| \leq \max\{\mu-1, 0\}, \quad L \geq 2.$$

Theorem 4.6. *Consider the semilinear equation*

$$\text{Op}(a)u = F[u],$$

with a and F as above, and assume that $u \in H_\tau(\mathbb{R}^d)$, $\tau > \frac{d}{2} + \max\{|l_k|\}$ is a solution. In the case $m = 0$ assume further that $\langle x \rangle^{\epsilon_0} u(x) \in L^2(\mathbb{R}^d)$ for some $\epsilon_0 > 0$. Then, u extends to a holomorphic function in the sector of \mathbb{C}^d given by

$$\{z = x + iy \in \mathbb{C}^d : |y| \leq c(1 + |x|)\},$$

for some $c > 0$, satisfying there the estimate $|u(x)| \lesssim e^{-\epsilon|x|}$ for some $\epsilon > 0$.

The novelty with respect to Theorem 4.5 is represented by the non-linear term (see also M. Capiello, T. Gramchev, L. Rodino [13]), and by the extension of the solutions to a sector of \mathbb{C}^d . As an example where solutions are known in closed form, consider the generalized Korteweg-de Vries equation

$$v_t + v_x + v^l v_x + v_{xxx} = 0,$$

where $l \geq 1$ is a positive integer. The solitary wave solutions have the form $v(t, x) = u(x - Vt)$, where $V > 1$ and u satisfies the equation

$$Au = \frac{1}{l+1} v^{l+1}.$$

Here A is the operator with **SG**-elliptic symbol

$$a(x, \xi) = \xi^2 + V - 1 \in \mathbf{ASG}_1^{0,2,1}(\mathbb{R}).$$

Explicit solutions are given by

$$u(x) = \sqrt{\frac{(l+1)(l+2)(V-1)}{2}} \cosh^{-1/l} \left(\frac{\sqrt{V-1}}{2} lx \right),$$

which has poles at the points $z = i \frac{(2k+1)\pi}{l\sqrt{V-1}}$, $k \in \mathbb{Z}$. Also, the exponential decay in sectors containing the real axes predicted by Theorem 4.6 is confirmed.

To conclude the section, we observe that **SG** wave-front sets in the framework of Gel'fand-Shilov classes were considered by M. Capiello and L. Rodino [16], and **SG** Fourier integral operators in the same context can be found in M. Capiello [11]. Also, we would like to mention other papers concerning the action on Gel'fand-Shilov classes of different types of pseudo-differential operators: A. Khrennikov, B. Nilsson, S. Nordebo, J. Toft [54], J. Toft [77, 78], B. Prangoski [69], M. Capiello, S. Pilipović, B. Prangoski [14]. See also M. Capiello, R. Schulz [17] for a different notion of Gel'fand-Shilov wave-front set. Finally, we would like also to observe that Gel'fand-Shilov classes have been recently used in connection with the study of the Boltzmann equation, see N. Lerner, Y. Morimoto, K. Pravda-Starov, C.-J. Xu [56], Y. Morimoto, K. Pravda-Starov, C.-J. Xu [63].

5. Spectral asymptotics for **SG**-elliptic selfadjoint operators on \mathbb{R}^d

In this section we will be concerned with the subclass of **SG**-operators given by those elements $A \in \mathbf{L}^{m,\mu}(\mathbb{R}^d)$, $m, \mu \in \mathbb{R}$, which are **SG**-classical, that is, $A = \text{Op}(a)$ with $a \in \mathbf{SG}_{\text{cl}}^{m,\mu}(\mathbb{R}^d) \subset \mathbf{SG}^{m,\mu}(\mathbb{R}^d)$, see below for the precise definition. In particular, we illustrate the behaviour for $\lambda \rightarrow +\infty$ of the counting function $N_A(\lambda)$ of a **SG**-classical elliptic selfadjoint operator $A \in \mathbf{L}_{\text{cl}}^{m,\mu}(\mathbb{R}^d)$ with $m, \mu > 0$.

5.1. SG-classical operators on \mathbb{R}^d

We begin by recalling some basic definitions and results (see, e.g., [38, 59] for additional details and proofs). In Definitions 5.1 and 5.2 below, a 0-excision function χ is an element of $C^\infty(\mathbb{R}^d)$ such that $\chi(t) = 0$ when t belongs to a (fixed) neighborhood of the origin.

Definition 5.1. i) A symbol $a(x, \xi)$ belongs to the class $\mathbf{SG}_{\text{cl}(x)}^{m, \mu}(\mathbb{R}^d)$ if there exist $a_{m-j, \cdot}(x, \xi) \in C^\infty((\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d)$, $j = 0, 1, \dots$, positively homogeneous functions of order $m-j$ with respect to the variable x , such that, for a 0-excision function χ ,

$$a(x, \xi) - \sum_{j=0}^{N-1} \chi(x) a_{m-j, \cdot}(x, \xi) \in \mathbf{SG}^{m-N, \mu}(\mathbb{R}^d), \quad N = 1, 2, \dots$$

ii) A symbol $a(x, \xi)$ belongs to the class $\mathbf{SG}_{\text{cl}(\xi)}^{m, \mu}(\mathbb{R}^d)$ if there exist $a_{\cdot, \mu-k}(x, \xi) \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$, $k = 0, 1, \dots$, positively homogeneous functions of order $\mu - k$ with respect to the variable ξ , such that, for a 0-excision function χ ,

$$a(x, \xi) - \sum_{k=0}^{N-1} \chi(\xi) a_{\cdot, \mu-k}(x, \xi) \in \mathbf{SG}^{m, \mu-N}(\mathbb{R}^d), \quad N = 1, 2, \dots$$

Definition 5.2. A symbol $a \in \mathbf{SG}^{m, \mu}(\mathbb{R}^d)$ is **SG-classical**, and we write $a \in \mathbf{SG}_{\text{cl}(x, \xi)}^{m, \mu}(\mathbb{R}^d) = \mathbf{SG}_{\text{cl}}^{m, \mu}(\mathbb{R}^d) = \mathbf{SG}_{\text{cl}}^{m, \mu}$, if

i) there exist $a_{m-j, \cdot}(x, \xi) \in C^\infty((\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d)$, $j = 0, 1, \dots$, positively homogeneous functions of order $m-j$ with respect to the variable x , such that, for a 0-excision function χ , $\chi(x) a_{m-j, \cdot}(x, \xi) \in \mathbf{SG}_{\text{cl}(\xi)}^{m-j, \mu}(\mathbb{R}^d)$ and

$$a(x, \xi) - \sum_{j=0}^{N-1} \chi(x) a_{m-j, \cdot}(x, \xi) \in \mathbf{SG}^{m-N, \mu}(\mathbb{R}^d), \quad N = 1, 2, \dots;$$

ii) there exist $a_{\cdot, \mu-k}(x, \xi) \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$, $k = 0, 1, \dots$, positively homogeneous functions of order $\mu - k$ with respect to the variable ξ , such that, for a 0-excision function χ , $\chi(\xi) a_{\cdot, \mu-k}(x, \xi) \in \mathbf{SG}_{\text{cl}(x)}^{m, \mu-k}(\mathbb{R}^d)$ and

$$a(x, \xi) - \sum_{k=0}^{N-1} \chi(\xi) a_{\cdot, \mu-k}(x, \xi) \in \mathbf{SG}^{m, \mu-N}(\mathbb{R}^d), \quad N = 1, 2, \dots$$

We set $\mathbf{L}_{\text{cl}(x, \xi)}^{m, \mu}(\mathbb{R}^d) = \mathbf{L}_{\text{cl}}^{m, \mu} = \text{Op}(\mathbf{SG}_{\text{cl}}^{m, \mu})$.

Notice that the definition of **SG-classical** symbol implies a condition of compatibility for the terms of the expansions with respect to x and ξ . In fact,

defining σ_e^{m-j} and $\sigma_e^{\mu-k}$ on $\mathbf{SG}_{\text{cl}(x)}^{m,\mu}$ and $\mathbf{SG}_{\text{cl}(\xi)}^{m,\mu}$, respectively, as

$$\begin{aligned}\sigma_e^{m-j}(a)(x, \xi) &= a_{m-j,\cdot}(x, \xi), \quad j = 0, 1, \dots, \\ \sigma_\psi^{\mu-k}(a)(x, \xi) &= a_{\cdot, \mu-k}(x, \xi), \quad k = 0, 1, \dots,\end{aligned}$$

it is possibile to prove that

$$\begin{aligned}a_{m-j, \mu-k} &= \sigma_{\psi_e}^{\mu-k, m-j}(a) = \sigma_e^{m-j}(\sigma_\psi^{\mu-k}(a)) = \sigma_\psi^{\mu-k}(\sigma_e^{m-j}(a)), \\ & \quad j = 0, 1, \dots, \quad k = 0, 1, \dots\end{aligned}$$

Moreover, the composition of two \mathbf{SG} -classical operators is still classical. For $A = \text{Op}(a) \in \mathbf{L}_{\text{cl}}^{m,\mu}$ the triple

$$\sigma(A) = (\sigma_\psi(A), \sigma_e(A), \sigma_{\psi_e}(A)) = (a_{\cdot, \mu}, a_{m, \cdot}, a_{m, \mu}) = (a_\psi, a_e, a_{\psi_e})$$

is called the *principal symbol of A*. The three components are also called the ψ -, e - and ψ_e -principal symbol, respectively. This definition keeps the usual multiplicative behaviour, that is, for any $R \in \mathbf{L}_{\text{cl}}^{m,\mu}$, $S \in \mathbf{L}_{\text{cl}}^{t,\tau}$, $m, \mu, t, \tau \in \mathbb{R}$, $\sigma(RS) = \sigma(S)\sigma(T)$, with componentwise product in the right-hand side. We also set

$$\begin{aligned}(5.1) \quad \text{Sym}_p(A)(x, \xi) &= \text{Sym}_p(a)(x, \xi) = \\ &= a_{\mathbf{m}, \boldsymbol{\mu}}(x, \xi) = \chi(\xi)a_\psi(x, \xi) + \chi(x)(a_e(x, \xi) - \chi(\xi)a_{\psi_e}(x, \xi)),\end{aligned}$$

for a fixed 0-excision function χ . Note that, for $a \in \mathbf{SG}_{\text{cl}}^{m,\mu}$, $a = a_{\mathbf{m}, \boldsymbol{\mu}} \bmod \mathbf{SG}_{\text{cl}}^{m-1, \mu-1}$. Theorem 5.3 below allows us to express the ellipticity of \mathbf{SG} -classical operators in terms of their principal symbol:

Theorem 5.3. *An operator $A \in \mathbf{L}_{\text{cl}}^{m,\mu}$ is elliptic if and only if each element of the triple $\sigma(A)$ is non-vanishing on its domain of definition.*

5.2. Weyl formulae

We first refer to the papers by L. Maniccia, P. Panarese [58] and L. Maniccia, E. Schrohe, J. Seiler [59], and begin by recalling facts concerning the spectrum of \mathbf{SG} -elliptic operators. For $m, \mu > 0$, an \mathbf{SG} -elliptic $A \in \mathbf{L}^{m,\mu}$ is considered as an unbounded operator

$$A: H_{m,\mu}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d),$$

which has dense domain and turns out to be closed. By $\rho(A)$ we denote the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ maps the domain $H_{m,\mu}(\mathbb{R}^d)$ of A bijectively onto $L^2(\mathbb{R}^d)$. The spectrum of A is then given by $\mathbb{C} \setminus \rho(A)$.

Theorem 5.4. *Given an \mathbf{SG} -elliptic operator $A \in \mathbf{L}^{m,\mu}(\mathbb{R}^d)$ with $m, \mu > 0$, only one of the following properties holds:*

1. *the spectrum of A is the whole complex plane \mathbb{C} ;*

2. the spectrum of A is a countable set, without any limit point.

Theorem 5.5. *Let $A \in \mathbf{L}_{\text{cl}}^{m,\mu}(\mathbb{R}^d)$, $m, \mu > 0$, be **SG**-elliptic and selfadjoint. Then, the following properties hold true.*

1. $(\lambda I - A)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$ for every $\lambda \in \rho(A)$. More precisely, $(\lambda I - A)^{-1}$ is an extension by continuity from $\mathcal{S}(\mathbb{R}^d)$, or a restriction from $\mathcal{S}'(\mathbb{R}^d)$, of an **SG**-elliptic operator belonging to $\mathbf{L}^{-m,-\mu}(\mathbb{R}^d)$.
2. The spectrum of A consists of a sequence of real isolated eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ with finite multiplicities, clustering at infinity; the orthonormal system of eigenfunctions $\{e_j\}_{j \in \mathbb{N}}$, is complete in $L^2(\mathbb{R}^d)$, and $e_j \in \mathcal{S}(\mathbb{R}^d)$, $j \in \mathbb{N}$.
3. $-A$ is the infinitesimal generator of an analytic semigroup of bounded operators on $L^2(\mathbb{R}^d)$, $H(t) = e^{-tA}$, $t \geq 0$, called the heat semigroup, with kernel

$$H(t, x, y) = \sum_{j \in \mathbb{N}} e^{-t\lambda_j} e_j(x) e_j(y).$$

4. $H(t)$ is trace class when $t > 0$ and

$$\text{Trace } H(t) = \int_{\mathbb{R}^d} H(t, x, x) dx = \sum_j e^{-t\lambda_j}.$$

As a consequence, denoting by $\{\lambda_j\}_{j \in \mathbb{N}}$ the sequence of eigenvalues of an **SG**-elliptic, selfadjoint, positive operator A , ordered so that $j \leq k \Rightarrow \lambda_j \leq \lambda_k$, with each eigenvalue repeated accordingly to its multiplicity, the counting function

$$N_A(\lambda) = \sum_{\lambda_j \leq \lambda} 1$$

is well-defined, see, e.g., [3, 58, 65]. A first result concerning the asymptotic behavior of $N_A(\lambda)$ for $\lambda \rightarrow +\infty$ was proven by L. Maniccia and P. Panarese in the aforementioned paper [58], for the subclass $\mathbf{ECL}^{m,\mu} \subset \mathbf{L}^{m,\mu}$, given by all those $A = \text{Op}(a) \in \mathbf{L}^{m,\mu}$ defined on manifolds with ends such that $a = a_{\mathbf{m},\mu} \bmod \mathbf{SG}^{m-1,\mu-1}$, with $a_{\mathbf{m},\mu}$ given as in (5.1). The result was obtained by means of the so-called *heat kernel method*, combined with the Karamata Tauberian Theorem. The next Theorem 5.6 is the corresponding formulation for **SG**-classical operators on \mathbb{R}^d .

Theorem 5.6. *Let $A \in \mathbf{L}_{\text{cl}}^{m,\mu}(\mathbb{R}^d)$, $m, \mu > 0$, be **SG**-elliptic, selfadjoint and positive. Then, for $\lambda \rightarrow +\infty$,*

$$(5.2) \quad N_A(\lambda) = \begin{cases} C_1 \lambda^{\frac{d}{m}} + o\left(\lambda^{\frac{d}{m}}\right) & \text{if } m < \mu, \\ C_2 \lambda^{\frac{d}{\mu}} + o\left(\lambda^{\frac{d}{\mu}}\right) & \text{if } m > \mu, \\ C_0^1 \lambda^{\frac{d}{m}} \log \lambda + o\left(\lambda^{\frac{d}{m}} \log \lambda\right) & \text{if } m = \mu. \end{cases}$$

It is remarkable that the constants C_1, C_2, C_0^1 , appearing in (5.2), depend only on the principal symbol $\sigma(A)$. This implies that, in the analogous formulae which hold for operators on manifolds with ends, such quantities have a geometrical (that is, invariant) meaning, see, e.g., [3]. The remainder estimates in (5.2) were then further improved, as explained in the next two results. We first recall the definition of Λ -ellipticity in the context of **SG**-operators.

Definition 5.7. Let Λ be a closed sector of the complex plane with vertex at the origin. A symbol $a(x, \xi) \in \mathbf{SG}^{m, \mu}(\mathbb{R}^d)$ and the corresponding operator $A = \text{Op}(a)$ are called Λ -elliptic if there exist constants $C, R > 0$ such that

1. $a(x, \xi) - \lambda \neq 0$, for any $\lambda \in \Lambda$ and (x, ξ) satisfying $|x| + |\xi| \geq R$;
2. $|a(x, \xi) - \lambda|^{-1} \leq C \langle x \rangle^{-m} \langle \xi \rangle^{-\mu}$ for any $\lambda \in \Lambda$ and (x, ξ) satisfying $|x| + |\xi| \geq R$.

Theorem 5.8. Let $A \in \mathbf{L}_{\text{cl}}^{m, \mu}(\mathbb{R}^d)$ with m, μ positive integers be **SG**-elliptic and selfadjoint. Assume also that

1. A is Λ -elliptic with respect to a closed sector Λ of the complex plane with vertex at the origin;
2. A is invertible as an operator from $L^2(\mathbb{R}^d)$ to itself;
3. the spectrum of A does not intersect the real interval $(-\infty, 0)$.

Then, for certain $\delta_j > 0$, $j = 0, 1, 2$, when $\lambda \rightarrow +\infty$,

$$(5.3) \quad N_A(\lambda) = \begin{cases} C_1 \lambda^{\frac{d}{m}} + O(\lambda^{\frac{d}{m} - \delta_1}) & \text{if } m < \mu, \\ C_2 \lambda^{\frac{d}{\mu}} + O(\lambda^{\frac{d}{\mu} - \delta_2}) & \text{if } m > \mu, \\ C_0^1 \lambda^{\frac{d}{m}} \log \lambda + C_0^2 \lambda^{\frac{d}{m}} + O(\lambda^{\frac{d}{m} - \delta_0}) & \text{if } m = \mu. \end{cases}$$

The cases $m \neq \mu$ in (5.3) were originally obtained on \mathbb{R}^d by F. Nicola [65], see also P. Boggiatto, F. Nicola [8]. In the case $m = \mu$, he also proved that, for $\lambda \rightarrow +\infty$,

$$N_A(\lambda) = C_0^1 \lambda^{\frac{d}{m}} \log \lambda + O(\lambda^{\frac{d}{m}}).$$

Compared to Theorem 5.6, in addition to the better estimate of the remainder term, the constants C_1, C_2, C_0^1 were expressed in terms of certain trace operators, defined on suitable subalgebras of $\mathbf{L}_{\text{cl}}^{m, \mu}$ and applied to A . The explicit expression of the second term of the Weyl formula (5.3) in the case $m = \mu$ was obtained by U. Battisti and S. Coriasco in [3], together with the value of C_0^2 , again in terms of suitable trace operators. In [3], the definition of such trace operators has been extended to the case of operators on manifold with ends, by an approach which differs from the one adopted in [65]. The ζ -function of operators in $\mathbf{L}_{\text{cl}}^{m, \mu}$ has also been thoroughly investigated in [3], both on \mathbb{R}^d as well as on manifolds with ends. Incidentally, we also remark that the analysis performed in [65] and [3] allowed to extend the concept of Wodzicki residue to the operators belonging to the **SG**-classes (see also U. Battisti, S. Coriasco [4]).

The remainder estimate for the Weyl formula has been further improved by S. Coriasco and L. Maniccia in [31] in the cases $m \neq \mu$, as described in the following Theorem 5.9. The techniques used in its proof involve the theory of Fourier integral operators in the **SG**-classes, see Section 6 below.

Theorem 5.9. *Let $A \in \mathbf{L}_{\text{cl}}^{m,\mu}(\mathbb{R}^d)$, $m, \mu > 0$, be **SG**-elliptic, selfadjoint and positive. Then, for $\lambda \rightarrow +\infty$,*

(5.4)

$$N_A(\lambda) = \begin{cases} C_1 \lambda^{\frac{d}{m}} + O(\lambda^{\frac{d}{\mu}}) + O(\lambda^{\frac{d}{m} - \frac{1}{\mu}}) = C_1 \lambda^{\frac{d}{m}} + O(\lambda^{\frac{d}{m} - \delta_1}) & \text{if } m < \mu, \\ C_2 \lambda^{\frac{d}{\mu}} + O(\lambda^{\frac{d}{m}}) + O(\lambda^{\frac{d}{\mu} - \frac{1}{m}}) = C_2 \lambda^{\frac{d}{\mu}} + O(\lambda^{\frac{d}{\mu} - \delta_2}) & \text{if } m > \mu, \end{cases}$$

where $\delta_1 = \min \left\{ \frac{1}{\mu}, d \left(\frac{1}{m} - \frac{1}{\mu} \right) \right\}$ and $\delta_2 = \min \left\{ \frac{1}{m}, d \left(\frac{1}{\mu} - \frac{1}{m} \right) \right\}$.

The order of the remainder terms in (5.4) is then determined by the ratio of m and μ and the dimension d of the base space, since

$$\begin{aligned} \frac{d}{m} - \frac{1}{\mu} \leq \frac{d}{\mu} \text{ for } m < \mu &\Leftrightarrow 1 < \frac{\mu}{m} \leq 1 + \frac{1}{d}, \\ \frac{d}{\mu} - \frac{1}{m} \leq \frac{d}{m} \text{ for } m > \mu &\Leftrightarrow 1 < \frac{m}{\mu} \leq 1 + \frac{1}{d}. \end{aligned}$$

In particular, when $\frac{\max\{m,\mu\}}{\min\{m,\mu\}} \geq 2$, the remainder is always $O(\lambda^{\frac{d}{\max\{m,\mu\}}})$. It is conjectured that the orders of the remainder in Theorem 5.9 are the best possible ones.

Examples of operators on which the analysis described in this section can be applied include operators of Schrödinger type on \mathbb{R}^d , that is $A = -\Delta_g + V$, Δ_g the Laplace-Beltrami operator in \mathbb{R}^d associated with a suitable metric g , V a smooth potential that grows as $\langle x \rangle^\mu$, with an appropriate $\mu > 0$ related to g (see, e.g., [29]).

6. Concluding remarks

The literature on the study of the regularity, wave-front set propagation, and eigenvalue asymptotics of elliptic operators on \mathbb{R}^d is vast, and we also refer the reader to the bibliographies of the quoted papers and books. We conclude with just a few remarks on some extension of the results above to the manifold case, as well as to the study of **SG**-hyperbolic operators.

For what concerns the wave-front sets theory in the environment of general modulation spaces, it is well-known that such functional spaces cannot be invariantly defined in full generality on manifolds. Some examples of modulation spaces, namely, the Sobolev-Kato spaces $H_{t,\tau}$, can be invariantly defined, e.g., on the manifolds with ends. In such cases, the results described above can be extended to manifolds, cf. R. Melrose [60] and coauthors.

An interesting open problem is a definition of Gel'fand-Shilov spaces on manifolds with ends, and a generalization of the results in Section 4 to this setting.

For what concerns the spectral asymptotics, well-known results, in the case of compact manifolds, were proved by L. Hörmander [51] and V. Guillemin [46], see also the book by H. Kumano-go [55]. On the other hand, for operators globally defined on \mathbb{R}^d , see P. Boggiatto, E. Buzano, L. Rodino [6], B. Helffer [47], L. Hörmander [52], A. Mohammed [62], M. A. Shubin [75]. Many other situations have been considered, see the book by V.J. Ivrii [53]. On manifolds with ends, in addition to the papers cited above, we also mention the paper [18] by T. Christiansen and M. Zworski, who studied the Laplace-Beltrami operator associated with a scattering metric (see the corresponding list of references, for more details on the scattering calculus and its applications to the spectral theory). In analogy with, e.g., the works by B. Helffer, D. Robert [48, 49], a main tool in the proof of the illustrated results from [31] has been the theory of **SG** Fourier integral operators, initially developed by S. Coriasco in [24], see also, e.g., M. Cappiello [11], M. Ruzhansky, M. Sugimoto [73], E. Cordero, F. Nicola, L. Rodino [21], and the recent works by S. Coriasco and M. Ruzhansky [34], S. Coriasco and R. Schulz [35, 36]. Such theory has been applied also to the study of hyperbolic problems in the **SG**-environment. Since here we focus on the elliptic **SG**-theory, for this subject we just refer the reader to, e.g., [1, 2, 11, 25, 29, 32, 33].

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