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PSEUDO-DIFFERENTIAL OPERATORS AND LOCALIZATION OPERATORS ON $S^{\mu}_{\nu}(\mathbb{R})$ SPACE INVOLVING FRACTIONAL FOURIER TRANSFORM

Jitendra Kumar Dubey[1], Anuj Kumar[2] and S.K. Upadhyay[3]

Dedicated to Professor Stanković on the occasion of his 90^{th} birthday and to Professor Vickers on the on the occasion of his 60^{th} birthday.

Abstract. In this paper, some properties of pseudo-differential operators, SG-elliptic partial differential equations with polynomial coefficients and localization operators on space $S^{\mu}_{\nu}(\mathbb{R})$, are studied by using fractional Fourier transform.

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1. Introduction:

Almeida [1], Namias [5] and others introduced the fractional Fourier transform which is a generalization of the Fourier transform. Fractional Fourier transform is the most important tool, which is frequently used in signal processing and other branches of mathematical sciences and engineerings.

The pseudo-differential operator is a generalization of the partial differential operator, and played an important role in study of the properties of Sobolev spaces, partial differential equations and localization operators. Zaidmann [11], Wong [10] and Cappiello et al. [2, 3] discussed the properties of pseudo-differential operator on $S(\mathbb{R}^n)$ space and certain types of Gelfand-Shilov space by using Fourier transformation.

Pathak et al. [6], Prasad and Kumar [7] studied the properties of pseudo-differential operator on the Schwartz space $S(\mathbb{R})$ involving fractional Fourier transformation. Motivated by Cappiello et al. [2], our main aim in this paper is to study the properties of pseudo-differential operators and localization operators on $S^{\mu}_{\nu}(\mathbb{R})$ space involving fractional Fourier transformation.

Now, from [2, 4, 6] we recall definitions and properties which are useful for our further investigations:

[1] DST-CIMS, Banaras Hindu University, Varanasi - 221005, India, email: jitendradubey1989@gmail.com

[2] DST-CIMS, Banaras Hindu University, Varanasi - 221005, India, email: anujk743@gmail.com

[3] DST-CIMS, Department of Mathematical Sciences, Indian Institute of Technology, Banaras Hindu University, Varanasi - 221005, India, email: sk_upadhyay2001@yahoo.com

Definition 1.1. The fractional Fourier transform with rotational angle α of a function $\phi \in L^1(\mathbb{R})$, is defined by

$$\hat{\phi}_\alpha(\xi) = (F_\alpha \phi)(\xi) = \int_{\mathbb{R}} K_\alpha(x,\xi)\phi(x)dx, \tag{1.1}$$

where,

$$K_\alpha(x,\xi) = \begin{cases} C_\alpha e^{\frac{i(x^2+\xi^2)\cot\alpha}{2} - ix\xi\csc\alpha} & \text{if } \alpha \neq n\pi \\ \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-ix\xi} & \text{if } \alpha = \frac{\pi}{2}, \end{cases} \qquad \forall n \in \mathbb{Z},$$

and

$$C_\alpha = (2\pi i \sin\alpha)^{\frac{-1}{2}} e^{\frac{i\alpha}{2}} = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}. \tag{1.2}$$

Definition 1.2. The inversion formula of the fractional Fourier transform is given by

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{K_\alpha(x,\xi)}\hat{\phi}_\alpha(\xi)d\xi, \tag{1.3}$$

where

$$\overline{K_\alpha(x,\xi)} = \begin{cases} C'_\alpha e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2} + ix\xi\csc\alpha} & \text{if } \alpha \neq n\pi \\ \frac{1}{(2\pi)^{\frac{1}{2}}} e^{ix\xi} & \text{if } \alpha = \frac{\pi}{2}, \end{cases} \qquad \forall n \in \mathbb{Z},$$

and

$$C'_\alpha = \frac{(2\pi i \sin\alpha)^{\frac{1}{2}}}{\sin\alpha} e^{\frac{-i\alpha}{2}} = \sqrt{2\pi(1 + i\cot\alpha)}. \tag{1.4}$$

Definition 1.3. Let $m \in (-\infty, \infty)$. Define the symbol class $\Gamma^m_{\nu,\mu}(\mathbb{R})$ to be the set of all functions $p(x,\xi) \in C^\infty(\mathbb{R}\times\mathbb{R})$ such that for any two non-negative integers α and β, there exists a positive constant C, which satisfies the following estimates

$$\left|D^\alpha_\xi D^\beta_x p(x,\xi)\right| \leq C^{\alpha+\beta+1}(\alpha!)^\nu(\beta!)^\mu\langle\xi\rangle^{m_1-\alpha}\langle x\rangle^{m_2-\beta}, \tag{1.5}$$

for every $(x,\xi) \in \mathbb{R}\times\mathbb{R}$ and μ,ν are non-negative indices.

Definition 1.4. The generalized pseudo-differential operator P_α associated with $p \in \Gamma^m_{\nu,\mu}(\mathbb{R})$ is defined by

$$P_\alpha\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{K_\alpha(x,\xi)}p(x,\xi)\hat{\phi}_\alpha(\xi)d\xi, \tag{1.6}$$

where $\hat{\phi}_\alpha(\xi)$ and $\overline{K_\alpha(x,\xi)}$ are given in (1.1) and (1.4).

Definition 1.5. The ultra-differentiable test space $S^{\mu}_{\nu}(\mathbb{R})$ is defined to be the set of all functions $f \in C^{\infty}(\mathbb{R})$, such that

$$|\partial_x^{\alpha} f(x)| \leq C^{\alpha+1}(\alpha!)^{\mu} e^{-\epsilon x^{\frac{1}{\nu}}}, \tag{1.7}$$

where C and ϵ are suitable positive constants in $\mathbb{R}$. The space S^{μ}_{ν} is non-trivial, if and only if $\mu + \nu \geq 1$ and $\mu, \nu > 0$.

Definition 1.6. An infinitely differential complex-valued function ϕ is a member of the Schwartz space $S(\mathbb{R})$ iff for every choice of β and ν of non-negative integers, it satisfies

$$\gamma_{\beta,\nu}(\phi) = \sup_{x \in \mathbb{R}} \left| x^{\beta} D^{\nu} \phi(x) \right| < \infty. \tag{1.8}$$

Definition 1.7. For a fixed non-zero function ϕ, called "window", the short time Fourier transform (STFT) of f with respect to ϕ is given by

$$(V_{\phi} f)(x, \xi) = \langle f, M_{\xi} T_x \phi \rangle = \int_{\mathbb{R}} f(t) \overline{\phi(t - x)} e^{-i\xi t} dt. \tag{1.9}$$

Theorem 1.8. *Let* $g \in S_w(\mathbb{R}^d)$ *and* $f \in S'_w(\mathbb{R}^d)$. *Then* $V_{\phi} f$ *is continuous, and satisfies the following inequality*

$$|(V_{\phi} f)(x, \xi)| \leq C e^{\lambda \omega(x) + \mu \omega(\xi)}, \tag{1.10}$$

for every $x, \xi \in \mathbb{R}^d$ *and constants* $C, \lambda, \mu > 0$.

A self-adjoint differential operator $P(x, D)$ with polynomial coefficient $C_{\alpha\beta}$ is given by

$$P(x, D) = \sum_{\substack{\alpha \leq m_1 \\ \beta \leq m_2}} C_{\alpha\beta} x^{\beta} D_x^{\alpha} \tag{1.11}$$

satisfying global ellipticity condition of the form

$$|p(x, \xi)| \geq C_0 (\xi)^{m_1} (x)^{m_2}, \tag{1.12}$$

for $|x| + |\xi| \geq R > 0$.
If $\mu' \geq \mu, \nu' \geq \nu$, then $S^{\mu}_{\nu}(\mathbb{R}) \to S^{\mu'}_{\nu'}(\mathbb{R})$ and Fourier transform acts as an isomorphism interchanging the indices μ and ν

$$F : S^{\mu}_{\nu}(\mathbb{R}) \to S^{\nu}_{\mu}(\mathbb{R}). \tag{1.13}$$

Furthermore, Let P be a self-adjoint SG-elliptic operator of the form (1.11) and let $\lambda \in \mathbb{C} \backslash spec(P)$. Then on SG pseudo-differential calculus, if $m_1 \geq 0, m_2 \geq 0$ the operator $(P - \lambda)$ is also SG-elliptic and

$$(P - \lambda)^{-1} \circ x^q \partial_x^p : H^s(\mathbb{R}) \to H^s(\mathbb{R}), \tag{1.14}$$

is continuous for all $p, q \in \mathbb{Z}_+, p \leq m_1, q \leq m_2$ and for every $s \in \mathbb{R}$, it satisfies the following norm inequality

(1.15)
$$\left\|(P-\lambda)^{-1}[P, x^\beta \partial_x^\alpha]\phi\right\|_s \leq (\alpha!)^\mu (\beta!)^\nu \sum_{\substack{\gamma\leq\alpha,\sigma\leq\beta \\ (\gamma+\sigma)\leq(\alpha+\beta)}} \frac{(C_1)^{(\alpha-\gamma)+(\beta-\sigma)}}{(\gamma!)^\mu(\sigma!)^\nu} \left\|x^\sigma \partial_x^\gamma \phi\right\|_s,$$

for $\alpha, \beta \in \mathbb{Z}_+$ and positive constant C_1 independent of α, β.

From Gröchenig [4], continuity of a linear functional f means that there exist constants $C, \lambda, \mu > 0$ and integers $M, N \geq 0$ such that for all $g \in S_w(\mathbb{R}^d)$,

$$|\langle f, g\rangle| \leq C\Big(\sum_{\|p\|_1\leq M} \left\|e^{\lambda\omega} D^p g\right\|_{L^\infty} + \sum_{\|q\|_1\leq N} \|e^{\mu\omega} D^q \hat{g}\|_{L^\infty} \Big). \tag{1.16}$$

In Section 2 of this paper we use the Leibnitz' formula which is given below:

$$\partial^n(x^m f(x)) = \sum_{j\leq n, j\leq m} \binom{n}{j}\binom{m}{j} j! x^{m-j} (\partial^{n-j} f)(x). \tag{1.17}$$

Now, this paper is divided into three sections:

Section 1 is introductory, the various definitions and properties related to this paper are given.

In Section 2, properties of pseudo-differential operators associated with symbol $p \in \Gamma^m_{\nu,\mu}(\mathbb{R})$ on $S^\mu_\nu(\mathbb{R})$ space are studied using fractional Fourier transformation, and commutator identities for differential operators on Schwartz space $S(\mathbb{R})$ are obtained using the same transformation.

Finally in Section 3, the properties of localization operators involving fractional Fourier transform on $S^\mu_\nu(\mathbb{R})$ space are studied. We are able to find the solution of heat equation in form of pseudo-differential operator by exploiting fractional Fourier transform technique.

2. Pseudo-differential operators and commutator identities for differential operator

In this section, we find the action of pseudo-differential operators on the $S^\mu_\nu(\mathbb{R})$ space, $\mu \geq 1, \nu \geq 1$ and obtain the relation of composition of operator and parametrices. Commutator identities associated with fractional Fourier transform on $S(\mathbb{R})$ space are studied. These commutator identities are useful to show the assumption (1.15) in case of SG-elliptic differential operators with polynomial coefficients.

Theorem 2.1. *Given $p \in \Gamma^m_{\nu,\mu}$, the operator P_α defined by* (1.6) *is linear and continuous from $S^{\mu'}_{\nu'}$ into itself for any μ', ν' with $\mu' \geq \mu, \nu' \geq \nu$. Furthermore, P_α can be extended to a linear and continuous map from $(S^{\mu'}_{\nu'})'$ into itself.*

Proof. By Definition 1.4 and for $\beta, \gamma \in \mathbb{Z}_+$ and any positive integer N, we can write

$$\begin{aligned}
&D^{\beta}_x P_\alpha \phi(x)\\
&= \frac{1}{2\pi} C'_\alpha \int_{\mathbb{R}} D^{\beta}_x \big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha} p(x,\xi)\big)\hat{\phi}_\alpha(\xi)d\xi\\
&= \frac{1}{2\pi} C'_\alpha \int_{\mathbb{R}} \sum_{\beta_1\le\beta}\binom{\beta}{\beta_1} D^{\beta_1}_x \big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big) D^{\beta_2}_x\big(p(x,\xi)\big)\hat{\phi}_\alpha(\xi)d\xi\\
&= \frac{1}{2\pi} C'_\alpha \sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\int_{\mathbb{R}}\sum_{\beta'_1\le\beta_1}\binom{\beta_1}{\beta'_1}\\
&\quad\times D^{\beta'_1}_x\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}}\big) D^{\beta''_1}_x\big(e^{ix\xi\csc\alpha}\big) D^{\beta_2}_x\big(p(x,\xi)\big)\hat{\phi}_\alpha(\xi)d\xi\\
&= \frac{1}{2\pi} C'_\alpha \sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\beta'_1\le\beta_1}\binom{\beta_1}{\beta'_1}\int_{\mathbb{R}} D^{\beta'_1}_x\big(e^{\frac{-ix^2\cot\alpha}{2}}\big)e^{\frac{-i\xi^2\cot\alpha}{2}}\\
&\quad\times\big[(i\xi\csc\alpha)^{\beta''_1}e^{ix\xi\csc\alpha}\big] D^{\beta_2}_x\big(p(x,\xi)\big)\hat{\phi}_\alpha(\xi)d\xi\\
&= \frac{1}{2\pi} C'_\alpha \sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\beta'_1\le\beta_1}\binom{\beta_1}{\beta'_1}\int_{\mathbb{R}}\big(e^{\frac{-ix^2\cot\alpha}{2}}\big)P_{\beta'_1}(x,\frac{-i\cot\alpha}{2})e^{\frac{-i\xi^2\cot\alpha}{2}}\\
&\quad\times\big[(i\xi\csc\alpha)^{\beta''_1}e^{ix\xi\csc\alpha}\big] D^{\beta_2}_x\big(p(x,\xi)\big)\hat{\phi}_\alpha(\xi)d\xi\\
&= \frac{1}{2\pi} C'_\alpha \sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\beta'_1\le\beta_1}\binom{\beta_1}{\beta'_1}\int_{\mathbb{R}}\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)\sum_{\eta=0}^{\beta'_1}a_\eta x^\eta\cot\alpha\\
&\quad\times\big[(i\xi\csc\alpha)^{\beta''_1}e^{ix\xi\csc\alpha}\big] D^{\beta_2}_x\big(p(x,\xi)\big)\hat{\phi}_\alpha(\xi)d\xi\\
&= \frac{1}{2\pi} C'_\alpha \sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\beta'_1\le\beta_1}\binom{\beta_1}{\beta'_1}(i\csc\alpha)^{\beta''_1}\sum_{\eta=0}^{\beta'_1}a_\eta x^\eta\cot\alpha\\
&\quad\times\int_{\mathbb{R}}\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)\xi^{\beta''_1}D^{\beta_2}_x\big(p(x,\xi)\big)\hat{\phi}_\alpha(\xi)d\xi.
\end{aligned}$$

Then,

$$\begin{aligned}
&x^\gamma D^{\beta}_x P_\alpha\phi(x)\\
&\quad= \frac{1}{2\pi} C'_\alpha \sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\beta'_1\le\beta_1}\binom{\beta_1}{\beta'_1}(i\csc\alpha)^{\beta''_1}\big(\sum_{\eta=0}^{\beta'_1}a_\eta x^\eta x^\gamma\cot\alpha\big)\langle x\rangle^{-2N}\\
&\qquad\times\int_{\mathbb{R}}\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)(1-\Delta_\xi)^N\big[\xi^{\beta''_1}D^{\beta_2}_x\big(p(x,\xi)\big)\hat{\phi}_\alpha(\xi)\big]d\xi.
\end{aligned}$$

Choosing $\langle x\rangle^\delta = \left|\sum_{\eta=0}^{\beta'_1}a_\eta x^{\eta+\gamma}\right|$ and $N = \left[\frac{\delta+m_2}{2}\right]+1$, using (1.13) and by standard factorial inequality, we obtain

$$\langle x\rangle^{\delta-2N}\left|(1-\Delta_\xi)^N\big[\xi^{\beta''_1}D^{\beta_2}_x\big(p(x,\xi)\big)\hat{\phi}_\alpha(\xi)\big]\right|\le C^{\delta+\beta'+1}(\delta!)^\nu(\beta''_1!)^{\mu'}(\beta_2!)^\mu e^{-a\xi^{\frac{1}{\mu'}}},$$

for $\beta' = \beta_1'' + \beta_2$ and some positive contants C, a.
Then, by the conditions $\mu' \geq \mu, \nu' \geq \nu$, it follows that P_α is continuous from $S_{\nu'}^{\mu'}$ into itself.
Next, for $\phi, \psi \in S_{\nu'}^{\mu'}(\mathbb{R})$

$$\begin{aligned}
&\int_{\mathbb{R}} \big(P_\alpha \phi(x)\big)\psi(x)dx \\
&= \int_{\mathbb{R}} \big[\frac{1}{2\pi} C_\alpha' \int_{\mathbb{R}} \big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2} + ix\xi\csc\alpha} p(x,\xi)\big)\hat{\phi}_\alpha(\xi)d\xi\big]\psi(x)dx \\
&= \int_{\mathbb{R}} \hat{\phi}_\alpha(\xi)\big[\frac{1}{2\pi} C_\alpha' \int_{\mathbb{R}} \big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2} + ix\xi\csc\alpha} p(x,\xi)\big)\psi(x)dx\big]d\xi \\
&= \int_{\mathbb{R}} \hat{\phi}_\alpha(\xi)p_\psi(\xi)d\xi
\end{aligned}$$

$$i.e. \int_{\mathbb{R}} \big(P_\alpha \phi(x)\big)\psi(x)dx = \int_{\mathbb{R}} \hat{\phi}_\alpha(\xi)p_\psi(\xi)d\xi,$$

where $p_\psi(\xi) = \frac{1}{2\pi} C_\alpha' \int_{\mathbb{R}} \big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2} + ix\xi\csc\alpha} p(x,\xi)\big)\psi(x)dx$.
Furthermore, by the same argument as in the first part of the proof, it follows that the map $\psi \to p_\psi$ is linear and continuous from $S_{\nu'}^{\mu'}$ to $S_{\mu'}^{\nu'}$.
Then, for $\phi \in (S_{\nu'}^{\mu'})'(\mathbb{R})$, we can define

$$P_\alpha \phi(\psi) = \hat{\phi}(p_\psi); \quad \psi \in S_{\nu'}^{\mu'}(\mathbb{R})$$

This is a linear continuous map from $(S_{\nu'}^{\mu'})'$ into itself and it extends P_α. □

Lemma 2.2. *Let $\alpha, \beta, \rho, \sigma \in \mathbb{Z}_+$ and $u \in S(\mathbb{R})$. Then the following identity holds:*

$$x^\beta \partial_x^\alpha (x^\sigma \partial_x^\rho u) = \sum_{\substack{\gamma \leq \alpha \\ \gamma \leq \sigma}} \frac{\alpha!}{(\alpha-\gamma)!} \binom{\sigma}{\gamma} x^{\sigma-\gamma} x^\beta \partial_x^\rho (\partial_x^{\alpha-\gamma} u). \tag{2.1}$$

Proof. See [2, p.307]. □

Lemma 2.3. *Let $\beta, \beta', \rho, \delta \in \mathbb{Z}_+$ and $u \in S(\mathbb{R})$. Then the following identity holds:*

$$x^\beta \partial_x^\rho u = \sum_{\substack{k \leq \beta' \\ k \leq \delta}} \frac{\beta'!}{(\beta'-k)!} \binom{\delta}{k} (-1)^k \partial_x^{\delta-k} (x^{\beta'-k} u). \tag{2.2}$$

Proof. Using Definition 1.2, the inversion fractional Fourier transform for $u \in S(\mathbb{R})$, we have

$$u(x) = \frac{1}{2\pi} C_\alpha' \int_{\mathbb{R}} \big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2} + ix\xi\csc\alpha}\big)\hat{u}_\alpha(\xi)d\xi. \tag{2.3}$$

For $\rho \in \mathbb{Z}_+$, we calculate

$$\begin{aligned}
\partial_x^\rho u(x) &= \frac{1}{2\pi}C_\alpha^{'}\int_{\mathbb{R}}\partial_x^\rho\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)\hat{u}_\alpha(\xi)d\xi\\
&= \frac{1}{2\pi}C_\alpha^{'}\int_{\mathbb{R}}e^{\frac{-i\xi^2\cot\alpha}{2}}\partial_x^\rho\big(e^{\frac{-ix^2\cot\alpha}{2}}\times e^{ix\xi\csc\alpha}\big)\hat{u}_\alpha(\xi)d\xi.
\end{aligned}$$

Using (1.17), we have

$$\begin{aligned}
&\partial_x^\rho u(x)\\
&=\frac{1}{2\pi}C_\alpha^{'}\int_{\mathbb{R}}e^{\frac{-i\xi^2\cot\alpha}{2}}\sum_{j\le\rho}\binom{\rho}{j}\partial_x^{\rho-j}\big(e^{\frac{-ix^2\cot\alpha}{2}}\big)\partial_x^j\big(e^{ix\xi\csc\alpha}\big)\hat{u}_\alpha(\xi)d\xi\\
&=\frac{1}{2\pi}C_\alpha^{'}\sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)\int_{\mathbb{R}}(i\xi\csc\alpha)^j\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)\hat{u}_\alpha(\xi)d\xi.
\end{aligned}$$

Then,

$$\begin{aligned}
&x^\beta\partial_x^\rho u(x)\\
&= \frac{1}{2\pi}C_\alpha^{'}\sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^j(i\csc\alpha)^{-\beta}\\
&\quad\times\int_{\mathbb{R}}\xi^j e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}}e^{ix\xi\csc\alpha}(ix\csc\alpha)^\beta\hat{u}_\alpha(\xi)d\xi\\
&= \frac{1}{2\pi}C_\alpha^{'}\sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}e^{\frac{-ix^2\cot\alpha}{2}}\\
&\quad\times\int_{\mathbb{R}}\partial_\xi^\beta\big[(e^{ix\xi\csc\alpha})\big(\xi^j e^{\frac{-i\xi^2\cot\alpha}{2}}\hat{u}_\alpha(\xi)\big)\big]d\xi\\
&= \frac{1}{2\pi}C_\alpha^{'}\sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}e^{\frac{-ix^2\cot\alpha}{2}}(-1)^\beta\\
&\quad\times\int_{\mathbb{R}}e^{ix\xi\csc\alpha}\partial_\xi^\beta\big[\big(\xi^j e^{\frac{-i\xi^2\cot\alpha}{2}}\hat{u}_\alpha(\xi)\big)\big]d\xi\\
&= \frac{1}{2\pi}C_\alpha^{'}\sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}e^{\frac{-ix^2\cot\alpha}{2}}(-1)^\beta\sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\\
&\quad\times\int_{\mathbb{R}}e^{ix\xi\csc\alpha}\partial_\xi^{\beta_1}\big(e^{\frac{-i\xi^2\cot\alpha}{2}}\big)\partial_\xi^{\beta'}\big(\xi^j\hat{u}_\alpha(\xi)\big)d\xi\\
&= \frac{1}{2\pi}C_\alpha^{'}\sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^\beta\sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\\
&\quad\times\int_{\mathbb{R}}\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)P_{\beta_1}\big(\xi,\frac{-i\cot\alpha}{2}\big)\partial_\xi^{\beta'}\big(\xi^j\hat{u}_\alpha(\xi)\big)d\xi\\
&= \frac{1}{2\pi}C_\alpha^{'}\sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^\beta\sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\eta\le\beta_1}a_\eta\cot\alpha\\
&\quad\times\int_{\mathbb{R}}\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)\xi^\eta\partial_\xi^{\beta'}\big(\xi^j\hat{u}_\alpha(\xi)\big)d\xi
\end{aligned}$$

$$
\begin{aligned}
(2.4)\qquad & x^\beta \partial_x^\rho u(x) \\
= \quad & \frac{1}{2\pi} C_\alpha^{'} \sum_{j\le\rho} \binom{\rho}{j} P_{\rho-j}\big(x, \frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^\beta \\
& \times \sum_{\beta_1\le\beta} \binom{\beta}{\beta_1} \sum_{\eta\le\beta_1} (a_\eta \cot\alpha) \frac{(\eta+\beta')!}{\eta!} \\
& \times \int_{\mathbb{R}} \big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2} + ix\xi\csc\alpha}\big) \partial_\xi^{\beta'} \big(\xi^j \xi^{\eta+\beta'} \hat{u}_\alpha(\xi)\big) d\xi.
\end{aligned}
$$

We assume that $j+\eta+\beta' = \delta \in \mathbb{Z}_+$, then we have

$$
\begin{aligned}
(2.5)\qquad & x^\beta \partial_x^\rho u(x) \\
= \quad & \sum_{j\le\rho} \binom{\rho}{j} P_{\rho-j}\big(x, \frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^\beta \\
& \times \sum_{\beta_1\le\beta} \binom{\beta}{\beta_1} \sum_{\eta\le\beta_1} (a_\eta \cot\alpha) \frac{(\eta+\beta')!}{\eta!} \\
& \times \frac{1}{2\pi} C_\alpha^{'} \int_{\mathbb{R}} \big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2} + ix\xi\csc\alpha}\big) \partial_\xi^{\beta'} \big(\xi^\delta \hat{u}_\alpha(\xi)\big) d\xi.
\end{aligned}
$$

By using (2.3), (2.5) becomes

$$
\begin{aligned}
(2.6)\qquad & x^\beta \partial_x^\rho u(x) \\
= \quad & \sum_{j\le\rho} \binom{\rho}{j} P_{\rho-j}\big(x, \frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^\beta \\
& \times \sum_{\beta_1\le\beta} \binom{\beta}{\beta_1} \sum_{\eta\le\beta_1} (a_\eta \cot\alpha) \frac{(\eta+\beta')!}{\eta!} \\
& \times F_\alpha^{-1} \big[\partial_\xi^{\beta'} \big(\xi^\delta \hat{u}_\alpha(\xi)\big)\big].
\end{aligned}
$$

Applying Leibnitz rule in (2.5), we get

$$\begin{aligned} & x^{\beta}\partial_x^{\rho}u(x) \\ = & \sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^{\beta} \\ & \times\sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\eta\le\beta_1}(a_\eta\cot\alpha)\frac{(\eta+\beta')!}{\eta!} \\ & \times\frac{1}{2\pi}C_\alpha^{'}\sum_{k\le\beta'}\binom{\beta'}{k}\int_{\mathbb{R}}\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)\partial_\xi^{k}(\xi^{\delta})\partial_\xi^{\beta'-k}\big(\hat{u}_\alpha(\xi)\big)d\xi \\ = & \sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^{\beta} \\ & \times\sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\eta\le\beta_1}(a_\eta\cot\alpha)\frac{(\eta+\beta')!}{\eta!} \\ & \times\frac{1}{2\pi}C_\alpha^{'}\sum_{k\le\beta'}\frac{\beta'!}{k!(\beta'-k)!}\,\frac{\delta(\delta-1)\dots(\delta-k+1)(\delta-k)\dots2.1}{(\delta-k)!} \\ & \times\int_{\mathbb{R}}\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)\xi^{\delta-k}\partial_\xi^{\beta'-k}\big(\hat{u}_\alpha(\xi)\big)d\xi \end{aligned}$$

$$\begin{aligned} & x^{\beta}\partial_x^{\rho}u(x) \\ = & \sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^{\beta} \\ & \times\sum_{\beta_1\le\beta}\binom{\beta}{\beta_1}\sum_{\eta\le\beta_1}(a_\eta\cot\alpha)\frac{(\eta+\beta')!}{\eta!} \\ & \times\sum_{k\le\beta'}\frac{\beta'!}{(\beta'-k)!}\binom{\delta}{k} \\ & \big[\frac{1}{2\pi}C_\alpha^{'}\int_{\mathbb{R}}\big(e^{\frac{-i(x^2+\xi^2)\cot\alpha}{2}+ix\xi\csc\alpha}\big)\xi^{\delta-k}\partial_\xi^{\beta'-k}\big(\hat{u}_\alpha(\xi)\big)d\xi\big]. \end{aligned}$$

From (2.3), we have

$$\begin{aligned} & x^{\beta}\partial_x^{\rho}u(x) \\ = & \sum_{k\le\beta'}\frac{\beta'!}{(\beta'-k)!}\binom{\delta}{k} \\ & \times\big[\sum_{j\le\rho}\binom{\rho}{j}P_{\rho-j}\big(x,\frac{-i\cot\alpha}{2}\big)(i\csc\alpha)^{j-\beta}(-1)^{\beta}\sum_{\beta_1\le\beta}\binom{\beta}{\beta_1} \\ & \times\sum_{\eta\le\beta_1}(a_\eta\cot\alpha)\frac{(\eta+\beta')!}{\eta!}\big]F_\alpha^{-1}\big[\xi^{\delta-k}\partial_\xi^{\beta'-k}\big(\hat{u}_\alpha(\xi)\big)d\xi\big]. \end{aligned} \tag{2.7}$$

Using (2.6) in (2.7), we get

$$x^\beta \partial_x^\rho u(x) = \sum_{\substack{k\le\beta' \\ k\le\delta}} \frac{\beta'!}{(\beta'-k)!}\binom{\delta}{k}(-1)^k \partial_x^{\delta-k}(x^{\beta'-k}u(x)).$$

□

As an consequence of Lemma 2.2 and Lemma 2.3, we are able to show the following commutator identity.

Lemma 2.4. *Let $\alpha, \beta, \rho, \sigma, \beta', \delta \in \mathbb{Z}_+$. Then the following commutator decomposition is true*

(2.8)

$$\begin{aligned}&\left[x^\beta \partial_x^\alpha, x^\sigma \partial_x^\rho\right]u \\ &= \sum_{\substack{\gamma\le\alpha \\ \gamma\le\sigma}}{}^{'} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} \sum_{\substack{k\le\beta' \\ k\le\delta}}{}^{'} \frac{\beta'!}{(\beta'-k)!}\binom{\delta}{k}(-1)^k x^{\sigma-\gamma}\partial_x^{\delta-k}\left(x^{\beta'-k}\partial_x^{\alpha-\gamma}u\right),\end{aligned}$$

where $\sum' \sum'$ *means that* $\gamma + k > 0$.

Proof. We know that

$$\left[x^\beta \partial_x^\alpha, x^\sigma \partial_x^\rho\right]u = x^\beta \partial_x^\alpha (x^\sigma \partial_x^\rho u) - x^\sigma \partial_x^\rho (x^\beta \partial_x^\alpha u). \tag{2.9}$$

Using Lemma 2.2, (2.9) yields

$$\begin{aligned}&\left[x^\beta \partial_x^\alpha, x^\sigma \partial_x^\rho\right]u \\ &\quad= \sum_{\substack{\gamma\le\alpha \\ \gamma\le\sigma}} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} x^{\sigma-\gamma}x^\beta \partial_x^{\alpha-\gamma+\rho}u \\ &\qquad- \sum_{\substack{k\le\rho \\ k\le\beta}} \frac{\rho!}{(\rho-k)!}\binom{\beta}{k} x^{\beta-k}x^\sigma \partial_x^{\rho-k+\alpha}u \\ &\quad= \sum_{\substack{\gamma\le\alpha \\ \gamma\le\sigma}} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} x^{\sigma-\gamma}\left(x^\beta \partial_x^\rho(\partial_x^{\alpha-\gamma}u)\right) \\ &\qquad- \sum_{\substack{k\le\rho \\ k\le\beta}} \frac{\rho!}{(\rho-k)!}\binom{\beta}{k} x^{\beta-k}\left(x^\sigma \partial_x^\alpha(\partial_x^{\rho-k}u)\right).\end{aligned}$$

Applying Lemma 2.3 in above, we get

$$\begin{aligned}
&\big[x^{\beta}\partial_x^{\alpha}, x^{\sigma}\partial_x^{\rho}\big]u \\
&= \sum_{\substack{\gamma\le\alpha\\ \gamma\le\sigma}} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} x^{\sigma-\gamma}\Big[\sum_{\substack{k\le\beta'\\ k\le\delta}} \frac{\beta'!}{(\beta'-k)!}\binom{\delta}{k}(-1)^k\partial_x^{\delta-k}(x^{\beta'-k}\partial_x^{\alpha-\gamma}u)\Big] \\
&\quad -\sum_{\substack{k\le\rho\\ k\le\beta}} \frac{\rho!}{(\rho-k)!}\binom{\beta}{k} x^{\beta-k}\Big[\sum_{\substack{\gamma\le\sigma'\\ \gamma\le\delta}} \frac{\sigma'!}{(\sigma'-\gamma)!}\binom{\delta}{\gamma}(-1)^{\gamma}\partial_x^{\delta-\gamma}(x^{\sigma'-\gamma}\partial_x^{\rho-k}u)\Big].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\big[x^{\beta}\partial_x^{\alpha}, x^{\sigma}\partial_x^{\rho}\big]u \\
&= \sum_{\substack{\gamma\le\alpha\\ \gamma\le\sigma}} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} \sum_{\substack{k\le\beta'\\ k\le\delta}} \frac{\beta'!}{(\beta'-k)!}\binom{\delta}{k}(-1)^k x^{\sigma-\gamma}\partial_x^{\delta-k}(x^{\beta'-k}\partial_x^{\alpha-\gamma}u) \\
&\quad -\sum_{\substack{\gamma\le\delta\\ \gamma\le\sigma'}} \frac{\delta!}{(\delta-\gamma)!}\binom{\sigma'}{\gamma} \sum_{\substack{k\le\beta\\ k\le\rho}} \frac{\beta!}{(\beta-k)!}\binom{\rho}{k}(-1)^{\gamma} x^{\beta-k}\partial_x^{\delta-\gamma}(x^{\sigma'-\gamma}\partial_x^{\rho-k}u),
\end{aligned}$$

which gives (2.8). □

Proposition 2.5. *Let P be a self-adjoint SG-elliptic operator of the form* (1.11) *with $m_1 > 0, m_2 > 0$ and let $\lambda \in \mathbb{C}\backslash spec(P)$. Then, for every $\alpha, \beta \in \mathbb{Z}_+$ and for every $s \in \mathbb{R}$ the condition* (1.15) *holds with $\mu = \nu = 1$.*

Proof. By using Lemma 2.4 and (1.11), we can write

$$\begin{aligned}
(P-\lambda)^{-1}[P, x^{\beta}\partial_x^{\alpha}]u &= \sum_{\substack{\rho\le m_1\\ \sigma\le m_2}} \widetilde{C}_{\rho\sigma} \sideset{}{'}\sum_{\substack{\gamma\le\alpha\\ \gamma\le\sigma}} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} \sideset{}{'}\sum_{\substack{k\le\beta'\\ k\le\delta}} \frac{\beta'!}{(\beta'-k)!}\binom{\delta}{k}(-1)^k \\
&\quad \times (P-\lambda)^{-1}x^{\sigma-\gamma}\partial_x^{\delta-k}\big(x^{\beta'-k}\partial_x^{\alpha-\gamma}u\big),
\end{aligned}$$

with $\widetilde{C}_{\rho\sigma} = (-1)^{\rho}C_{\rho\sigma}$. □

From (1.14), the operator $(P-\lambda)^{-1}x^{\sigma-\gamma}\partial_x^{\delta-k}$ is bounded on $H^s(\mathbb{R})$ for every $s \in \mathbb{R}$. Then

$$\begin{aligned}
&\big\|(P-\lambda)^{-1}[P, x^{\beta}\partial_x^{\alpha}]u\big\|_s \\
&\le C \sum_{\substack{\rho\le m_1\\ \sigma\le m_2}} \sideset{}{'}\sum_{\substack{\gamma\le\alpha\\ \gamma\le\sigma}} \frac{\alpha!}{(\alpha-\gamma)!}\binom{\sigma}{\gamma} \sideset{}{'}\sum_{\substack{k\le\beta'\\ k\le\delta}} \frac{\beta'!}{(\beta'-k)!}\binom{\delta}{k} \left\|x^{\beta'-k}\partial_x^{\alpha-\gamma}u\right\|_s \\
&\le C_1 \sideset{}{'}\sum_{\gamma\le\alpha} \frac{\alpha!}{(\alpha-\gamma)!} \sideset{}{'}\sum_{k\le\beta'} \frac{\beta'!}{(\beta'-k)!} \left\|x^{\beta'-k}\partial_x^{\alpha-\gamma}u\right\|_s,
\end{aligned}$$

which gives (1.15).

3. Localization operators and Weyl operators using fractional Fourier transform:

In this Section, by fractional Fourier transform tool we study the properties of localization operators and Weyl operators on $S^{\mu}_{\mu}(\mathbb{R})$ space and its dual space $(S^{\mu}_{\mu})'(\mathbb{R})$ in terms of time-frequency representation.
For parameter α and $t, x, w \in \mathbb{R}$:
Translation:

$$T_x\phi(t) = \phi(t-x).$$

Modulation:

$$M_{w,\alpha}\phi(t) = e^{i[\frac{(w^2+t^2)\cot\alpha}{2} - wt\csc\alpha]}\phi(t).$$

Then,

$$\begin{aligned} M_{w,\alpha}T_x\phi(t) &= M_{w,\alpha}\phi(t-x) \\ &= e^{i[\frac{(w^2+t^2)\cot\alpha}{2} - wt\csc\alpha]}\phi(t-x), \end{aligned} \tag{3.1}$$

for a fixed non-zero function ϕ called the "window".

Regarding to Definition 1.7, the short-time fractional Fourier transform (STFRFT) of function f with respect to ϕ is defined by

$$\begin{aligned} V_{\phi,\alpha}f(x,w) &= \langle f, M_{w,\alpha}T_x\phi\rangle \\ &= \int_{\mathbb{R}} f(t)\overline{\phi(t-x)}e^{-i[\frac{(w^2+t^2)\cot\alpha}{2} - wt\csc\alpha]}dt. \end{aligned} \tag{3.2}$$

For ϕ_1, ϕ_2 with $\langle\phi_1, \phi_2\rangle = 1$ the following inversion formula is valid:

$$f(t) = \int_{\mathbb{R}} V_{\phi_1,\alpha}f(x,w)M_{w,\alpha}T_x\phi_2(t)dxdw. \tag{3.3}$$

For given function $a(x,w)$, we define the localization operator using fractional Fourier transform $A^{\phi_1,\phi_2}_{a,\alpha}$ associated with symbol a, parameter α and windows ϕ_1, ϕ_2; it is given by

$$A^{\phi_1,\phi_2}_{a,\alpha}f(t) = \int_{\mathbb{R}} a(x,w)V_{\phi_1,\alpha}f(x,w)M_{w,\alpha}T_x\phi_2(t)dxdw. \tag{3.4}$$

In view of (3.2) and (3.4), we get

$$\langle A^{\phi_1,\phi_2}_{a,\alpha}f, g\rangle = \langle a, \overline{V_{\phi_1,\alpha}f}V_{\phi_2,\alpha}g\rangle, \tag{3.5}$$

where f, g, ϕ and ϕ_1, ϕ_2 are elements of $S(\mathbb{R})$. Since $V_{\phi,\alpha}$ is a continuous map from $S(\mathbb{R})$ to $S(\mathbb{R}\times\mathbb{R})$, then from (3.5), $a \in S'(\mathbb{R})$.

Using the arguments of [2, p.309], we prove the following proposition:

Proposition 3.1. *Let $\phi_1, \phi_2 \in S^{\mu}_{\mu}(\mathbb{R}), \mu \geq \frac{1}{2}$ and let $a \in (S^{\mu}_{\mu})^{'}(\mathbb{R}^2)$. Then* (3.5) *defines a linear map*

$$A^{\phi_1,\phi_2}_{a,\alpha} : S^{\mu}_{\mu}(\mathbb{R}) \longrightarrow (S^{\mu}_{\mu})^{'}(\mathbb{R}), \tag{3.6}$$

where $A^{\phi_1,\phi_2}_{a,\alpha}$ is defined in (3.4).

Proof. In view of Toft [9], we take the window $\phi \in S^{\mu}_{\mu}(\mathbb{R}), \mu \geq \frac{1}{2}$ and linear functional $f \in S^{\mu}_{\mu}(\mathbb{R})$, then we estimate (3.2) in the following way

$$\begin{aligned} |V_{\phi,\alpha} f(x,w)| &= \left| \int_{\mathbb{R}} f(t)\overline{\phi(t-x)} e^{-i[\frac{(w^2+t^2)\cot\alpha}{2} - wt\csc\alpha]} dt \right| \\ &\leq \left| \int_{\mathbb{R}} f(t)\overline{\phi(t-x)} dt \right| \\ &= \left| \langle f(t), \phi(t-x) \rangle \right|. \end{aligned} \tag{3.7}$$

Using (1.16) the above expression becomes

$$|V_{\phi,\alpha} f(x,w)| \leq C\Big(\sum_{\|p\|_1 \leq M} \left\| e^{\lambda\omega} D^p \phi \right\|_{L^\infty} + \sum_{\|q\|_1 \leq N} \left\| e^{\mu\omega} D^q \hat{\phi} \right\|_{L^\infty} \Big), \tag{3.8}$$

for integers $M, N \geq 0$ and constants $C, \lambda, \mu > 0$. □

The above equation (3.8) shows that STFRFT $V_{\phi,\alpha}$ maps continuously $S^{\mu}_{\mu}(\mathbb{R})$ into $S^{\mu}_{\mu}(\mathbb{R} \times \mathbb{R})$, for windows $\phi \in S^{\mu}_{\mu}(\mathbb{R}); \mu \geq \frac{1}{2}$.

Since $S^{\mu}_{\mu}(\mathbb{R})$ is an algebra, therefore, $\overline{V_{\phi_1,\alpha} f} V_{\phi_2,\alpha} g \in S^{\mu}_{\mu}(\mathbb{R} \times \mathbb{R})$.

By L^2-duality, the right hand side of (3.5) is well defined for $a \in (S^{\mu}_{\mu})^{'}(\mathbb{R} \times \mathbb{R})$. So, we get

$$A^{\phi_1,\phi_2}_{a,\alpha} : S^{\mu}_{\mu}(\mathbb{R}) \longrightarrow (S^{\mu}_{\mu})^{'}(\mathbb{R}).$$

Definition 3.2. For parameter α, Weyl pseudo-differential operator associated with fractional Fourier transform $L_{\sigma,\alpha}$ with symbol σ is defined by

$$L_{\sigma,\alpha} f(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(\frac{x+y}{2}, w) f(y) e^{i\left[\frac{((x-y)^2+w^2)\cot\alpha}{2} - (x-y)w\csc\alpha\right]} dydw. \tag{3.9}$$

Definition 3.3. Winger transform by using fractional Fourier transform associated with functions f, g is defined by

$$W_\alpha(f,g)(x,w) = \int_{\mathbb{R}} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})} e^{-i[\frac{(w^2+t^2)\cot\alpha}{2} - wt\csc\alpha]} dt. \tag{3.10}$$

Lemma 3.4. *For Weyl pseudo-differential operator $L_{\sigma,\alpha}$ and Winger transform $W_\alpha(f,g)(x,w)$ associated with fractional Fourier transform defined in* (3.9) *and* (3.10), *respectively, we find the following relation:*

$$\langle L_{\sigma,\alpha} f, g \rangle = \langle \sigma, W_\alpha(f,g) \rangle. \tag{3.11}$$

Proof.

$$\begin{aligned}
&\langle \sigma, W_\alpha(f,g)\rangle \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,w)\overline{W_\alpha(f,g)(x,w)}dxdw \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,w)\overline{[\int_{\mathbb{R}} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-i[\frac{(w^2+t^2)\cot\alpha}{2}-wt\csc\alpha]}dt]}dxdw \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,w)dxdw\int_{\mathbb{R}} \overline{f(x+\frac{t}{2})}g(x-\frac{t}{2})e^{i[\frac{(w^2+t^2)\cot\alpha}{2}-wt\csc\alpha]}dt,
\end{aligned}$$

putting $x - \frac{t}{2} = u$, then the right hand side of the above equation becomes

$$\begin{aligned}
&\langle \sigma, W_\alpha(f,g)\rangle \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(x,w)dxdw[\frac{-1}{2}\int_{\infty}^{-\infty} \overline{f(2x-u)}g(u)e^{i[\frac{(w^2+(2(x-u))^2)\cot\alpha}{2}-w(2(x-u))\csc\alpha]}du] \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(x,w)\overline{f(2x-u)}g(u)e^{i[\frac{(w^2+(2(x-u))^2)\cot\alpha}{2}-w(2(x-u))\csc\alpha]}dudxdw,
\end{aligned}$$

putting $2x - u = z$ and omitting constant term, we have

$$\begin{aligned}
&\langle \sigma, W_\alpha(f,g)\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(\frac{u+z}{2},w)\overline{f(z)}g(u)e^{i[\frac{(w^2+(z-u)^2)\cot\alpha}{2}-w(z-u)\csc\alpha]}dudzdw \\
&= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(\frac{u+z}{2},w)g(u)e^{i[\frac{(w^2+(z-u)^2)\cot\alpha}{2}-w(z-u)\csc\alpha]}dudw]\overline{f(z)}dz \\
&= \int_{-\infty}^{\infty} \big(L_{\sigma,\alpha}g(z)\big)\overline{f(z)}dz \\
&= \langle L_{\sigma,\alpha}f, g\rangle.
\end{aligned}$$

Hence,

$$\langle L_{\sigma,\alpha}f, g\rangle = \langle \sigma, W_\alpha(f,g)\rangle.$$

□

Proposition 3.5. *If* $\sigma \in (S^\mu_\mu)'(\mathbb{R}), \mu \geq \frac{1}{2}$, *then*

$$L_{\sigma,\alpha} : S^\mu_\mu(\mathbb{R}) \longrightarrow (S^\mu_\mu)'(\mathbb{R}). \tag{3.12}$$

Proof. If $f, g \in S^\mu_\mu(\mathbb{R})$, then in similar way as in Theorem 3.8 of [8], we can find $W_\alpha(f,g)(x,w) \in S^\mu_\mu(\mathbb{R}\times\mathbb{R})$.

Using Definition 3.3, we can write

$$\begin{aligned} |W_\alpha(f,g)(x,w)| &= \left|\int_{\mathbb{R}} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-i[\frac{(w^2+t^2)\cot\alpha}{2}-wt\csc\alpha]}dt\right| \\ &\le \left|\int_{\mathbb{R}} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}dt\right|, \end{aligned}$$

let $x+\frac{t}{2}=u$, then the above equation becomes

$$\begin{aligned} |W_\alpha(f,g)(x,w)| &\le \left|\int_{\mathbb{R}} f(u)\overline{g(u-t)}2du\right| \\ &\le \left|\int_{\mathbb{R}} f(u)\overline{g(u-t)}du\right| \\ |W_\alpha(f,g)(x,w)| &\le |\langle f(u), g(u-t)\rangle| . \end{aligned} \tag{3.13}$$

Using (1.16), (3.13) becomes

$$|W_\alpha(f,g)(x,w)| \le C\Big(\sum_{\|p\|_1\le M}\left\|e^{\lambda\omega}D^p g(t-x)\right\|_{L^\infty}+\sum_{\|q\|_1\le N}\left\|e^{\mu\omega}D^q\hat{g}(t-x)\right\|_{L^\infty}\Big), \tag{3.14}$$

for integers $M,N\ge 0$ and constants $C,\lambda,\mu>0$.
This shows that Winger transform associated with fractional Fourier transform maps $S^{\mu}_{\mu}(\mathbb{R})\otimes S^{\mu}_{\mu}(\mathbb{R})$ continuously into $S^{\mu}_{\mu}(\mathbb{R}\times\mathbb{R})$. □

By L^2-duality, the right hand side of (3.9) is well defined for $\sigma\in(S^{\mu}_{\mu})'(\mathbb{R}\times\mathbb{R})$.
So, we get

$$L_{\sigma,\alpha}: S^{\mu}_{\mu}(\mathbb{R}) \longrightarrow (S^{\mu}_{\mu})'(\mathbb{R}).$$

In view of Proposition 3.1 and Proposition 3.5, we obtain,

$$A^{\phi_1,\phi_2}_{a,\alpha} = L_{\sigma,\alpha}.$$

Example 3.6. Consider the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \quad u(0,x)=f(x), \quad u\in S^{\mu}_{\nu}(\mathbb{R}), \tag{3.15}$$

where,

$$\Delta = (\frac{d}{dx}-ix\cot\alpha)^2.$$

Applying fractional Fourier transform on (3.15) and using Definition 1.2, we get

$$\frac{\partial\hat{u}_\alpha}{\partial t} = \frac{1}{2\pi}C'_\alpha\int_{\mathbb{R}} e^{\frac{i(x^2+\xi^2)\cot\alpha}{2}-ix\xi\csc\alpha}\Delta u d\xi.$$

From [7, p.357], we have

$$\begin{aligned} \frac{\partial\hat{u}_\alpha}{\partial t} &= \frac{1}{2\pi}C'_\alpha\int_{\mathbb{R}}(\frac{d}{dx}-ix\cot\alpha)^2 e^{\frac{i(x^2+\xi^2)\cot\alpha}{2}-ix\xi\csc\alpha}u(\xi)d\xi \\ &= (-i\xi\csc\alpha)^2\frac{1}{2\pi}C'_\alpha\int_{\mathbb{R}} e^{\frac{i(x^2+\xi^2)\cot\alpha}{2}-ix\xi\csc\alpha}u(\xi)d\xi. \end{aligned}$$

Hence

$$\frac{\partial \hat{u}_\alpha}{\partial t} = -(\xi \csc \alpha)^2 \hat{u}_\alpha, \tag{3.16}$$

and

$$\hat{u}_\alpha(0,\xi) = \hat{f}_\alpha(\xi).$$

Taking the inverse fractional Fourier transform of both sides on (3.16), we find the solution of the heat equation,

$$u(t,x) = \frac{1}{2\pi} C_\alpha^{'} \int_{\mathbb{R}} e^{-t(\xi \csc \alpha)^2} \hat{f}_\alpha(\xi) e^{\frac{i(x^2+\xi^2)\cot\alpha}{2} - ix\xi \csc\alpha} d\xi. \tag{3.17}$$

Since $\hat{f}_\alpha(\xi) \in S_\nu^\mu(\mathbb{R})$ and $e^{-t(\xi \csc \alpha)^2} \in \Gamma_{\nu,\mu}^m(\mathbb{R})$, therefore from Theorem 2.1 $u(t,x) \in S_\nu^\mu(\mathbb{R})$. Hence we get the solution of the heat equation in a form of a pseudo-differential operator with symbol $a(t\xi csc\alpha) = e^{-t(\xi \csc \alpha)^2}$ and it is an element of the space $S_\nu^\mu(\mathbb{R})$.

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