Novi Sad J. Math.
Vol. 45, No. 1, 2015, 253-265

ON A SYSTEM OF NONLINEAR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS[1]

László Simon[2]

Abstract. We consider a system of a semilinear hyperbolic functional differential equation (where the lower order terms contain functional dependence on the unknown functions) with initial and boundary conditions and a quasilinear elliptic functional differential equation (containing t as a parameter) with boundary conditions. Existence of solutions for $t \in (0, T)$ will be shown and some examples will be formulated.

AMS Mathematics Subject Classification (2010): 35M33, 35R10

Key words and phrases: partial functional-differential equations, nonlinear systems of partial differential equations, nonlinear systems of mixed type.

1. Introduction

In the present paper we consider weak solutions of the following system of equations:

$$u''(t) + Q(u(t)) + \varphi(x)h'(u(t)) + H(t, x; u, z) + \psi(x)u'(t) = F_1(t, x; z), \tag{1.1}$$

$$-\sum_{j=1}^{n} D_j[a_j(t, x, Dz(t), z(t); u)] + a_0(t, x, Dz(t), z(t); u, z) = F_2(t, x; u) \tag{1.2}$$

$$(t, x) \in Q_T = (0, T) \times \Omega$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and we use the notations $u(t) = u(t, x)$, $u' = D_t u$, $u'' = D_t^2 u$, $z(t) = z(t, x)$, $Dz = \left(\frac{\partial z}{\partial x_1}, \dots \frac{\partial z}{\partial x_n}\right)$, Q may be e.g. a linear second order symmetric elliptic differential operator in the variable x; h is a C^2 function having certain polynomial growth, H contains nonlinear functional (non-local) dependence on u and z, with some polynomial growth and F_1 contains some functional dependence on z. Further, the functions a_j define a quasilinear elliptic differential operator in x (for fixed t) with functional dependence on u for $i = 1, \dots, n$ and on u, z for $i = 0$, respectively. Finally, F_2 may non-locally depending on u. The system (1.1), (1.2) consists of a semilinear

[1]This work was supported by the Hungarian National Foundation for Scientific Research under grant OTKA K 81403.

[2]Institute of Mathematics, Faculty of Science, L. Eötvös University of Budapest, e-mail: simonl@cs.elte.hu

hyperbolic functional equation and an elliptic functional equation (containing the time t as a parameter).

This paper was motivated by some problems which were modelled by systems consisting of (functional) differential equations of different types (see [11]. In [3] S. Cinca investigated a model, consiting of an elliptic, a parabolic and an ordinary nonlinear differential equation, which arise when modelling diffusion and transport in porous media with variable porosity. In [5] J.D. Logan, M.R. Petersen and T.S. Shores considered and numerically studied a similar system which describes reaction-mineralogy-porosity changes in porous media with one-dimensional space variable. J. H. Merkin, D.J. Needham and B.D. Sleeman considered in [6] a system, consisting of a nonlinear parabolic and an ordinary differential equation, as a mathematical model for the spread of morphogens with density dependent chemosensitivity. In [2], [7], [8] the existence of solutions of such systems were studied.

In Section 2 the existence of weak solutions will be proved for $t \in (0, T)$, in Section 3 some examples will be shown. In a separate paper we shall prove existence and certain properties of solutions for $t \in (0, \infty)$.

2. Solutions in $(0, T)$

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0, T) \times \Omega$. Denote by $W^{1,p}(\Omega)$ the Sobolev space of real valued functions with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^n |D_j u|^p + |u|^p \right) dx \right]^{1/p} \qquad (2 \le p < \infty, \quad D_j u = \frac{\partial u}{\partial x_j}).$$

The number q is defined by $1/p + 1/q = 1$. Further, let $V_1 \subset W^{1,2}(\Omega)$ and $V_2 \subset W^{1,p}(\Omega)$ be closed linear subspaces containing $C_0^\infty(\Omega)$), $V_j^\star$ the dual spaces of V_j, the duality between $V_j^\star$ and V_j will be denoted by $\langle \cdot, \cdot \rangle$, the scalar product in $L^2(\Omega)$ will be denoted by $(\cdot, \cdot)$. Finally, denote by $L^p(0, T; V_j)$ the Banach space consisting of the set of measurable functions $u : (0, T) \to V_j$ with the norm

$$\|u\|_{L^p(0,T;V_j)} = \left[\int_0^T \|u(t)\|_{V_j}^p dt \right]^{1/p}$$

and $L^\infty(0, T; V_j)$, $L^\infty(0, T; L^2(\Omega))$ the set of measurable functions $u : (0, T) \to V_j$, $u : (0, T) \to L^2(\Omega)$, respectively, with the $L^\infty(0, T)$ norm of the functions $t \mapsto \|u(t)\|_{V_j}$, $t \mapsto \|u(t)\|_{L^2(\Omega)}$, respectively.

Now we formulate the assumptions on the functions in (1.1), (1.2).

(A_1). $Q : V_1 \to V_1^\star$ is a linear continuous operator such that

$$\langle Qu, v \rangle = \langle Qv, u \rangle, \quad \langle Qu, u \rangle \ge c_0 \|u\|_{V_1}^2$$

for all $u, v \in V_1$ with some constant $c_0 > 0$.

(A_2). $\varphi, \psi : \Omega \to \mathbb{R}$ are measurable functions satisfying with constants c_1, c_2

$$0 < c_1 \le \varphi(x) \le c_2, \quad c_1 \le \psi(x) \le c_2 \text{ for a.a. } x \in \Omega.$$

(A_3). $h : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function satisfying

$$h(\eta) \ge 0, \quad |h''(\eta)| \le \text{const}|\eta|^{\lambda-1} \text{ for } |\eta| > 1 \text{ where}$$

$$1 < \lambda \le \lambda_0 = \frac{n}{n-2} \text{ if } n \ge 3, \quad 1 < \lambda < \infty \text{ if } n = 2.$$

(A_4). $H : Q_T \times L^2(Q_T) \times L^p(0,T;V_2) \to \mathbb{R}$ is a function for which $(t,x) \mapsto H(t,x;u,z)$ is measurable for all fixed $u \in L^2(Q_T)$, $z \in L^p(0,T;V_2)$, H has the Volterra property, i.e. for all $t \in [0,T]$, $H(t,x;u,z)$ depends only on the restriction of u and z to Q_t (i.e. it does not depend on $u(\tau,x), z(\tau,x)$ for $\tau > t$). Further, the following inequality holds for all $t \in [0,T]$ and $u \in L^2(\Omega)$, $z \in L^p(0,T;V_2)$:

$$\begin{aligned} &\int_\Omega |H(t,x;u,z)|^2 dx \\ &\le \text{const}\left[\|z\|^2_{L^p(0,T;V_2)} + 1\right]\left[\int_0^t \int_\Omega h(u)dxd\tau + \int_\Omega h(u)dx + 1\right]; \end{aligned}$$

and for all fixed number $K > 0$, there exists a bounded (nonlinear) operator $z \mapsto M(K,z) \in \mathbb{R}^+$, $z \in L^p(0,T;V_2)$ such that

$$\begin{aligned} &\int_0^t \left[\int_\Omega |H(\tau,x;u_1,z) - H(\tau,x;u_2,z)|^2 dx\right] d\tau \\ &\le M(K,z)\int_0^t \left[\int_\Omega |u_1 - u_2|^2 dx\right] d\tau \text{ if } \|u_j\|_{L^\infty(0,T;V_1)} \le K. \end{aligned}$$

(The last inequality means that $H(t,x;u,z)$ is locally Lipschitz in u and the Lipschitz constant is bounded if z is bounded in $L^p(0,T;V_2)$.)

Finally, $(z_k) \to z$ in $L^p(0,T;V_2)$ implies

$$H(t,x;u_k,z_k) - H(t,x;u_k,z) \to 0 \text{ in } L^2(Q_T) \text{ uniformly if } \|u_k\|_{L^2(Q_T)} \le \text{const}.$$

(A_5). $F_1 : Q_T \times L^p(0,T;V_2) \to \mathbb{R}$ is a function satisfying $(t,x) \mapsto F_1(t,x;z) \in L^2(Q_T)$ for all fixed $z \in L^p(0,T;V_2)$ and $(z_k) \to z$ in $L^p(0,T;V_2)$ implies that $F_1(t,x;z_k) \to F_1(t,x;z)$ in $L^2(Q_T)$.

Further,

$$\int_0^T \|F_1(\tau,x;z)\|^2_{L^2(\Omega)} d\tau \le \text{const}\left[1 + \|z\|^{\beta_1}_{L^p(0,T;V_2)}\right]$$

with some constant $\beta_1 > 0$.

(B_1) The functions

$$a_j : Q_T \times \mathbb{R}^{n+1} \times L^2(Q_T) \to \mathbb{R} \quad (j = 1, \dots n),$$

$$a_0 : Q_T \times \mathbb{R}^{n+1} \times L^2(Q_T) \times L^p(0,T;V_2) \to \mathbb{R}$$

are such that $a_j(t,x,\xi;u)$, $a_0(t,x,\xi;u,z)$ are measurable functions of variables $(t,x) \in Q_T$ for all fixed $\xi \in \mathbb{R}^{n+1}$, $u \in L^2(Q_T)$, $z \in L^p(0,T;V_2)$ and continuous functions of variable $\xi \in \mathbb{R}^{n+1}$ for all fixed $u \in L^2(Q_T)$, $z \in L^p(0,T;V_2)$ and a.a. fixed $(t,x) \in Q_T$.

Further, if $(u_k) \to u$ in $L^2(Q_T)$ then for all $z \in L^p(0,T;V_2)$, $\xi \in \mathbb{R}^{n+1}$, a.a. $(t,x) \in Q_T$, for a subsequence

$$a_j(t,x,\xi;u_k) \to a_j(t,x,\xi;u) \quad (j = 1,\dots,n),$$

$$a_0(t,x,\xi;u_k,z) \to a_0(t,x,\xi;u,z).$$

(B_2) For $j = 1,\dots,n$

$$|a_j(t,x,\xi;u)| \le g_1(u)|\xi|^{p-1} + [k_1(u)](t,x)$$

where $g_1 : L^2(Q_T) \to \mathbb{R}^+$ is a bounded, continuous (nonlinear) operator,

$$k_1 : L^2(Q_T) \to L^q(Q_T) \text{ is continuous and}$$

$$\|k_1(u)\|_{L^q(Q_T)} \le \text{const}(1 + \|u\|^\gamma_{L^2(Q_T)});$$

$$|a_0(t,x,\xi;u,z)| \le g_2(u,z)|\xi|^{p-1} + [k_2(u,z)](t,x)$$

where

$$g_2 : L^2(Q_T) \times L^p(0,T;V_2) \to \mathbb{R}^+ \text{ and } k_2 : L^2(Q_T) \times L^p(0,T;V_2) \to L^q(Q_T)$$

are continuous bounded operators such that

$$\|k_2(u,z)\|_{L^q(Q_T)} \le \text{const}\left[1 + \|u\|^\gamma_{L^2(Q_T)}\right]$$

with some constant $\gamma \ge 0$.

(B_3) The following inequality holds for all $t \in [0,T]$ with some constants $c_2 > 0$, $\beta > 0$ (not depending on t):

$$\int_{Q_T} \sum_{j=1}^n [a_j(t,x,Dz(t),z(t);u) - a_j(t,x,Dz^\star(t),z^\star(t);u)][D_j z(t) - D_j z^\star(t)]dxdt +$$

$$\int_{Q_T} [a_0(t,x,Dz(t),z(t);u,z) - a_0(t,x,Dz^\star(t),z^\star(t);u,z^\star)][z(t) - z^\star(t)]dxdt \ge$$

$$\frac{c_2}{1 + \|u\|^\beta_{L^2(Q_T)}} \|z - z^\star\|^p_{L^p(0,T;V_2)}.$$

(B_4) For all fixed $u \in L^2(Q_T)$ the function

$$F_2 : Q_T \times L^2(Q_T) \to \mathbb{R} \text{ satisfies } (t,x) \mapsto F_2(t,x;u) \in L^q(Q_T),$$

$$\|F_2(t,x;u)\|_{L^q(Q_T)} \le \text{const}\left[1 + \|u\|^\gamma_{L^2(Q_T)}\right]$$

(see (B_2)) and

$$(u_k) \to u \text{ in } L^2(Q_T) \text{ implies } F_2(t,x;u_k) \to F_2(t,x;u) \text{ in } L^q(Q_T).$$

Finally,

$$\frac{\beta_1}{2}\frac{\beta+\gamma}{p-1} < 1.$$

Theorem 2.1. *Assume* (A_1) – (A_5) *and* (B_1) – (B_4). *Then for all* $u_0 \in V_1$, $u_1 \in L^2(\Omega)$ *there exist* $u \in L^\infty(0,T;V_1)$, $z \in L^p(0,T;V_2)$ *such that*

$$u' \in L^\infty(0,T;L^2(\Omega)), \quad u'' \in L^2(0,T;V_1^\star),$$

u, z *satisfy (1.1) in the sense: for a.a.* $t \in [0,T]$, *all* $v \in V_1$

$$\langle u''(t), v\rangle + \langle Q(u(t)), v\rangle + \int_\Omega \varphi(x)h'(u(t))vdx + \int_\Omega H(t,x;u,z)vdx + \int_\Omega \psi(x)u'(t)vdx = \int_\Omega F_1(t,x;z)v)dx \tag{2.1}$$

and the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.2}$$

Further, u, z *satisfy (1.2) in the sense: for a.a.* $t \in (0,T)$, *all* $w \in V_2$

$$\int_\Omega \left[\sum_{j=1}^n a_j(t,x,Dz(t),z(t);u)\right] D_j w dx + \int_\Omega a_0(t,x,Dz(t),z(t);u,z)wdx = \int_\Omega F_2(t,x;u)wdx. \tag{2.3}$$

Proof. The proof is based on the results of [10], the theory of monotone operators (see, e.g., [4], [9], [12]) and Schauder's fixed point theorem as follows.

Consider the problem (2.1), (2.2) for u with an arbitrary fixed $z = \tilde{z} \in L^p(0,T;V_2)$. According to [10] assumptions $(A_1) - (A_5)$ imply that there exists a unique solution $u = \tilde{u} \in L^\infty(0,T;V_1)$ with the properties $\tilde{u}' \in L^\infty(0,T;L^2(\Omega))$, $\tilde{u}'' \in L^2(0,T;V_1^\star)$ satisfying (2.1) and the initial condition (2.2). Then consider problem (2.3) for z with the above $u = \tilde{u}$. According to the theory of monotone operators there exists a unique solution $z \in L^p(0,T;V_2)$ of (2.3). By using the notation $S(\tilde{z}) = z$, we shall show that the operator $S : L^p(0,T;V_2) \to L^p(0,T;V_2)$ satisfies the assumptions of Schauder's fixed point theorem: it is continuous, compact and there exists a closed ball $\overline{B_R(0)} \subset L^p(0,T;V_2)$ such that

$$S(\overline{B_R(0)}) \subset \overline{B_R(0)}. \tag{2.4}$$

Then Schauder's fixed point theorem will imply that S has a fixed point $z^\star \in L^p(0,T;V_2)$. Defining $u^\star$ by the solution of (2.1), (2.2) with $z = z^\star$, functions $u^\star$, $z^\star$ satisfy (2.1) – (2.3). □

Lemma 2.2. *The operator $S : L^p(0,T;V_2) \to L^p(0,T;V_2)$, defined by $S(\tilde{z}) = z$ is compact.*

Proof. Let $(\tilde{z}_k)$ be a bounded sequence in $L^p(0,T;V_2)$ and consider the (unique) solution $\tilde{u}_k$ of (2.1), (2.2) with fixed $z = \tilde{z}_k$. We show that $(\tilde{u}_k)$ is bounded in $L^\infty(0,T;V_1)$ and $(\tilde{u}_k')$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Indeed, applying the arguments in the proof of Theorem 2.1 in [10], one gets the unique solutions $\tilde{u}_k$ of (2.1), (2.2) as the (weak) limit of Galerkin approximations

$$\tilde{u}_{mk}(t) = \sum_{l=1}^{m} g_{lm}^k(t) w_l, \text{ where } g_{lm}^k \in W^{2,2}(0,T)$$

and $w_1, w_2, \dots$ is a linearly independent system in V_1 such that the linear combinations are dense in V_1, further, the functions $\tilde{u}_{mk}$ satisfy (for $j = 1, \dots, m$)

$$\langle \tilde{u}_{mk}''(t), w_j \rangle + \langle Q(\tilde{u}_{mk}(t)), w_j \rangle + \int_\Omega \varphi(x) h'(\tilde{u}_{mk}(t)) w_j dx + \tag{2.5}$$

$$\int_\Omega H(t,x;\tilde{u}_{mk},\tilde{z}_k) w_j dx + \int_\Omega \psi(x) \tilde{u}_{mk}'(t) w_j dx = \int_\Omega F_1(t,x;\tilde{z}_k) w_j dx,$$

$$\tilde{u}_{mk}(0) = u_{m0}, \quad \tilde{u}_{mk}'(0) = u_{m1}, \tag{2.6}$$

where u_{m0}, u_{m1} $(m = 1,2,\dots)$ are linear combinations of $w_1, w_2, \dots w_m$, satisfying $(u_{m0}) \to u_0$ in V_1 and $(u_{m1}) \to u_1$ in $L^2(\Omega)$ as $m \to \infty$.

Multiplying (2.5) by $(g_{lm}^k)'(t)$, summing with respect to j and integrating over $(0,t)$, by Young's inequality we find

$$\frac{1}{2}\|\tilde{u}_{mk}'(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\langle Q(\tilde{u}_{mk}(t)), \tilde{u}_{mk}(t)\rangle + \int_\Omega \varphi(x) h(\tilde{u}_{mk}(t)) dx + \tag{2.7}$$

$$\int_0^t \left[\int_\Omega H(\tau,x;\tilde{u}_{mk},\tilde{z}_k)\tilde{u}_{mk}'(\tau) dx\right] d\tau + \int_0^t \left[\int_\Omega \psi(x) |\tilde{u}_{mk}'(\tau)|^2 dx\right] d\tau =$$

$$\int_0^t \left[\int_\Omega F_1(\tau,x;\tilde{z}_k)\tilde{u}_{mk}'(\tau) dx\right] d\tau + \frac{1}{2}\|\tilde{u}_{mk}'(0)\|_{L^2(\Omega)}^2 + \frac{1}{2}\langle Q(\tilde{u}_{mk}(0)), \tilde{u}_{mk}(0)\rangle +$$

$$\int_\Omega \varphi(x) h(\tilde{u}_{mk}(0)) dx \le \frac{1}{2}\int_0^T \|F_1(\tau,x;\tilde{z}_k)\|_{L^2(\Omega)}^2 d\tau + \frac{1}{2}\int_0^T \|\tilde{u}_{mk}'(\tau)\|_{L^2(\Omega)}^2 + \text{const},$$

where the constant does not depend on m, k, t. (See [10].)

By using (A_2), (A_4), (A_5) and the Cauchy-Schwarz inequality, we obtain from (2.7)

$$\frac{1}{2}\|\tilde{u}_{mk}'(t)\|_{L^2(\Omega)}^2 + \frac{c_0}{2}\|\tilde{u}_{mk}(t))\|_{V_1}^2 + c_1 \int_\Omega h(\tilde{u}_{mk}(t)) dx \le \tag{2.8}$$

$$\int_0^T \|F_1(\tau,x;\tilde{z}_k)\|_{L^2(\Omega)}^2 d\tau +$$

$$\text{const}\left\{1+\int_0^t \|\tilde{u}'_{mk}(\tau)\|^2_{L^2(\Omega)}d\tau+\int_0^t\left[\int_\Omega h(\tilde{u}_{mk}(\tau))dx\right]d\tau\right\}.$$

Consequently,

$$\|\tilde{u}'_{mk}(t)\|^2_{L^2(\Omega)}+\int_\Omega h(\tilde{u}_{mk}(t))dx \le$$

$$\text{const}\left\{1+\int_0^t\left[\|\tilde{u}'_{mk}(\tau)\|^2_{L^2(\Omega)}+\int_\Omega h(\tilde{u}_{mk}(\tau))dx\right]d\tau\right\},$$

where the constant does not depend on k, m, t. Thus by Gronwall's lemma

$$\|\tilde{u}'_{mk}(t)\|^2_{L^2(\Omega)}+\int_\Omega h(\tilde{u}_{mk}(t))dx \le \text{const} \tag{2.9}$$

and so by (A_1) and (2.8)

$$\|\tilde{u}_{mk}(t)\|_{V_1} \le \text{const}, \tag{2.10}$$

where the constants do not depend on k, m, t. The inequalities (2.9), (2.10) imply that the weak limits $\tilde{u}_k$, $\tilde{u}'_k$ of $(\tilde{u}_{mk})$ and $(\tilde{u}'_{mk})$, respectively, are bounded in $L^\infty(0,T;V_1)$, $L^\infty(0,T;L^2(\Omega))$, respectively.

Consequently, by the well known compact embedding theorem (see [4]) there is a subsequence of $(\tilde{u}_k)$, again denoted by $(\tilde{u}_k)$, for simplicity, which is convergent in $L^2(Q_T)$ to some $\tilde{u}$ and $(\tilde{u}_k) \to \tilde{u}$ a.e. in Q_T.

Now we show that the sequence of solutions z_k of (2.3) with $u = \tilde{u}_k$ converges in $L^p(0,T;V_2)$ to the solution of (2.3) with $u = \tilde{u}$. By (B_3)

$$\frac{c_2}{1+\|\tilde{u}_k\|^\beta_{L^2(Q_T)}}\|z_k - z\|^p_{L^p(0,T;V_2)} \le \tag{2.11}$$

$$\int_{Q_T}\sum_{j=1}^n[a_j(t,x,Dz_k,z_k;\tilde{u}_k)-a_j(t,x,Dz,z;\tilde{u}_k)](D_jz_k-D_jz)dtdx+$$

$$\int_{Q_T}[a_0(t,x,Dz_k,z_k;\tilde{u}_k,z_k)-a_0(t,x,Dz,z;\tilde{u}_k,z)](z_k-z)dtdx=$$

$$\int_{Q_T}[F_2(t,x;\tilde{u}_k)-F_2(t,x;\tilde{u})](z_k-z)dtdx-$$

$$\int_{Q_T}\sum_{j=1}^n[a_j(t,x,Dz,z;\tilde{u}_k)-a_j(t,x,Dz,z;\tilde{u})](D_jz_k-D_jz)dtdx-$$

$$\int_{Q_T}[a_0(t,x,Dz,z;\tilde{u}_k,z)-a_0(t,x,Dz,z;\tilde{u},z)](z_k-z)dtdx.$$

By using Hölder's inequality, it is not difficult to show that all the terms on the right hand side of (2.11) converge to 0 as $k \to \infty$. Indeed, by (B_4)

$$\lim_{k\to\infty}\|F_2(t,x;\tilde{u}_k)-F_2(t,x;\tilde{u})\|_{L^q(Q_T)}=0 \tag{2.12}$$

and $z_k - z$ is bounded in $L^p(0,T;V_2)$ and thus in $L^p(Q_T)$, since (B_3) implies

$$\int_{Q_T} \sum_{j=1}^{n} [a_j(t,x,Dz_k,z_k;\tilde{u}_k) - a_j(t,x,0,0;\tilde{u}_k)] D_j z_k dtdx + \tag{2.13}$$

$$\int_{Q_T} [a_0(t,x,Dz_k,z_k;\tilde{u}_k,z_k) - a_0(t,x,0,0;\tilde{u}_k,0)] z_k dtdx \geq$$

$$\frac{c_2}{1+\|\tilde{u}_k\|^{\beta}_{L^2(Q_t)}} \|z_k\|^p_{L^p(0,T;V_2)}$$

and for the left hand side of (2.13) we have by Hölder's inequality and (B_2)

$$\int_{Q_T} \sum_{j=1}^{n} [a_j(t,x,Dz_k,z_k;\tilde{u}_k) - a_j(t,x,0,0;\tilde{u}_k)] D_j z_k dtdx + \tag{2.14}$$

$$\int_{Q_T} [a_0(t,x,Dz_k,z_k;\tilde{u}_k,z_k) - a_0(t,x,0,0;\tilde{u}_k,0)] z_k dtdx =$$

$$\int_{Q_T} F_2(t,x;\tilde{u}_k) z_k dtdx -$$

$$\int_{Q_T} \left[\sum_{j=1}^{n} a_j(t,x,0,0;\tilde{u}_k) D_j z_k + a_0(t,x,0,0;\tilde{u}_k,0) z_k \right] dtdx$$

and the absolute value of the right hand side of (2.14) can be estimated by

$$\left\{ \|F_2(t,x;\tilde{u}_k)\|_{L^q(Q_T)} + \text{const} \left[\|k_1(\tilde{u}_k)\|_{L^q(Q_T)} + c(\tilde{u}_k) \right] \right\} \|z_k\|_{L^p(0,T;V_2)}$$

and so (2.13), (2.14) and $p > 1$ imply that $\|z_k\|_{L^p(0,T;V_2)}$ is bounded.

The further terms on the right hand side of (2.11) can be estimated similarly, by using Hölder's inequality. E.g.

$$\int_{Q_T} |a_0(t,x,Dz,z;\tilde{u}_k,z) - a_0(t,x,Dz,z;\tilde{u},z)|^p dtdx \to 0 \tag{2.15}$$

because by (B_1) the integrand converges to 0 a.e. in Q_T for a subsequence and by (B_2) the sequence of the integrands is equiintegrable, so Vitali's theorem implies (2.15) for a subsequence, which holds for the original sequence, too, by Cantor's trick.

Consequently, from (2.11) one obtains

$$\lim_{k\to\infty} \|z_k - z\|_{L^p(0,T;V_2)} = 0. \tag{2.16}$$

□

Lemma 2.3. *The operator $S : L^p(0,T;V_2) \to L^p(0,T;V_2)$ is continuous.*

Proof. Assume that

$$(\tilde{z}_k) \to \tilde{z} \text{ in } L^p(0,T;V_2). \tag{2.17}$$

Now we show that for the solutions $\tilde{u}_k$ of (2.1), (2.2) with $z = \tilde{z}_k$

$$(\tilde{u}_k) \to \tilde{u} \text{ in } L^2(Q_T) \tag{2.18}$$

and a.e. in Q_T for a subsequence where $\tilde{u}$ is the solution of (2.1), (2.2) with $z = \tilde{z}$. Then from the second part of the proof of Lemma 2.2 we shall obtain

$$(z_k) \to z \text{ in } L^p(0,T;V_2) \tag{2.19}$$

for the original sequence (by using Cantor's trick) where z_k and z are the solutions of (2.3) with $u = \tilde{u}_k$ and $u = \tilde{u}$, respectively.

In the proof of (2.18) we use the (uniqueness) Theorem 4.1 of [10]. Since $(\tilde{z}_k)$ is bounded in $L^p(0,T;V_2)$, $(\tilde{u}_k)$ is bounded in $L^2(Q_T)$ (see the proof of Lemma 2.2). Further, $\tilde{u}$ and $\tilde{u}_k$ are weak solutions of (1.1) (i.e. of (2.1)) with $z = \tilde{z}$ and $z = \tilde{z}_k$, respectively and satisfy the initial conditions (2.2), thus

$$\tilde{u}''(t) + Q(\tilde{u}(t)) + \varphi(x)h'(\tilde{u}(t)) + H(t,x;\tilde{u},\tilde{z}) + \psi(x)\tilde{u}'(t) = F_1(t,x;\tilde{z}), \tag{2.20}$$

$$\tilde{u}_k''(t) + Q(\tilde{u}_k(t)) + \varphi(x)h'(\tilde{u}_k(t)) + H(t,x;\tilde{u}_k,\tilde{z}) + \psi(x)\tilde{u}_k'(t) = F_1(t,x;\tilde{z}_k) + H(t,x;\tilde{u}_k,\tilde{z}) - H(t,x;\tilde{u}_k,\tilde{z}_k). \tag{2.21}$$

Theorem 4.1 of [10] implies that for the solutions $\tilde{u}$ of (2.20) and $\tilde{u}_k$ of (2.21) we have for any $s \in [0,T]$ an estimation of the form

$$\|\tilde{u}_k(s) - \tilde{u}(s)\|^2_{L^2(\Omega)} \le \text{const} \int_{Q_T} \left| \int_0^t [F_1(\tau,x;\tilde{z}_k) - F_1(\tau,x;\tilde{z})]d\tau \right|^2 dtdx +$$

$$\text{const} \int_{Q_T} \left| \int_0^t [H(\tau,x;\tilde{u}_k,\tilde{z}_k) - H(\tau,x;\tilde{u}_k,\tilde{z})]d\tau \right|^2 dtdx,$$

where the right hand side is converging to 0 as $k \to \infty$ by (A_4), (A_5).

So, we have proved (2.18) which completes the proof of Lemma 2.3. □

Lemma 2.4. *There is a closed ball*

$$\overline{B_R(0)} = \{z \in L^p(0,T;V_2) : \|z\|_{L^p(0,T;V_2)} \le R\}$$

such that $S(\overline{B_R(0)}) \subset \overline{B_R(0)}$.

Proof. According to (2.8) we have for the sequence $(\tilde{u}_m)$ of Galerkin approximation of the solution of (2.1), (2.2) (with $z = \tilde{z}$)

$$\frac{1}{2}\|\tilde{u}_m'(t)\|^2_{L^2(\Omega)} + \frac{c_0}{2}\|\tilde{u}_m(t)\|^2_{V_1} + c_1\int_\Omega h(\tilde{u}_m(t))dx \le \tag{2.22}$$

$$\int_0^T \|F_1(\tau, x; \tilde{z})\|^2_{L^2(\Omega)}d\tau +$$

$$\text{const}\left\{1 + \int_0^t \|\tilde{u}_m'(\tau)\|^2_{L^2(\Omega)}d\tau + \int_0^t\left[\int_\Omega h(\tilde{u}_m(\tau))dx\right]d\tau\right\}$$

where the constants do not depend on $m, t, \tilde{z}$. Hence, by Gronwall's lemma one obtains

$$\|\tilde{u}_m'(t)\|^2_{L^2(\Omega)} + \int_\Omega h(\tilde{u}_m(t))dx \le \text{const}\int_0^T \|F_1(\tau, x; \tilde{z})\|^2_{L^2(\Omega)}d\tau + \tag{2.23}$$

$$\text{const}\int_0^t\left[\int_0^T [1 + \|F_1(\tau, x; \tilde{z})\|^2_H d\tau]\cdot \mathrm{e}^{t-s}\right]ds =$$

$$\text{const}\int_0^T \|F_1(\tau, x; \tilde{z})\|^2_H d\tau + \text{const},$$

where the constants are independent of $m, t, \tilde{z}$. Thus by (2.22) and (A_5) we find

$$\|\tilde{u}_m(t)\|^2_{V_1} \le \text{const}\left[1 + \int_0^T \|F_1(\tau, x; \tilde{z})\|^2_H d\tau\right] \le \text{const}\left[1 + \|\tilde{z}\|^{\beta_1}_{L^p(0,T;V_2)}\right],$$

which implies (for the limit of $(\tilde{u}_m)$)

$$\|\tilde{u}\|^2_{L^2(Q_T)} \le \text{const}\left[1 + \|\tilde{z}\|^{\beta_1}_{L^p(0,T;V_2)}\right]. \tag{2.24}$$

On the other hand, by (2.13), (2.14) we have for the solution z of (2.3) with $u = \tilde{u}$

$$\frac{c_2}{1 + \|\tilde{u}\|^\beta_{L^2(Q_T)}}\|z\|^p_{L^p(0,T;V_2)} \le \|F_2(t, x; \tilde{u})\|_{L^q(Q_T)}\|z\|_{L^p(0,T;V_2)} + \tag{2.25}$$

$$\text{const}\left[\|k_1(\tilde{u})\|_{L^q(Q_T)} + c(\tilde{u})\right]\|z\|_{L^p(0,T;V_2)},$$

where the first constant does not depend on $\tilde{u}$, further, by (B_2)

$$\|k_1(\tilde{u})\|_{L^q(Q_T)} \le \text{const}\left[1 + \|\tilde{u}\|^\gamma_{L^2(Q_T)}\right] \quad \text{and} \tag{2.26}$$

$$c(\tilde{u}) \le \text{const}\left[1 + \|\tilde{u}\|^\gamma_{L^2(Q_T)}\right].$$

The inequalities (2.25), (2.26) imply

$$\|z\|^{p-1}_{L^p(0,T;V_2)} \le \tag{2.27}$$

$$\text{const}\left[1+\|\tilde{u}\|^{\beta}_{L^2(Q_T)}\right]\cdot\left[\|F_2(t,x;\tilde{u})\|_{L^q(Q_T)}+1+\|\tilde{u}\|^{\gamma}_{L^2(Q_T)}\right]$$

thus by (2.24) and (B_4)

$$\|z\|_{L^p(0,T;V_2)} \le \text{const}\left[1+\|\tilde{u}\|^{\frac{\beta+\gamma}{p-1}}_{L^2(Q_T)}\right] \le \text{const}\left[1+\|\tilde{z}\|^{\frac{\beta_1(\beta+\gamma)}{2(p-1)}}_{L^p(0,T;V_2)}\right], \tag{2.28}$$

where the constants do not depend on $\tilde{u}$ and $\tilde{z}$.

According to the assumption (B_4)

$$\frac{\beta_1(\beta+\gamma)}{2(p-1)} < 1, \tag{2.29}$$

thus for sufficiently large R

$$\tilde{z}\in\overline{B_R(0)}=\left\{\tilde{z}\in L^p(0,T;V_2),\quad \|\tilde{z}\|_{L^p(0,T;V_2)}\le R\right\}$$

implies

$$\|z\|_{L^p(0,T;V_2)}\le R,\ \text{i.e. }\ z\in\overline{B_R(0)}.$$

So the proof of Lemma 2.4 is completed. □

Finally, Lemmas 2.2 - 2.4 and Schauder's fixed point theorem imply that ST has a fixed point and, consequently, there exists a solution of (2.1), (2.3).

3. Examples

Let the operator Q be defined by

$$\langle Qu, v\rangle = \int_\Omega\left[\sum_{j,l=1}^{n} a_{jl}(x)(D_l u)(D_j v) + d(x)uv\right]dx,$$

where $a_{jl}, d \in L^\infty(\Omega)$, $a_{jl}=a_{lj}$, $\sum_{j,l=1}^{n} a_{jl}(x)\xi_j\xi_l \ge c_0|\xi|^2$, $d \ge c_0$ with some positive constant c_0. Then, clearly, assumption (A_1) is satisfied.

If h is a C^2 function such that $h(\eta)=|\eta|^{\lambda+1}$ if $|\eta|>1$ then $(A_3)$is satisfied.

The condition (A_4) is satisfied e.g. if

$$H(t,x;u,z)=\chi(t,x)g_1(L_1 z)g_2(L_2 u) \text{ where } \chi\in L^\infty(Q_T),$$

$$L_1 : L^p(0,T;V_2)\to L^2(Q_T),\quad L_2 : L^2(Q_T)\to L^2(Q_T)$$

are continuous linear operators (having the Volterra property); g_1 is a globally Lipschitz bounded function, g_2 is a globally Lipschitz function. The operator $F_1 : Q_T \times L^p(0,T;V_2)\to\mathbb{R}$ may have the form $F_1(t,x;z)=f_1(t,x,L_3 z)$, where $f_1(t,x,\mu)$ is measurable in (t,x), continuous in μ and

$$|f_1(t,x,\mu)| \le \text{const}|\mu|^{\beta_1/2} + \tilde{f}_1(t,x), \text{ where}$$

$$0\le\beta_1\le 2,\quad \tilde{f}_1\in L^2(Q_T),\quad L_3 : L^p(0,T;V_2)\to L^2(Q_T)$$

is a linear continuous operator. Then (A_5) is fulfilled.

Now we formulate examples for a_j satisfying (B_1) – (B_3):

$$a_j(t,x,\xi;u) = \alpha(t,x,L_4u)\xi_j|\zeta|^{p-2}, \quad j = 1,\dots,n \text{ where } \zeta = (\xi_1,\dots,\xi_n),$$

$\alpha(t,x,\nu)$ is measurable in (t,x), continuous in ν and satisfies

$$\frac{\text{const}}{1+|\nu|^\beta} \le \alpha(t,x,\nu) \le \text{const}(1+|\nu|^\gamma)$$

with some positive constants, $L_4, L_5 : L^2(Q_T) \to L^\infty(Q_T)$ are continuous linear operators;

$$a_0(t,x,\xi;u,z) = \alpha_0(t,x,L_5u)\xi_0|\xi_0|^{p-2} + cz + (\text{sg}c)\alpha_1(L_6z),$$

where $\alpha_0(t,x,\nu_1)$ is measurable in (t,x), continuous in ν_1, $c \ge 0$ is a constant and

$$\frac{\text{const}}{1+|\nu_1|^\beta} \le \alpha_0(t,x,\nu_1) \le \text{const}(1+|\nu_1|^\gamma)$$

with some positive constants, $L_6 : L^2(Q_T) \to L^2(Q_T)$ is a continuous linear operator and α_1 is a bounded globally Lipschitz function with sufficiently small Lipschitz constant. If the values of α, α_0 are between two positive constants then L_4, L_5 may be $L^2(Q_T) \to L^2(Q_T)$ continuous linear operators.

Finally, the function $F_2 : Q_T \times L^2(Q_T) \to \mathbb{R}$ may have the form $F_2(t,x;u) = f_2(t,x,L_7u)$ where $f_2(t,x,\mu)$ is measurable in (t,x), continuous in μ and

$$|f_2(t,x,\mu)| \le \text{const}|\mu|^\gamma + \tilde{f}_2(t,x),$$

$$0 \le \gamma \le 1, \quad \tilde{f}_2 \in L^2(Q_T) \text{ and } L_7 : L^2(Q_T) \to L^2(Q_T)$$

is a continuous linear operator. Then (B_4) is satisfied.

References

[1] Adams, R.A., Sobolev spaces. New York, San Francisco and London: Academic Press 1975.

[2] Besenyei, Á., On nonlinear systems containing nonlocal terms. PhD Thesis, Eötvös Loránd University, Budapest, 2008.

[3] Cinca, S., Diffusion und Transport in porösen Medien bei veränderlichen Porosität, Diplomawork, Univ. Heidelberg, 2000.

[4] Lions, J.L., Quelques méthodes de résolution des problemes aux limites non linéaires. Paris: Dunod Gauthier-Villars 1969.

[5] Logan, J.D., Petersen, M.R., Shores, T.S., Numerical study of reaction-mineralogy-porosity changes in porous media. Appl. Math. Comput. 127 (2002), 149–164.

[6] Merkin, J.H., Needham, D.J., Sleeman, B.D., A mathematical model for the spread of morphogens with density dependent chemosensitivity. Nonlinearity 18 (2005), 2745–2773.

[7] Simon, L., On some singular systems of parabolic functional equations. Math. Bohem. 135 (2010), 123–132.

[8] Simon, L., On singular systems of parabolic functional equations. Operator Theory: Advances and Applications 216 (2011), 317–330.

[9] Simon, L., Application of monotone type operators to parabolic and functional parabolic PDE's. Handbook of Differential Equations, Evolutionary Equations Vol. 4, Elsevier 2008, 267–321.

[10] Simon, L., Semilinear hyperbolic functinal equations. Banach Center Publications 101 (2014), 205–222.

[11] Wu, J., Theory and Applications of Partial Functional Differential Equations, New York, Berlin, Heidelberg: Springer 1996.

[12] E. Zeidler, Nonlinear functional analysis and its applications II A and II B, Springer, 1990.

Received by the editors December 17, 201?