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A MASSERA TYPE THEOREM IN HYPERFUNCTIONS IN THE REFLEXIVE LOCALLY CONVEX VALUED CASE

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Abstract. We continue our study on Massera type theorems in hyperfunctions from [11] and [12]. In the latter, we gave a result in hyperfunctions with values in a reflexive Banach space. In this article, we report its generalization to the case of hyperfunctions with values in a reflexive locally convex space.

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1. Introduction

In [9], Massera studied the existence of a periodic solution to a periodic ordinary differential equation, and gave the result that for a periodic linear ordinary differential equation of normal form, the existence of a bounded solution in the future implies the existence of a periodic solution.

Theorem 1.1 ([9, Theorem 4]). *Consider an equation*

$$\frac{dx}{dt} = A(t)x + f(t),$$

where $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ *and* $f : \mathbb{R} \to \mathbb{R}^m$ *are* 1*-periodic and continuous. Then, the existence of a bounded solution in the future (i.e., a solution defined and bounded on a set* $\{t > t_0\}$ *with some* t_0*) implies the existence of a* 1*-periodic solution.*

Note that the inverse implication follows from the boundedness of a periodic C^1-function and therefore we have the equivalence between the existence of a bounded solution in the future and that of a 1-periodic solution.

There appeared many generalizations of Theorem 1.1, and there arose a question whether such phenomena appear commonly in periodic linear equations. For example, we refer to Chow-Hale [1] and Hino-Murakami [3] for functional differential equations with finite or infinite delay, to Shin-Naito [16] and Naito-Nguyen-Miyazaki-Shin [10] for Banach valued cases, and to Zubelevich

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[17] for discrete dynamical systems in reflexive Banach spaces and those in sequentially complete locally convex spaces with the sequential Montel property. See also the references therein.

Being interested in these results, we have studied such phenomena in the framework of hyperfunctions, and gave a Massera type theorem for hyperfunctions in [11] and its reflexive Banach valued variant in [12]. In this article, we report a generalization of the latter result to the case of hyperfunctions with values in a reflexive locally convex space.

The plan of this paper is as follows. In the section 2, we prepare some notions and related terminologies on bounded hyperfunctions at infinity and operators of type K introduced in [11, 12], and give our main result Theorem 2.3. In the section 3, we recall and study duality and compactness results of the spaces of holomorphic functions taking values in a reflexive locally convex space. In the last section 4, we give the proof of our main result.

2. Main result

In this section, we recall some notions and terminologies briefly and state our main result, Theorem 2.3. As for the preparation part, we follow [12, §2] and refer to [11, §2 and §3] for details. See Sato [13, 14], Kawai [5], Sato-Kawai-Kashiwara [15], and Kaneko [4], for original hyperfunctions, Fourier hyperfunctions, and related topics.

2.1. Bounded hyperfunctions and classes of operators

Let us first recall the notion of bounded hyperfunctions at infinity.

We take a compactification $\mathbb{D}^1 := \mathbb{R} \sqcup \{\pm\infty\}$ of $\mathbb{R}$, and by considering the diagram:

$$\begin{array}{ccc} \mathbb{C} = \mathbb{R} + i\mathbb{R} & \subset & \mathbb{D}^1 + i\mathbb{R} \\ \cup & & \cup \\ \mathbb{R} = \,]-\infty, +\infty[& \subset & \mathbb{D}^1 = [-\infty, +\infty] \end{array}$$

we identify $\mathbb{C}$ with an open subset of $\mathbb{D}^1 + i\mathbb{R}$.

Let E be a sequentially complete Hausdorff locally convex space. We denote by $\mathcal{N}(E)$ the family of continuous semi-norms of E, and by ${}^E\mathscr{O}$ the sheaf of E-valued holomorphic functions on $\mathbb{C}$.

Definition 2.1. (1) The sheaf ${}^E\mathscr{O}_{L^\infty}$ of *E-valued bounded holomorphic functions at infinity* on $\mathbb{D}^1 + i\mathbb{R}$ is defined by

$$ {}^E\mathscr{O}_{L^\infty}(U) = \{f \in {}^E\mathscr{O}(U \cap \mathbb{C}) \mid \forall L \Subset U, f \text{ is bounded on } L \cap \mathbb{C}\}$$

for any open set $U \subset \mathbb{D}^1 + i\mathbb{R}$.

(2) The sheaf $\mathscr{B}_{L^\infty}$ of *E-valued bounded hyperfunctions at infinity* on $\mathbb{D}^1$ is defined as the sheaf associated with the presheaf

$$\mathbb{D}^1 \supset \Omega \mapsto \varinjlim_{U} \frac{{}^E\mathscr{O}_{L^\infty}(U \setminus \Omega)}{{}^E\mathscr{O}_{L^\infty}(U)},$$

for any open set $\Omega \subset \mathbb{D}^1$. Here U runs through complex neighborhoods of Ω, that is, open sets in $\mathbb{D}^1 + i\mathbb{R}$, including Ω as a closed subset.

The space ${}^E\mathscr{O}_{L^\infty}(U)$ is endowed with a natural locally convex topology by the family of semi-norms

$$f \mapsto \sup_{w \in L \cap \mathbb{C}} p(f(w)),$$

where L runs through compact subsets in U and p runs through continuous semi-norms of E. In the scalar case ($E = \mathbb{C}$), we use abbreviations $\mathscr{O}_{L^\infty}$ and $\mathscr{B}_{L^\infty}$ instead of ${}^{\mathbb{C}}\mathscr{O}_{L^\infty}$ and ${}^{\mathbb{C}}\mathscr{B}_{L^\infty}$, respectively. We also use the abbreviation ${}^E\mathscr{B}$ of ${}^E\mathscr{B}_{L^\infty}|_{\mathbb{R}}$.

We list up some properties of bounded hyperfunctions. Refer to [11, §2] for the precise statements and the proofs.

- $\mathscr{B}_{L^\infty}$ is an extension of $\mathscr{B}$ to $\mathbb{D}^1$. That is, $\mathscr{B}_{L^\infty}|_{\mathbb{R}} = \mathscr{B}$.
- $\mathscr{B}_{L^\infty}$ is flabby. (In general, vector valued variants are not.)
- A section in $\mathscr{B}_{L^\infty}(]a, +\infty])$ admits a boundary value representation.
- There exists a natural embedding $L^\infty(]a, +\infty[) \hookrightarrow \mathscr{B}_{L^\infty}(]a, +\infty])$.
- The space $\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ of the global sections of our sheaf $\mathscr{B}_{L^\infty}$ (in scalar valued case) can be identified with the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions (in 1-dimensional case) due to Chung-Kim-Lee [2].

Let us next recall a class of operators acting on bounded hyperfunctions at infinity.

Let $K = [a, b]$ be a closed interval in $\mathbb{R}$ (including the case $K = \{a\}$), and U an open set in $\mathbb{D}^1 + i\mathbb{R}$. Consider a family $P = \{P_V\}_{V \subset U}$ for open subsets $V \subset U$ of continuous linear maps

$$P_V : {}^E\mathscr{O}_{L^\infty}(V + K) \to {}^E\mathscr{O}_{L^\infty}(V).$$

Note that the vectorial sum $V + K := \{w + t \mid w \in V, t \in K\}$ is well-defined even in case $V \not\subset \mathbb{C}$ under the convention $w + t = w$ for $w = \pm\infty + is \in V \setminus \mathbb{C}$ and $t \in K$.

Definition 2.2 (Operators of type K)**.** P is said to be *an operator of type* K for ${}^E\mathscr{O}_{L^\infty}$ on U, if the diagram below commutes for any pair of open sets $V_1 \supset V_2$ in U.

$$\begin{tikzcd}
{}^E\mathscr{O}_{L^\infty}(V_1 + K) \arrow[r, "P_{V_1}"] \arrow[d, "\text{restriction}"'] & {}^E\mathscr{O}_{L^\infty}(V_1) \arrow[d, "\text{restriction}"] \\
{}^E\mathscr{O}_{L^\infty}(V_2 + K) \arrow[r, "P_{V_2}"] & {}^E\mathscr{O}_{L^\infty}(V_2)
\end{tikzcd}$$

An operator P of type K for ${}^E\mathscr{O}_{L^\infty}$ on U induces a family of linear maps

$$P_\Omega : {}^E\mathscr{B}_{L^\infty}(\Omega + K) \to {}^E\mathscr{B}_{L^\infty}(\Omega), \quad \text{for open sets } \Omega \subset \mathbb{D}^1 \cap U,$$

commuting with restrictions. An operator of type $K = \{0\}$ corresponds to a local operator, while an operator of type $K = [-r, 0]$ corresponds to an operator of finite delay r.

2.2. Massera type theorem in the reflexive valued case

Let us also recall the terminologies of ${}^E\mathscr{O}_{L^\infty}$-solutions and ${}^E\mathscr{B}_{L^\infty}$-solutions to equations, and the notion of ω-periodicity for bounded hyperfunctions and for operators of type K.

Let P be an operator of type $K = [a, b] \subset \mathbb{R}$ for ${}^E\mathscr{O}_{L^\infty}$ on U. For an open set $V \subset U$ and a section $f \in {}^E\mathscr{O}_{L^\infty}(V)$, we say that u is an ${}^E\mathscr{O}_{L^\infty}$-solution to the equation $Pu = f$ on V, or an ${}^E\mathscr{O}_{L^\infty}(V)$-solution to $Pu = f$, if u belongs to ${}^E\mathscr{O}_{L^\infty}(V + K)$ and satisfies $P_V u = f$. Similarly, for an open set $\Omega \subset \mathbb{D}^1 \cap U$ and $f \in {}^E\mathscr{B}_{L^\infty}(\Omega)$, an ${}^E\mathscr{B}_{L^\infty}$-solution to $Pu = f$ on Ω is a section $u \in {}^E\mathscr{B}_{L^\infty}(\Omega + K)$ satisfying $P_\Omega u = f$. Moreover, when f is a germ of ${}^E\mathscr{B}_{L^\infty}$ at $+\infty$ (that is, $f \in ({}^E\mathscr{B}_{L^\infty})_{+\infty}$), it makes sense to consider an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution to the equation $Pu = f$.

The ω-translation operator $T_\omega : u \mapsto u(\cdot + \omega)$ for a positive constant ω, is an operator of type $\{\omega\}$, and the ω-difference operator $T_\omega - 1$ is an operator of type $[0, \omega]$. A section $u \in {}^E\mathscr{B}_{L^\infty}(\Omega + [0, \omega])$ is called ω-periodic if it is an ${}^E\mathscr{B}_{L^\infty}(\Omega)$-solution to the equation $(T_\omega - 1)u = 0$, and an operator P of type K is called ω-periodic if $P_V \circ T_\omega = T_\omega \circ P_{V+\omega}$ holds as maps ${}^E\mathscr{O}_{L^\infty}(V + K + \omega) \to {}^E\mathscr{O}_{L^\infty}(V)$ for any V.

Then, we have,

- Every ω-periodic hyperfunction $f \in {}^E\mathscr{B}(\mathbb{R})$ has the unique ω-periodic extension $\hat{f} \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$.
- Every ω-periodic bounded hyperfunction $f \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ admits an ω-periodic boundary value representation.
- An ω-periodic operator of type K preserves the ω-periodicity of its operands.

Now we can state our main result. Let P be an ω-periodic operator of type $K = [a, b]$ for ${}^E\mathscr{O}_{L^\infty}$ on a strip neighborhood $\mathbb{D}^1 + i]{-d}, d[$ of $\mathbb{D}^1$ with some $d > 0$, and $f \in {}^E\mathscr{B}(\mathbb{R})$ an ω-periodic E-valued hyperfunction. The unique ω-periodic extension of f in ${}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ is denoted also by the same symbol f by the abuse of the notation.

Theorem 2.3. *Assume that E is a reflexive locally convex space. Then, $Pu = f$ has an ω-periodic ${}^E\mathscr{B}(\mathbb{R})$-solution if and only if it has an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution.*

3. Duality and compactness for ${}^E\mathscr{O}(L)$

Throughout this section, E denotes a reflexive locally convex space over $\mathbb{C}$ and E' denotes its strong dual space. We recall the weak form of Köthe duality for ${}^E\mathscr{O}$ from [12, §3], and study a compactness result of ${}^E\mathscr{O}$.

3.1. A weak form of the Köthe duality

Consider the space ${}^{E}\mathscr{O}(L) := \varinjlim_{V \ni L} {}^{E}\mathscr{O}(V)$ endowed with the locally convex inductive limit topology for a compact set $L \subset \mathbb{C}$, where V runs through the open neighborhoods of L in $\mathbb{C}$. We give a weak form of the Köthe duality.

Let us cite two definitions ([12, Definition 3.1 and 3.2]).

Definition 3.1. For open neighborhoods V, $W \subset \mathbb{C}$ of L, we take a compact neighborhood M of L in $W \cap V$ whose boundary $\gamma := \partial M$ consists of finite piecewise smooth simple closed curves, and define a bilinear form

$$\langle \cdot, \cdot \rangle_L : {}^{E'}\mathscr{O}(W \setminus L) \times {}^{E}\mathscr{O}(V) \to \mathbb{C}$$

by

$$\langle F, f \rangle_L := \int_\gamma F(w)(f(w))dw \tag{3.1}$$

for $F \in {}^{E'}\mathscr{O}(W \setminus L)$ and $f \in {}^{E}\mathscr{O}(V)$. Here $F(w)(f(w))$ is a value of the continuous linear functional $F(w) \in E'$ evaluated at $f(w) \in E$.

Definition 3.2. Let L be a compact set in $\mathbb{C}$ and W an open neighborhood. We define linear maps $\alpha : ({}^{E}\mathscr{O}(L))' \to {}^{E'}\mathscr{O}(W \setminus L)$ and $\beta : {}^{E'}\mathscr{O}(W \setminus L) \to ({}^{E}\mathscr{O}(L))'$ by

$$\alpha(\varphi)(w)(x) := \varphi\Big(\frac{1}{2\pi i}\frac{1}{w - \cdot}x\Big) \in \mathbb{C}, \tag{3.2}$$

for $\varphi \in ({}^{E}\mathscr{O}(L))'$, $x \in E$ and $w \in W \setminus L$, and by

$$\beta(F)(f) := \langle F, f \rangle_L,$$

for $F \in {}^{E'}\mathscr{O}(W \setminus L)$ and $f \in {}^{E}\mathscr{O}(L)$. Here we regard $\frac{1}{2\pi i}\frac{1}{w-\cdot}x$ as an element of ${}^{E}\mathscr{O}(L)$ in the right hand side of (3.2).

Then, we can show the well-definedness of $\langle \cdot, \cdot \rangle_L$, α, β, and the continuity of α and β. Moreover, we can also show that $\beta \circ \alpha = \mathrm{id}_{({}^{E}\mathscr{O}(L))'}$ and that the range of $\alpha \circ \beta - \mathrm{id}_{({}^{E'}\mathscr{O}(W \setminus L))}$ is included in ${}^{E'}\mathscr{O}(W)$. These facts give the following results. (See Theorem 3.10 and Corollary 3.11 of [12].)

Theorem 3.3. *Let E be a reflexive locally convex space. The maps α and β induce the isomorphism between vector spaces*

$$({}^{E}\mathscr{O}(L))' \xrightarrow{\sim} {}^{E'}\mathscr{O}(W \setminus L)/{}^{E'}\mathscr{O}(W).$$

Corollary 3.4 (Köthe duality). *Let E be a reflexive locally convex space. The maps α and β also induce the isomorphism between vector spaces*

$$({}^{E}\mathscr{O}(L))' \xrightarrow{\sim} {}^{E'}\mathscr{O}^\circ(\mathbb{P}^1 \setminus L).$$

Here ${}^{E'}\mathscr{O}^\circ(\mathbb{P}^1 \setminus L)$ denotes the subspace $\{F \in {}^{E'}\mathscr{O}(\mathbb{C} \setminus L) \mid \lim_{|w| \to \infty} F(w) = 0\}$ of ${}^{E'}\mathscr{O}(\mathbb{C} \setminus L)$.

See Köthe [8, §27.3] for the classical Köthe duality.

3.2. Montel type lemma

Consider a compact set $L \subset \mathbb{C}$ and an open neighborhood V of L in $\mathbb{C}$. As before, $\langle \cdot, \cdot \rangle_L : {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L) \times {}^{E}\mathscr{O}(V) \to \mathbb{C}$ denotes the bilinear form given by (3.1), used in the Köthe duality.

Lemma 3.5 (Montel type lemma)**.** *Let E be a reflexive locally convex space, and $\{f_n\}_{n\in\mathbb{N}}$ a bounded sequence in ${}^{E}\mathscr{O}(V)$. Then, there exists $f \in {}^{E}\mathscr{O}(V)$ satisfying the following property: For any $F \in {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L)$, we can take a subsequence $\{n(k)\}_k$ such that*

$$\lim_{k\to\infty} \langle F, f_{n(k)} \rangle_L = \langle F, f \rangle_L.$$

This lemma reflects the fact that bounded sets in ${}^{E}\mathscr{O}(L)$ are precompact with respect to the weak topology.

Note that when E is a reflexive Banach space, the subsequence $\{n(k)\}_k$ can be taken independently of $\langle F, \cdot \rangle_L$. But we can not expect sequential precompactness in the general case. Since we want to use sequential convergence in the proof of Theorem 2.3, we gave a statement in terms of sequences.

Proof of Lemma 3.5. Note that since E is reflexive, any bounded set in E is weakly relatively compact. We denote by E_{w} the space E endowed with the weak topology. Also note that bounded sets in ${}^{E}\mathscr{O}(V)$ are equicontinuous as a family of maps from V to E, which follows from the Cauchy estimate. The proof consists of three steps (I), (II) and (III).

(I) the choice of f and its holomorphy.

For any $l \in \mathbb{N}$ and any compact $L \subset V$, the set $\{f_n(w) \mid n \geq l,\, w \in L\}$ in E is bounded, and therefore

$$A_{l,L} := \text{the weak closure of } \{f_n(w) \mid n \geq l,\, w \in L\} \tag{3.3}$$

is weakly compact (i.e., compact in the topology induced from E_{w}). We consider $B := \prod_{w\in V} A_{0,\{w\}} \subset (E_{\mathrm{w}})^V$ endowed with the direct product topology, that is, the topology of pointwise convergence. Then, B is compact (from Tychonoff's theorem). We also consider

$$B_l := \text{the closure of } \{f_n \mid n \geq l\} \text{ in } (E_{\mathrm{w}})^V \tag{3.4}$$

for $l \in \mathbb{N}$. Then they are non-empty compact subsets in B and decreasing in l, and they share a common element $f \in \bigcap_l B_l$.

Each B_l is equicontinuous as a family of maps from V to E_{w}, since the closure of an equicontinuous family with respect to the topology of pointwise convergence is also equicontinuous. (See, for example, Kelley-Namioka [7, Chap.2, 8.12].) Moreover, it follows from Kelley [6, Chap.7, Theorem 15] that the topology of pointwise convergence of the equicontinuous family B_l coincides with its topology of convergence on compact sets. Therefore, f is a uniform limit of some subnet of the sequence $\{f_n\}_n$ consisting of E_{w}-valued holomorphic functions on V, which implies that f itself is holomorphic as an E_{w}-valued map on V. Since the E-valued holomorphy and the E_{w}-valued holomorphy are equivalent for E-valued maps, f is holomorphic as a map $V \to E$, that is, $f \in {}^{E}\mathscr{O}(V)$.

(II) the choice of $\{n(k)\}_k$ according to L and F.

For given L and F, we take a contour γ and its compact neighborhood $\Gamma \subset V \setminus L$. The set $A_{0,\Gamma}$ (see (3.3) with $L = \Gamma$) is bounded in E as we have seen in the part (I), and the correspondence

$$E' \ni y \mapsto q(y) := \sup_{x \in A_{0,\Gamma}} |y(x)|$$

defines a continuous semi-norm q on E'. Since $\{F(w)\}_{w\in\Gamma}$ is compact in E', we have $M := \sup_{w\in\Gamma} q(F(w)) < +\infty$, which in particular implies

$$|F(w)(g(w))| \leq M, \text{ for any } g \in B_0 \text{ and } w \in \Gamma \tag{3.5}$$

with B_0 defined in (3.4). Moreover, since $w \mapsto F(w)(g(w))$ is holomorphic as was shown in [12, Lemma 3.2], $\{w \mapsto F(w)(g(w)) \mid g \in B_0\}$ becomes an equicontinuous family of functions on $\operatorname{Int}\Gamma$.

Take a dense and countable subset $C = \{w_1, w_2, \dots\}$ of γ, and define a neighborhood W_k of the origin in E_{w} by

$$W_k := \{x \in E \mid \sup_{1\leq j\leq k} |F(w_j)(x)| < 1/k\}$$

for any $k \geq 1$. Recall that f belongs to the closure of $\{f_n\}_{n\geq l}$ in $(E_{\mathrm{w}})^V$ with respect to the topology of pointwise convergence, for any $l \in \mathbb{N}$. Using again the coincidence of the topology of pointwise convergence and the topology of convergence on compact sets for an equicontinuous family, we can easily see that the set

$$\{f_n\}_{n\geq l} \cap \{g \in (E_{\mathrm{w}})^V \mid \forall w \in \gamma, g(w) - f(w) \in W_k\}$$

is non-empty. Therefore we can take $n(k)$ for each $k \geq 1$ satisfying $n(1) < n(2) < \cdots$ and

$$\forall w \in \gamma,\ f_{n(k)}(w) - f(w) \in W_k.$$

(III) the convergence of $\langle F, f_{n(k)}\rangle_L$ to $\langle F, f\rangle_L$ as $k \to \infty$.

We define $\mathbb{C}$-valued holomorphic functions h_k and h on $V \setminus L$ by

$$h_k(w) := F(w)(f_{n(k)}(w)), \quad h(w) := F(w)(f(w)),$$

and consider the sequence $\{h_k\}_{k\geq 1}$. Since each f_n belongs to B, it follows from (3.5) that

$$|h_k(w)| \leq |F(w)(f_{n(k)}(w))| \leq M$$

for $w \in \Gamma$ and $k \geq 1$. Therefore, as we have already seen, $\{h_k\}_k$ is an equicontinuous family on holomorphic functions on $\operatorname{Int}\Gamma$. Moreover, for $w_j \in C$, it follows from $f_{n(k)}(w_j) - f(w_j) \in W_k$ for $k \geq j$ that

$$|h_k(w_j) - h(w_j)| = |F(w_j)(f_{n(k)}(w_j) - f(w_j))| < 1/k.$$

This estimate shows that $h_k(w_j) \to h(w_j)$ as $k \to \infty$ for each $w_j \in C$. In other words, the sequence $\{h_k\}_k$ converges to h with respect to the topology of convergence on each point in C.

Note that on an equicontinuous family, the topology of convergence on each point in a given subset coincides with the topology of convergence on each point in the closure of the subset. (See Kelley-Namioka [7, Chap.2, 8.13].) Since C is dense in γ, we have that $h_k(w) \to h(w)$ as $k \to \infty$ on each point in γ.

Now we can use Lebesgue's bounded convergence theorem to show

$$\lim_{k\to\infty} \int_\gamma h_k(w)dw = \int_\gamma h(w)dw,$$

that is,

$$\lim_{k\to\infty} \langle F, f_{n(k)}\rangle_L = \langle F, f\rangle_L,$$

which concludes the proof. □

4. Proof of the main theorem

Using the preparation in the section 3, we can prove our main theorem.

Proof of Theorem 2.3. The necessity follows from [12, Corollary 2.6], and we shall prove the sufficiency.

Assume that $Pu = f$ has an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution u. In a parallel manner as in the proof of Theorem 4.4 of [12], we can take $\tilde{u} \in {}^E\mathscr{O}_{L^\infty}(\dot{U} + K)$, $\tilde{f} \in {}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)$ satisfying $(T_\omega - 1)\tilde{f} = 0$ and $g \in {}^E\mathscr{O}_{L^\infty}(U)$ for some $a \in \mathbb{R}$ and $d > 0$, such that

$$[\tilde{u}] = u \text{ on } \Omega, \quad [\tilde{f}] = f \text{ on } \mathbb{D}^1, \quad P_{\dot{U}}\tilde{u} - g = \tilde{f} \text{ on } \dot{U},$$

under the notations

$$\Omega :=]a, +\infty], \quad U :=]a, +\infty] + iB_d, \quad \dot{U} :=]a, +\infty] + i\dot{B}_d = U \setminus \mathbb{D}^1.$$

Also in the same way, we define

$$S_k\tilde{u} := \frac{1}{k}\sum_{j=0}^{k-1} T_{j\omega}\tilde{u}|_{\dot{U}+K} \in {}^E\mathscr{O}_{L^\infty}(\dot{U} + K), \quad S_k g := \frac{1}{k}\sum_{j=0}^{k-1} T_{j\omega} g|_U \in {}^E\mathscr{O}_{L^\infty}(U),$$

for $k \geq 1$. Then, we have that

$$P_{\dot{U}} S_k\tilde{u} - S_k g = \tilde{f} \text{ on } \dot{U} \text{ for any } k \geq 1, \tag{4.1}$$

and that $\{S_k \tilde{u}\}_{k\in\mathbb{N}} \subset {}^E\mathscr{O}_{L^\infty}((\dot{U}+K)\cap\mathbb{C})$ and $\{S_k g\}_{k\in\mathbb{N}} \subset {}^E\mathscr{O}_{L^\infty}(U\cap\mathbb{C})$ are bounded.

We consider each pair $(S_k\tilde{u}, S_k g)$ as an E-valued holomorphic function on the disjoint union $V := ((\dot{U}+K)\cap\mathbb{C}) \sqcup (U\cap\mathbb{C})$. (Although two open sets $(\dot{U}+K)\cap\mathbb{C}$ and $U\cap\mathbb{C}$ are not disjoint, we can reduce the problem to the case of two disjoint open sets by translation.) Applying Lemma 3.5 to the sequence $\{(S_k\tilde{u}, S_k g)\}_{k\in\mathbb{N}}$, we can get a pair $(v,h) \in {}^E\mathscr{O}((\dot{U}+K)\cap\mathbb{C}) \times {}^E\mathscr{O}(U\cap\mathbb{C})$ satisfying the following property:

(C) for any $L_1 \Subset (\dot{U}+K)\cap\mathbb{C}$, $L_2 \Subset U\cap\mathbb{C}$, $F_1 \in {}^{E'}\mathscr{O}^\circ(\mathbb{P}^1\setminus L_1)$, and $F_2 \in {}^{E'}\mathscr{O}^\circ(\mathbb{P}^1\setminus L_2)$, there exists a subsequence $\{k(l)\}_l$ such that

$$\lim_{l\to\infty}\langle F_1, S_{k(l)}\tilde{u}\rangle_{L_1} = \langle F_1, v\rangle_{L_1}, \quad \lim_{l\to\infty}\langle F_2, S_{k(l)}g\rangle_{L_2} = \langle F_2, h\rangle_{L_1}. \tag{4.2}$$

A priori v belongs to ${}^E\mathscr{O}((\dot{U}+K)\cap\mathbb{C})$ and h belongs to ${}^E\mathscr{O}(U\cap\mathbb{C})$. Now, we want to show

(1) $v \in {}^E\mathscr{O}_{L^\infty}(\dot{U}+K)$,

(2) $h \in {}^E\mathscr{O}_{L^\infty}(U)$,

(3) the equality $P_{\dot{U}}v - h = \tilde{f}$ in ${}^E\mathscr{O}_{L^\infty}(\dot{U})$, and

(4) the ω-periodicity of v.

For this purpose, it suffices to show (4) and

(5) the equality $P_{\dot{U}\cap\mathbb{C}}v - h = \tilde{f}$ in ${}^E\mathscr{O}(\dot{U}\cap\mathbb{C})$.

In fact, we can easily prove the implications (4) $\Rightarrow$ (1); (1), (5) $\Rightarrow$ (2); and (1), (2), (5) $\Rightarrow$ (3).

In order to show (5), we take an arbitrary compact set $L \Subset \dot{U}$ and an arbitrary $F \in {}^{E'}\mathscr{O}^\circ(\mathbb{P}^1\setminus L)$. Then, we have, from (4.1), that

$$\langle F, P_{\dot{U}}S_k\tilde{u} - S_k g\rangle_L = \langle F, \tilde{f}\rangle_L,$$

which implies

$$\langle P_L^*(F), S_k\tilde{u}\rangle_{L+K} - \langle F, S_k g\rangle_L = \langle F, \tilde{f}\rangle_L.$$

Here $P_L^* : {}^{E'}\mathscr{O}^\circ(\mathbb{P}^1\setminus L) \to {}^{E'}\mathscr{O}^\circ(\mathbb{P}^1\setminus(L+K))$ is the abstract adjoint operator of P_L whose existence is guaranteed by Corollary 3.4. We can apply the property (C) to the two terms on the left hand side in the case $L_1 := L+K$, $L_2 = L$, $F_1 = P_L^*(F)$, $F_2 = F$. Then by taking the limit in (4.2), we have

$$\langle P_L^*(F), v\rangle_{L+K} - \langle F, h\rangle_L = \langle F, \tilde{f}\rangle_L,$$

or

$$\langle F, P_{\dot{U}\cap\mathbb{C}}v - h\rangle_L = \langle F, \tilde{f}\rangle_L. \tag{4.3}$$

Since L and F are arbitrary, the equality (5) follows from (4.3). Here we used Corollary 3.4 again.

In order to show (4), we also take an arbitrary compact set $L \Subset (\dot{U}+K)\cap\mathbb{C}$ and an arbitrary $F \in {}^{E'}\mathscr{O}^{\circ}(\mathbb{P}^1 \setminus L)$. Using the equality

$$(T_\omega - 1)S_k\tilde{u} = \frac{1}{k}(T_{k\omega} - 1)\tilde{u}$$

and the boundedness of $\{(T_{k\omega} - 1)\tilde{u}\}_k$, we have

$$\lim_{k\to\infty} \langle F, (T_\omega - 1)S_k\tilde{u}\rangle_L \to 0.$$

Then, by taking the adjoint, by applying the property (C), and by taking the adjoint again, we can successively show the following

$$\begin{aligned} \lim_{k\to\infty} \langle (T_\omega - 1)^*(F), S_k\tilde{u}\rangle_{L+[0,\omega]} &\to 0,\\ \langle (T_\omega - 1)^*(F), v\rangle_{L+[0,\omega]} &= 0,\\ \langle F, (T_\omega - 1)v\rangle_L &= 0. \end{aligned} \tag{4.4}$$

Since L and F are arbitrary, (4.4) implies (4).

By virtue of the ω-periodicity, v has a unique ω-periodic extension in ${}^{E}\mathscr{O}_{L^\infty}(\mathbb{D}^1+i\dot{B}_d)$. Moreover, h has a unique ω-periodic extension in ${}^{E}\mathscr{O}_{L^\infty}(\mathbb{D}^1+iB_d)$. In fact, since $h = P_{\dot{U}}v - \tilde{f}$ is ω-periodic on $\dot{U}$, it is ω-periodic also in U, and can be extended.

Finally note that $[v] \in {}^{E}\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ gives an ω-periodic solution. □

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