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ALMOST AUTOMORPHIC GENERALIZED FUNCTIONS

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Abstract. The paper deals with a new algebra of generalized functions. This algebra contains Bochner almost automorphic functions and almost automorphic distributions. Properties of this algebra are studied.

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1. Introduction

The concept of almost automorphy is a generalization of Bohr almost periodicity, it has been introduced by S. Bochner, see [1] and [2]. For a general study of almost automorphic functions see Veech's paper [7]. There is a considerable amount of papers and books on almost periodic functions and also almost automrphic functions.

L. Schwartz introduced and studied in [6] almost periodic distributions. The study of almost automorphic Schwartz distributions is done in the work [4].

An algebra of generalized functions containing Bohr almost periodic functions as well as Schwartz almost periodic distributions has been introduced and studied in [3]. In this work, we introduce and study a new algebra of generalized functions containing not only Bochner almost automorphic functions and almost automorphic distributions, but also the algebra of almost periodic generalized functions of [3]. So, naturally this paper can be seen as a continuation of our works on almost periodic generalized functions and almost automorphic distributions.

2. Regular almost automorphic functions

We consider functions and distributions defined on the whole space $\mathbb{R}$. Denote by $\mathcal{C}_b$ the space of bounded and continuous complex valued functions on $\mathbb{R}$ endowed with the norm $\|.\|_{L^\infty}$ of uniform convergence on $\mathbb{R}$, the space $(\mathcal{C}_b, \|.\|_{L^\infty})$ is a Banach algebra.

For the definition and properties of almost automorphic functions see [1], [2] and [7].

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Definition 1. *A complex-valued function f, defined and continuous on $\mathbb{R}$, is called almost automorphic if for any sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ one can extract a subsequence $(s_{n_k})_k$ such that*

$$g(x) := \lim_{k\to+\infty} f(x+s_{n_k}) \text{ exists for every } x\in\mathbb{R}$$

and

$$\lim_{k\to+\infty} g(x-s_{n_k}) = f(x) \text{ for every } x\in\mathbb{R}.$$

Denote by $\mathcal{C}_{aa}$ the space of almost automorphic functions on $\mathbb{R}$.

Remark 1. *The space $\mathcal{C}_{aa}$ is a Banach subalgebra of $\mathcal{C}_b$.*

Remark 2. *The space of Bohr almost periodic functions is denoted by $\mathcal{C}_{ap}$. Every Bohr almost periodic function is an almost automorphic function, and we have $\mathcal{C}_{ap} \subsetneqq \mathcal{C}_{aa} \subsetneqq \mathcal{C}_b$.*

Let $p\in[1,+\infty]$ and recall the Fréchet space

$$\mathcal{D}_{L^p} = \left\{\varphi\in\mathcal{C}^\infty : \forall j\in\mathbb{Z}_+,\ \varphi^{(j)}\in L^p\right\}$$

endowed with the countable family of norms

$$|\varphi|_{k,p} = \sum_{j\le k} \left\|\varphi^{(j)}\right\|_{L^p},\ k\in\mathbb{Z}_+.$$

Definition 2. *The space of almost automorphic infinitely differentiable functions on $\mathbb{R}$, denoted by $\mathcal{B}_{aa}$, is*

$$\mathcal{B}_{aa} := \left\{\varphi\in\mathcal{C}^\infty : \forall j\in\mathbb{Z}_+,\ \varphi^{(j)}\in\mathcal{C}_{aa}\right\}.$$

Example 1. *The space $\mathcal{B}_{ap}$ of regular almost periodic functions, see [3], is a Frechet subalgebra of $\mathcal{B}_{aa}$.*

Some properties of $\mathcal{B}_{aa}$ are summarized in the following proposition.

Proposition 1. *1. $\mathcal{B}_{aa}$ is a subalgebra of $\mathcal{C}_{aa}$.*

2. $\mathcal{B}_{aa}$ is a Frechet subalgebra of $\mathcal{D}_{L^\infty}$.

3. $\mathcal{B}_{aa} = \mathcal{C}_{aa}\cap\mathcal{D}_{L^\infty}$.

4. $\mathcal{B}_{aa} * L^1 \subset \mathcal{B}_{aa}$.

Proof. See [4]. □

A consequence of Proposition 1 is the following result.

Corollary 1. *Let $u\in\mathcal{D}_{L^\infty}$, then the following statements are equivalent :*
(i) $u\in\mathcal{B}_{aa}$.
(ii) $u*\varphi\in\mathcal{C}_{aa}, \forall\varphi\in\mathcal{D}$.

3. Almost automorphic distributions

The spaces of L^p-distributions, introduced in [6] and denoted by $\mathcal{D}'_{L^p}$, are the topological dual spaces of $\mathcal{D}_{L^q}$, with $\frac{1}{p}+\frac{1}{q}=1$ and $1 \leq q < +\infty$. In particular, $\mathcal{D}'_{L^1}$ is the topological dual of the space $\dot{\mathcal{B}}$ defined as the closure in $\mathcal{D}_{L^\infty}$ of the space of smooth functions with compact support. A distribution in $\mathcal{D}'_{L^1}$ is called an integrable distribution and a distribution in $\mathcal{D}'_{L^\infty}$ is called a bounded distribution. L. Schwartz provided the following characterization of L^p-distributions.

Proposition 2. *Let $p \in [1,+\infty]$. A tempered distribution T belongs to $\mathcal{D}'_{L^p}$ if and only if there exists $(f_j)_{j\leq k} \subset L^p$ such that*

$$T = \sum_{j=0}^{k} f_j^{(j)}. \tag{3.1}$$

A study of almost automorphic Schwartz distributions is done in the work [4]. The following result gives characterizations of almost automorphic distributions.

Theorem 1. *Let $T \in \mathcal{D}'_{L^\infty}, T$ is said to be an almost automorphic distribution if it satisfies one of the following equivalent statements :*

1. $T * \varphi \in \mathcal{C}_{aa}, \forall\ \varphi \in \mathcal{D}$.
2. $\exists\ (f_j)_{j\leq k} \subset \mathcal{C}_{aa},\ T = \sum\limits_{j\leq k} f_j^{(j)}$.

Definition 3. *Denote by $\mathcal{B}'_{aa}$ the space of almost automorphic distributions.*

Example 2. *The space $\mathcal{B}'_{ap}$ of almost periodic distributions of Schwartz is a proper subspace of $\mathcal{B}'_{aa}$.*

Some properties of $\mathcal{B}'_{aa}$ are summarized in the following proposition.

Proposition 3.
1. *If $T \in \mathcal{B}'_{aa}$, then $\forall i \in \mathbb{Z}_+, T^{(i)} \in \mathcal{B}'_{aa}$.*
2. $\mathcal{B}_{aa} \times \mathcal{B}'_{aa} \subset \mathcal{B}'_{aa}$.
3. $\mathcal{B}'_{aa} * \mathcal{D}'_{L^1} \subset \mathcal{B}'_{aa}$.

Proof. See [4]. □

4. Almost automorphic generalized functions

Let $I =]0,1]$, and recall the algebra of bounded generalized functions, denoted by $\mathcal{G}_{L^\infty}$,

$$\mathcal{G}_{L^\infty} := \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}},$$

where

$$\mathcal{M}_{L^\infty} := \left\{ (u_\epsilon)_\epsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O\left(\epsilon^{-m}\right), \epsilon \to 0 \right\}$$

and

$$\mathcal{N}_{L^\infty} := \left\{ (u_\epsilon)_\epsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O\left(\epsilon^{m}\right), \epsilon \to 0 \right\}$$

Remark 3. *See [5] for the references on the introduction and the study of the algebras $\mathcal{G}_{L^p}$ constructed on the Banach spaces L^p.*

Definition 4. *The space of almost automorphic moderate elements is defined as*

$$\mathcal{M}_{aa} := \left\{ (u_\epsilon)_\epsilon \in (\mathcal{B}_{aa})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O\left(\epsilon^{-m}\right), \epsilon \to 0 \right\}$$

and the space of almost automorphic negligible elements by

$$\mathcal{N}_{aa} := \left\{ (u_\epsilon)_\epsilon \in (\mathcal{B}_{aa})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O\left(\epsilon^{m}\right), \epsilon \to 0 \right\}.$$

The main properties of $\mathcal{M}_{aa}$ and $\mathcal{N}_{aa}$ are given in the following proposition.

Proposition 4. *1. The space $\mathcal{M}_{aa}$ is a subalgebra of $(\mathcal{B}_{aa})^I$.*

2. The space $\mathcal{N}_{aa}$ is an ideal in $\mathcal{M}_{aa}$.

Proof. 1. Easy by the results on the algebra $\mathcal{B}_{aa}$, see Propostion 1.

2. Let $(w_\epsilon)_\epsilon \in \mathcal{M}_{aa}$, i.e.

$$\forall k \in \mathbb{Z}_+, \exists m_0 \in \mathbb{Z}_+, \exists c_0 > 0, \exists \epsilon_0 \in I, \forall \epsilon < \epsilon_0, |w_\epsilon|_{k,\infty} < c_0 \epsilon^{-m_0},$$

and $(v_\epsilon)_\epsilon \in \mathcal{N}_{aa}$, i.e.

$$\forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, \exists c_1 > 0, \exists \epsilon_1 \in I, \forall \epsilon < \epsilon_1, |v_\epsilon|_{k,\infty} < c_1 \epsilon^{m}.$$

By using the Leibniz formula, we find $c_k > 0$ such that

$$\begin{aligned} |w_\epsilon v_\epsilon|_{k,\infty} &\leq c_k |w_\epsilon|_{k,\infty} |v_\epsilon|_{k,\infty}, \\ &\leq c_k c_0 c_1 \epsilon^{-m_0+m}. \end{aligned}$$

Take $m \in \mathbb{Z}_+$ such that $-m_0 + m = m_1 \in \mathbb{Z}_+$, so we obtain $\forall k \in \mathbb{Z}_+, \forall m_1 \in \mathbb{Z}_+, \exists C = c_0 c_1 c_k > 0, \exists \epsilon_2 = \inf(\epsilon_0, \epsilon_1) \in I, , \forall \epsilon < \epsilon_2,$

$$|w_\epsilon v_\epsilon|_{k,\infty} < C \epsilon^{m_1},$$

which gives $(w_\epsilon v_\epsilon)_\epsilon \in \mathcal{N}_{aa}$. □

Following the well-known classical construction of algebras of generalized functions of Colombeau type, see [5], we introduce the algebra of almost automorphic generalized functions.

Definition 5. *The algebra of almost automorphic generalized functions is defined as the quotient*

$$\mathcal{G}_{aa} := \frac{\mathcal{M}_{aa}}{\mathcal{N}_{aa}}.$$

Notation 1. *If* $u \in \mathcal{G}_{aa}$, *then* $u = [(u_\epsilon)_\epsilon] = (u_\epsilon)_\epsilon + \mathcal{N}_{aa}$, *where* $(u_\epsilon)_\epsilon$ *is a representative of* u.

Remark 4. *The algebra of almost automorphic generalized functions* $\mathcal{G}_{aa}$ *is embedded into* $\mathcal{G}_{L^\infty}$ *canonically.*

The following characterization of elements of $\mathcal{G}_{aa}$ is similar to the result of Theorem 1-(1).

Proposition 5. *Let* $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{L^\infty}$. *Then the following statements are equivalent*

1. $u \in \mathcal{G}_{aa}$.
2. $u_\varepsilon * \varphi \in \mathcal{B}_{aa}, \forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$.

Proof. If $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{aa}$, then $u_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I$, and due to (4) of Proposition 1, $u_\epsilon * \varphi \in \mathcal{B}_{aa}, \forall \epsilon \in I, \forall \varphi \in \mathcal{D}$. Conversely, let $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{L^\infty}$ and $u_\epsilon * \varphi \in \mathcal{B}_{aa}, \forall \epsilon \in I, \forall \varphi \in \mathcal{D}$, so $u_\epsilon \in \mathcal{D}_{L^\infty}, \forall \epsilon \in I$, and $u_\epsilon * \varphi \in \mathcal{B}_{aa}, \forall \epsilon \in I, \forall \varphi \in \mathcal{D}$, Corollary 1 gives that $u_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I$. Since $u \in \mathcal{G}_{L^\infty}$, we have

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\epsilon|_{k,\infty} = O\left(\epsilon^{-m}\right),\ \epsilon \to 0,$$

consequentely $(u_\epsilon)_\epsilon \in \mathcal{M}_{aa}$ and thus $u \in \mathcal{G}_{aa}$. □

Remark 5. *The characterization 2. from the previous Proposition does not depend on representatives.*

The following result is easy to prove.

Proposition 6. *The algebra of almost periodic generalized functions* $\mathcal{G}_{ap}$ *of [3] is embedded canonically into* $\mathcal{G}_{aa}$.

The following result is well-known.

Lemma 1. *There exists* $\rho \in \mathcal{S}$ *satisfying*

$$\int_{\mathbb{R}} \rho(x)\,dx = 1 \text{ and } \int_{\mathbb{R}} x^k \rho(x)\,dx = 0,\ \forall k \geq 1. \tag{4.1}$$

Denote by Σ the set of of functions $\rho \in \mathcal{S}$ satisfying (4.1), and define $\rho_\epsilon(.) := \frac{1}{\epsilon}\rho\left(\frac{.}{\epsilon}\right),\ \epsilon > 0$.

By means of convolution with a mollifier from Σ, we embed the space of almost automorphic distributions $\mathcal{B}'_{aa}$ into the algebra $\mathcal{G}_{aa}$.

Proposition 7. *Let $\rho \in \Sigma$, the map*

$$\begin{array}{cccc} i_{aa}: & \mathcal{B}'_{aa} & \longrightarrow & \mathcal{G}_{aa} \\ & u & \longmapsto & (u * \rho_\epsilon)_\epsilon + \mathcal{N}_{aa} \end{array}$$

is a linear embedding which commutes with derivatives.

Proof. Let $u \in \mathcal{B}'_{aa}$, then $\exists\ (f_j)_{j \le p} \subset \mathcal{C}_{aa}$ and $u = \sum\limits_{j \le p} f_j^{(j)}$. Let us show that $(u * \rho_\epsilon)_\epsilon \in \mathcal{M}_{aa}$. By (4) of Proposition 1, $u * \rho_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I$. Moreover, we have

$$\left| \left(u^{(i)} * \rho_\epsilon \right)(x) \right| \le \sum_{j \le p} \frac{1}{\epsilon^{i+j}} \int_{\mathbb{R}} \left| f_j (x - \epsilon y)\, \rho^{(i+j)} (y) \right| dy,$$

then

$$\sup_{x \in \mathbb{R}} \left| \left(u^{(i)} * \rho_\epsilon \right)(x) \right| \le \sum_{j \le p} \frac{1}{\epsilon^{i+j}} \left\| f_j \right\|_{L^\infty} \int_{\mathbb{R}} \left| \rho^{(i+j)} (y) \right| dy,$$

consequently $\exists\ C > 0$ such that

$$|u * \rho_\epsilon|_{k,\infty} \le \frac{C}{\epsilon^{k+p}} \ .$$

So, $(u * \rho_\epsilon)_\epsilon \in \mathcal{M}_{aa}$. The linearity of i_{aa} follows from the linearity of convolution. If $(u * \rho_\epsilon)_\epsilon \in \mathcal{N}_{aa}$, then $\lim\limits_{\epsilon \longrightarrow 0} u * \rho_\epsilon = 0$ in $\mathcal{D}'_{L^\infty}$, but, it is easy to see that $\lim\limits_{\epsilon \longrightarrow 0} u * \rho_\epsilon = u$ in $\mathcal{D}'_{L^\infty}$, so $u = 0$, which means that i_{aa} is injective. Finally, $i_{aa}\left(u^{(j)}\right) = \left[\left(u^{(j)} * \rho_\epsilon \right)_\epsilon \right]_\epsilon = \left[(u * \rho_\epsilon)_\epsilon \right]^{(j)} = \left(i_{aa}(u) \right)^{(j)}.$ □

Defining the canonical embedding

$$\begin{array}{cccc} \sigma_{aa}: & \mathcal{B}_{aa} & \longrightarrow & \mathcal{G}_{aa} \\ & f & \longmapsto & (f)_\epsilon + \mathcal{N}_{aa} \end{array},$$

we have two ways to embed the space $\mathcal{B}_{aa}$ into $\mathcal{G}_{aa}$ by i_{aa} and also by σ_{aa}. The following result gives that we have the same result.

Proposition 8. *The following diagram*

$$\begin{array}{ccc} \mathcal{B}_{aa} & \longrightarrow & \mathcal{B}'_{aa} \\ & \sigma_{aa} \searrow & \downarrow i_{aa} \\ & & \mathcal{G}_{aa} \end{array}$$

commutes.

Proof. It suffices to show that for $f \in \mathcal{B}_{aa}$ we have $(f * \rho_\epsilon - f)_\epsilon \in \mathcal{N}_{aa}$. Applying Taylor's formula, we obtain, $\forall m \in \mathbb{Z}_+$,

$$\begin{aligned} \left(f^{(j)} * \rho_\epsilon \right)(x) - f^{(j)}(x) \ = \ & \int_{\mathbb{R}} \sum_{k=1}^{m} \frac{(-\epsilon y)^k}{k!} f^{(k+j)}(x)\, \rho(y)\, dy + \\ & \int_{\mathbb{R}} \frac{(-\epsilon y)^{m+1}}{(m+1)!} f^{(m+j+1)}(x - \theta(x)\, \epsilon y)\, \rho(y)\, dy. \end{aligned}$$

Since $f \in \mathcal{B}_{aa}$ and $\rho \in \Sigma$, then

$$\left\| f^{(j)} * \rho_\epsilon - f^{(j)} \right\|_{L^\infty} \leq \left\| f^{(m+j+1)} \right\|_{L^\infty} \left\| y^{m+1}\rho \right\|_{L^1} \frac{\epsilon^{m+1}}{(m+1)!},$$

consequentely $\forall k \in \mathbb{Z}_+,\ \forall m' = m+1,$

$$|f * \rho_\epsilon - f|_{k,\infty} = O(\epsilon^{m'}), \varepsilon \to 0,$$

i.e. $(f * \rho_\epsilon - f)_\epsilon \in \mathcal{N}_{aa}$. □

The algebra of tempered generalized functions on $\mathbb{C}$ is denoted by $\mathcal{G}_\mathcal{T}$, see [5] for the definition and properties of $\mathcal{G}_\mathcal{T}$.

Proposition 9. *Let* $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{aa}$ *and* $F = [(f_\epsilon)_\epsilon] \in \mathcal{G}_\mathcal{T}$, *then*

$$F \circ u := [(f_\epsilon \circ u_\epsilon)_\epsilon] + \mathcal{N}_{aa}$$

is a well-defined element of $\mathcal{G}_{aa}$.

Proof. Since $(f_\epsilon)_\epsilon \in \mathcal{M}_\mathcal{T}$ and $(u_\epsilon)_\epsilon \in \mathcal{M}_{aa}$, by the classical result of composition of almost automorphic function with continuous function, we have $f_\epsilon \circ u_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I$. The estimates

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |f_\epsilon \circ u_\epsilon|_{k,\infty} = O(\epsilon^{-m}), \varepsilon \to 0,$$

are obtained from the fact that $(u_\epsilon)_\epsilon \in \mathcal{M}_{aa}$ and $(f_\epsilon)_\epsilon$ is polynomially bounded. It is easy to prove that the composition is independent on representatives. □

The convolution of an almost automorphic distribution with an integrable distribution is an almost automorphic distribution. We extend this result to the case of almost automorphic generalized functions.

Proposition 10. *Let* $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{aa}$ *and* $v \in \mathcal{D}'_{L^1}$, *then the convolution* $u * v$ *defined by*

$$u * v := [(u_\epsilon * v)_\epsilon]$$

is a well defined element of $\mathcal{G}_{aa}$.

Proof. The characterization (3.1) of elements of $\mathcal{D}'_{L^1}$ gives that there exists $(f_j)_{j \leq p} \subset L^1$ such that $v = \sum\limits_{i \leq p} f_i^{(i)}$. Let $(u_\epsilon)_\epsilon \in \mathcal{M}_{aa}$ be a representative of u. Then $u_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I$, by Proposition 1,$u_\epsilon * v = \sum\limits_{i \leq p} u_\epsilon^{(i)} * f_i \in \mathcal{B}_{aa}, \forall \epsilon \in I$. Moreover, by Young inequality, we have

$$\left\| (u_\epsilon * v)^{(j)} \right\|_{L^\infty} \leq \sum_{i \leq p} \|f_i\|_{L^1} \left\| u_\epsilon^{(i+j)} \right\|_{L^\infty},$$

so the fact that $(u_\epsilon)_\epsilon \in \mathcal{M}_{aa}$ gives that

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\epsilon * v|_{k,\infty} = O(\epsilon^{-m}), \varepsilon \to 0,$$

consequently $(u_\epsilon * v)_\epsilon \in \mathcal{M}_{aa}$. Finally, one shows that the result is independent on representatives by obtaining the same estimates. □

We give an extension of the classical Bohl-Bohr theorem. First, we recall the definition of a primitive of a generalized function.

Definition 6. *Let* $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{aa}$ *and* $x_0 \in \mathbb{R}$, *a primitive of* u *is a generalized function* U *defined by*

$$U(x) = \left(\int_{x_0}^{x} u_\epsilon(t)\, dt \right)_\epsilon + \mathcal{N}[\mathbb{C}]$$

Proposition 11. *A primitive of an almost automorphic generalized function is almost automorphic if and only if it is a bounded generalized function.*

Proof. Let $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{aa}$, so $u_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I$. If U is a primitive of u and $U \in \mathcal{G}_{aa}$, then $U \in \mathcal{G}_{L^\infty}$ because $\mathcal{G}_{aa} \subset \mathcal{G}_{L^\infty}$. Conversely, if $U = [(U_\epsilon)_\epsilon] \in \mathcal{G}_{L^\infty}$, then $\forall \epsilon \in I, U_\epsilon = \int_{x_0}^{x} u_\epsilon(t)\, dt \in \mathcal{D}_{L^\infty}$, so U_ϵ is bounded primitive of $u_\epsilon \in \mathcal{C}_{aa}$. By the classical result of Bohl-Bohr we have $U_\epsilon \in \mathcal{C}_{aa}$, consequentely $U_\epsilon \in \mathcal{C}_{aa} \cap \mathcal{D}_{L^\infty}, \forall \epsilon \in I$. By Proposition 1, $U_\epsilon \in \mathcal{B}_{aa}, \forall \epsilon \in I$. Moreover $(U_\epsilon)_\epsilon \in \mathcal{M}_{L^\infty}$, i.e.

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |U_\epsilon|_{k,\infty} = O\left(\epsilon^{-m}\right), \epsilon \longrightarrow 0,$$

so $(U_\epsilon)_\epsilon \in \mathcal{M}_{aa}$ and $U \in \mathcal{G}_{aa}$. The result is independent on representatives. □

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