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ABEL TYPE THEOREMS FOR THE WAVELET TRANSFORM THROUGH THE QUASIASYMPTOTIC BOUNDEDNESS

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Dedicated to Academician Bogoljub Stanković for his 90th birthday

Abstract. We analyze quasiasymptotic boundedness of distributions and their wavelet transforms, in general, as well as for a class of $\alpha-$ exponentially bounded distributions and their wavelet transforms in particular. The main idea of this paper is to use, instead of the quasiasymptotic behaviour, the notion of quasiasymptotic boundedness. In this way we obtain new Abelian type theorems for the wavelet transform of distributions with different growth.

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1. Introduction

The wavelet transform is a powerful tool for studying local properties of functions and generalized functions. Usually, wavelet analysis presents two main important features [1, 2, 9], the wavelet transform as a time-frequency analysis tool, and wavelet analysis as part of approximation theory [5]. Here we consider the first feature of the theory. For a more general approach, we refer to [3, 4]. This paper is a part of the Master's thesis of the second author [12].

The quasiasymptotics is used to analyze the pointwise properties of tempered distributions (see [3, 4, 10] and references there). By the general theory of tempered distributions, we are limited to comparison by the functions of the form $k^\alpha L(k)$, resp., $\varepsilon^\alpha L(\varepsilon)$, where L is Karamata's regulary varying function [7]. Moreover, the limit distribution has to be the homogeneous of order α, (see [10]). We refer to [7] for the properties of regularly varying functions L.

Our main idea of this paper in obtaining the Abel type theorems is to use, instead of the quasiasymptotic behaviour, the notion of quasiasymptotic boundedness. This enables us to have more freedom in the sense that we should not consider only the space of tempered distributions and the comparison function $\rho(k) = k^\alpha L(k)$, resp., $\varepsilon^\alpha L(\varepsilon)$. In this way we obtain new Abel type theorems for the waverlet transform of distributions with different growth.

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2. Definitions

We use the standard notation (cf. [11, 10, 1, 8]): $\mathbb{N}$ is the set of positive integers including zero, $\mathbb{R}_+$ is a set of positive real numbers, its complement is $\overline{\mathbb{R}}_-$ (similarly one has $\overline{\mathbb{R}}_+$) and $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$. The Schwartz spaces of smooth compactly supported and rapidly decreasing test functions are denoted by $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, respectively; their dual spaces, the spaces of distributions, and tempered distributions are $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, respectively. We use the Fourier transform of a function $s \in L^1(\mathbb{R})$: $s \mapsto Fs$,

$$(Fs)(\omega) = \left\langle e^{i\omega t} \middle| s \right\rangle = \int_{-\infty}^{+\infty} e^{-i\omega t} s(t)\, dt, \omega \in \mathbb{C}.$$

A function $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\int_{-\infty}^{+\infty} g(t)\, dt = 0$ is called a wavelet. Recall, [1], the wavelet transform of a function $s \in L^p(\mathbb{R})$, $1 \le p \le \infty$, is defined as

$$\mathcal{W}_g s(b,a) = \langle g_{b,a} | s \rangle = \int_{-\infty}^{+\infty} \frac{1}{a} \overline{g}\left(\frac{t-b}{a}\right) s(t)\, dt, \qquad (b,a) \in \mathbb{H}.$$

The function g is usually called *mother wavelet* or *analyzing wavelet*, and functions $g_{b,a}, (b,a) \in \mathbb{H}$ are wavelets. For $g, s \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we can calculate the wavelet coefficients in the Fourier spaces as

$$\mathcal{W}_g s(b,a) = \langle g_{b,a} | s \rangle = \frac{1}{2\pi} \langle \hat{g}_{b,a} | \hat{s} \rangle, (b,a) \in \mathbb{H}.$$

Let $f(t) \in \mathcal{S}'(\mathbb{R})$ and $\rho(k)$ be a positive continous function for $k > 0$. We say that $f(t)$ has quasiasymptotic behavior at infinity (resp., at zero) related to a positive function $\rho(k)$ in $\mathcal{S}(\mathbb{R})$, in the sense of convergence in $\mathcal{S}'(\mathbb{R})$, if

$$\frac{1}{\rho(k)} f(kt) \to g(t), \qquad k \to \infty, \qquad \left(\frac{1}{\rho(\varepsilon)} f(\varepsilon t) \to g(t), \qquad \varepsilon \to 0^+\right).$$

Moreover, one can consider these notions in $\mathcal{D}'(\mathbb{R})$, (cf. [10]), see 3.1 below.

3. Abelian type theorems

The quasiasymptotic boundedness of tempered distributions and the wavelet transform of tempered distributions supported by $[0,\infty)$ was considered in [6, 9]. In the first subsection we consider the quasiasymptotic boundednes for tempered distributions on the whole real line. One can transfer all the results to the $n-$dimensional case $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$. This will not be considered in this paper.

3.1. Abelian theorem for elements of $\mathcal{D}'(\mathbb{R})$

Let $\mathcal{H}$ be a space of test functions on $\mathbb{R}$ with the convergence structure so that $\mathcal{D}(\mathbb{R})$ is dense in it and the inclusion mapping $\mathcal{D}(\mathbb{R}) \to \mathcal{H}$ is continuous. This implies that $\mathcal{H}'$ is a subspace of the space of distributions $\mathcal{D}'(\mathbb{R})$.

Let $f \in \mathcal{H}'$ and $\rho(k), k > a > 0$, $(\rho(\varepsilon), 0 < \varepsilon < \varepsilon_0 < 1)$, be a positive and continous function. We say that f is a quasiasymptotically bounded function with respect to ρ at infinity, (at zero) over $\mathcal{H}$ if

$$f(k\cdot)/\rho(k), (f(\varepsilon\cdot)/\rho(\varepsilon)) \text{ is bounded in } \mathcal{H}' \text{ in the weak sense.} \tag{3.1}$$

Our first theorem (see [12]) is a simple generalization of the very well known one for tempered distributions.

Theorem 3.1. *Let a function $f \in \mathcal{D}'(\mathbb{R})$ be a quasiasymptotically bounded at zero with respect to a continuous positive real valued function $c(\varepsilon)$ (resp. at infinity with $c(k)$), that is*

$$|\langle f(\varepsilon x), \phi(x)\rangle| \le C_\phi c(\varepsilon), \qquad \varepsilon \to 0$$

$$(resp.\, |\langle f(kx), \phi(x)\rangle| \le C_\phi c(k), \qquad k \to \infty).$$

where $\phi \in \mathcal{D}(\mathbb{R})$ and $C_\phi > 0$ depends of ϕ. Let $g \in \mathcal{D}(\mathbb{R})$ be a mother wavelet. Then the wavelet transform for f is a bounded function at 0 (respectively ∞), i.e., there is a $C = C(g)$ such that

$$\left|\frac{\mathcal{W}_g f(x,x)}{c(x)}\right| \le C, \qquad x \to 0,$$

$$(resp. \left|\frac{\mathcal{W}_g f(x,x)}{c(x)}\right| \le C, \qquad x \to \infty).$$

Proof. Note if $g \in \mathcal{D}$, then $t \to \bar{g}\left(\frac{t-\varepsilon x}{\varepsilon y}\right) \rangle|, t \in \mathbb{R}$ is in $\mathcal{D}$. According to the definition of the wavelet transform, we have

$$\frac{|\mathcal{W}_g f(\varepsilon x, \varepsilon y)|}{c(\varepsilon)} = \left|\left\langle f(t), \frac{1}{c(\varepsilon)\varepsilon y}\bar{g}\left(\frac{t-\varepsilon x}{\varepsilon y}\right)\right\rangle\right|, x, y \in \mathbb{H}.$$

If we put $t = \varepsilon s$, we obtain

$$\frac{|\mathcal{W}_g f(\varepsilon x, \varepsilon y)|}{c(\varepsilon)} = \left|\left\langle \frac{f(\varepsilon s)}{c(\varepsilon)}, \frac{1}{y}\bar{g}\left(\frac{s-x}{y}\right)\right\rangle\right| \le C.$$

For $x = 1, y = 1$ (and later $\varepsilon = x$), we have

$$\left|\frac{\mathcal{W}_g f(x,x)}{c(x)}\right| \le C.$$

In the case $k \to \infty$, we have the same proof. □

3.2. Abelian theorem for α-exponentially distributions

Let $\alpha \in (0,2)$. Our main objective is the quasiasymptotic boundednes over the dual space of the space of test functions with the $\alpha-$exponential decrease defined as follows.

$$\mathcal{S}_\alpha(\mathbb{R}) = \left\{ \varphi \in C^\infty(\mathbb{R}) \quad | \, \gamma_k(\varphi) = \sup_{r\le k, x\in\mathbb{R}} e^{k|x|^\alpha} \left|\varphi^{(r)}(x)\right| < \infty \text{ for every } k>0 \right\}.$$

Clearly $\mathcal{S}_\alpha \subset \mathcal{S}_\beta$ if $0<\alpha<\beta$.
The convergence structure is clear if one knows well the Schwartz space $\mathcal{S}(\mathbb{R})$. Namely, $\gamma_k, k\in\mathbb{N}$, is a sequence of norms which defines the topology. Clearly, $\mathcal{D}(\mathbb{R})$ is dense in it. This space is an example of the Gelfand - Shilov type spaces $\mathcal{K}_{M_p}(\mathbb{R})$ in the special case when M_p is a smooth function $M_p(x) \ge m>0, x\in\mathbb{R}$, and $M_p(x) = e^{p|x|^\alpha}$ for $|x|>1$.

The following structural theorem can be proved in a standard way (cf. [8], Theorem 8.2.8).

Theorem 3.2. *For every $f \in \mathcal{S}'_\alpha$ there exist $p>0, m\in\mathbb{N}$ and bounded functions $F_j, j=0,1,...,m$ so that*

$$f = \sum_{j=0}^{m} \left(e^{p|x|^\alpha} F_j(x)\right)^{(j)}.$$

Now we consider the boundednes of the elements of this space.

Theorem 3.3. *Every $f\in\mathcal{S}'_\alpha$ is quasiasymptotically bounded with respect to $\rho(k) = e^{p|k|^{\alpha+2}}$ over the space $\mathcal{S}_{\alpha+2}$.*

Proof. Let $\phi \in \mathcal{S}_{\alpha+2}$. Then

$$\begin{aligned}
\left\langle \frac{f(kx)}{e^{p|k|^{\alpha+2}}}, \phi(x) \right\rangle &= \left\langle \sum_{j=0}^{m} \frac{\left(e^{p|\cdot|^\alpha} F_j(\cdot)\right)^{(j)}(kx)}{e^{p|k|^{\alpha+2}}}, \phi(x) \right\rangle \\
&= \sum_{j=0}^{m} \left\langle \frac{\left(e^{p|\cdot|^\alpha} F_j(\cdot)\right)^{(j)}(x)}{e^{p|k|^{\alpha+2}}}, \frac{1}{k}\phi\left(\frac{x}{k}\right) \right\rangle \\
&= \sum_{j=0}^{m} (-1)^j \left\langle \frac{e^{p|x|^\alpha} F_j(x)}{e^{p|k|^{\alpha+2}}}, \frac{1}{k^{j+1}}\phi^{(j)}\left(\frac{x}{k}\right) \right\rangle \\
&= \sum_{j=0}^{m} (-1)^j \left\langle \frac{e^{p|kx|^\alpha} F_j(kx)}{k^j e^{p|k|^{\alpha+2}}}, \phi^{(j)}(x) \right\rangle .
\end{aligned}$$

There exist constants $C_j>0, j=1,...,m$, such that

$$\left| \left\langle \frac{f(kx)}{e^{p|k|^{\alpha+2}}}, \phi(x) \right\rangle \right| \le \sum_{j=0}^{m} C_j \int_{\mathbb{R}} \frac{e^{p|k|^\alpha |x|^\alpha} F_j(kx)}{k^j e^{p|k|^{\alpha+2}}} \cdot \left|\phi^{(j)}(x)\right| dx.$$

Dividing the last intgral into two parts, we obtain, with another $\tilde{C}_j > 0$,

$$\left|\left\langle \frac{f(kx)}{e^{p|k|^{\alpha+2}}}, \phi(x)\right\rangle\right| \leq \sum_{j=0}^{m} \tilde{C}_j \left(\int\limits_{|x|<1} \left|\phi^{(j)}(x)\right| dx + \int\limits_{|x|>1} \frac{e^{pk^{\alpha}|x|^{\alpha}}}{k^j e^{pk^{\alpha+2}}} \cdot \frac{e^{(p+1)|x|^{\alpha+2}} \left|\phi^{(j)}(x)\right|}{e^{p|x|^{\alpha+2}} e^{|x|^{\alpha+2}}} dx \right).$$

Since $r \cdot t \leq r^2 + t^2$ we obtain $k^{\alpha} |x|^{\alpha} \leq k^{2\alpha} + |x|^{2\alpha}$.

According to this inequality and assumption $\alpha \in (0, 2)$ it follows that

$$pk^{\alpha} |x|^{\alpha} \leq pk^{\alpha+2} + p|x|^{\alpha+2}. \tag{3.2}$$

Now, with sutable constants D_j, we finally obtain

$$\begin{aligned} \left\langle \frac{f(kx)}{e^{p|k|^{\alpha+2}}}, \phi(x)\right\rangle &\leq \sum_{j=0}^{m} D_j + D_j \int\limits_{|x|>1} \frac{e^{p\left(k^{\alpha+2}+|x|^{\alpha+2}\right)}}{e^{p\left(k^{\alpha+2}+|x|^{\alpha+2}\right)}} \cdot \frac{dx}{e^{|x|^{\alpha+2}}} \\ &\leq \sum_{j=0}^{m} D_j + D_j \int\limits_{|x|>1} \frac{dx}{e^{|x|^{\alpha+2}}} < \infty, \end{aligned}$$

which finishes the proof. □

Now we show that the wavelet transformation of any distribution from $\mathcal{S}'_{\alpha}$ is $\alpha + 2$-exponentially bounded.

Theorem 3.4. *Let $f \in \mathcal{S}'_{\alpha}$, $\alpha < 2$ and $g \in C^{\infty}$ such that $g \in \mathcal{S}_{\alpha+2}$ is a mother wavelet. Then*

$$\left|\frac{\mathcal{W}_g f(x, x)}{e^{p|x|^{\alpha+2}}}\right| < \infty, \qquad x > x_0 > 0.$$

Proof. It is known that f is quasiasyptotically bounded with $e^{p|x|^{\alpha+2}}$ for some $p > 0$, so from Theorem 3.3 we have

$$\left|\frac{\mathcal{W}_g f(x, x)}{e^{p|x|^{\alpha+2}}}\right| = \left|\left\langle \frac{f(t)}{e^{p|x|^{\alpha+2}}}, \frac{1}{x^n} \bar{g}\left(\frac{s-x}{x}\right)\right\rangle\right| < C,$$

i.e.

$$\left|\frac{\mathcal{W}_g f(x, x)}{e^{p|x|^{\alpha+2}}}\right| = \left|\left\langle \frac{f(t)}{e^{p|x|^{\alpha+2}}}, \frac{1}{x^n} \bar{g}\left(\frac{s-x}{x}\right)\right\rangle\right| < \infty.$$

□

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