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AVERAGE STRONG ASSOCIATION AND COMPARISON OF REGULARITIES

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I would like to dedicate this work to Professor Stanković who, during more than sixty years inspired and contributed in many branches of Analysis and to Professor Wickers whose contributions in the relation of Physics and Generalized Functions Theory were inspiring and important for its development.

Abstract. New families of generalized functions are introduced (with an "integral type" evaluation of the behavior of representatives). A strong "average" (on the parameter) association with distributions is introduced (along an analogous line to Cesaro convergence of sequences), and comparisons are made with usual regularities of distributions (only the case of $\mathcal{C}^\infty$ and $\mathcal{G}^\infty$ type regularities is treated in detail here).

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1. Introduction

An important topic in Colombeau theory is the comparison of regularities between a Colombeau generalized function g and a distribution T to which it is associated in some given sense. The first and oldest result along this line was obtained by M. Oberguggenberger [2], who proved that if we embed (in some canonical way) distributions into a Colombeau algebra then the intersection of the image of the space of distributions with the subspace of $\mathcal{G}^\infty$ generalized functions is the image of $\mathcal{C}^\infty$ functions.

Soon it was proved [5, 1] that if a $\mathcal{G}^\infty$-generalized function is strongly associated to a distribution then the distribution in question is a $\mathcal{C}^\infty$-function (here strong association means that the effect of the difference of a representative of the generalized function and the distribution on a test function decreases as a positive power of the parameter ε).

Later analogous results were obtained concerning real analytic regularity [3], Zygmund type regularities [4], etc.

A natural question suggested by J. F. Colombeau is the following: If the action of $T - g_\varepsilon$ on test functions does not converge to zero but converges to zero, only in some "average sense", what can we conclude for the regularity of the distribution T in the case that we have information of "regularity" for the

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generalized function g? In other words can interesting properties be proved although the generalized function has wild oscillations on the parameter ε?

This question leads naturally to the definition of a new class of generalized functions along analog lines to the usual Colombeau generalized functions, but where the notion of "average on parameter" makes sense, i.e. we need to impose additional specifications on representatives, such as measurability of the parameter, local integrability and so on. In view of such investigations we have to present in a new way some of the usual definitions and give some new definitions adapted to our "averaging" investigations.

2. Definitions

Let us revisit the standard definition for usual simplified Colombeau generalized functions algebra:

A net g_ε of $\mathcal{C}^\infty$ functions in an open set $\Omega \subseteq \mathbb{R}^n$, with $\varepsilon \in (0,1]$, is said to be moderate if for any compact subset K of Ω and any multi-index α, there exists some real number a such that $sup_{x\in K}|\partial^\alpha g_\varepsilon(x)| = o(\varepsilon^a)$.

If, moreover, such a property holds for all reals a we say that the net is negligible.

The above properties can be expressed equivalently by asking that

$$sup_{\varepsilon\leq\varepsilon_0} sup_{x\in K}|\varepsilon^{-a}\partial^\alpha g_\varepsilon(x)|$$

is bounded for some $\varepsilon_0 > 0$.

If now, in this "new" reading of the definition, we replace the supremum on ε norm by any other norm, for example an L^p norm (provided that we impose the necessary specifications so that this makes sense), we obtain new kinds of Colombeau generalized functions. In our case where we use L^p norm, the corresponding spaces will be noted by replacing $\mathcal{G}$ with $\mathcal{G}_p$. Let us notice that the usual embeddings of distributions into Colombeau algebras yield elements with representatives depending continuously on ε and thus define also an element of the corresponding new Colombeau spaces $\mathcal{G}_p$.

Let us now give a precise definition of strong average association:

Definition 2.1. *Let $[g_\varepsilon]$ be a "new" Colombeau generalized function or even a usual one but with a representative g_ε measurable on ε and everywhere locally integrable on (x,ε). We say that it is strongly average associated to a distribution T if the following properties hold:*

a) for any test function φ, the limit of

$$\int_a^b \langle g_\eta - T, \varphi\rangle d\eta,$$

when a tends to zero, exists and defines a distribution denoted by

$$\int_0^b \langle g_\eta - T, \varphi\rangle d\eta;$$

b) there exists a strictly positive real number b such that

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\langle g_\eta - T, \varphi\rangle d\eta = o(\varepsilon^b).$$

For instance, the generalized function $\left[sin\left(\frac{1}{\varepsilon}\right)\right]$ is strongly average associated to zero, but not associated to any distribution.

In order to give the "regularity" definitions in the new frames we just have to replace the "sup on ε" norm by the corresponding norm used in the definition of the considered space. In a joint paper to appear with S. Pilipović, J. Vindas and H. Vernaeve, we study in detail the new Colombeau spaces and treat also the case of real analytic regularity as well as the case of Besov type regularity. Here, only the simpler, but fundamental, case of $\mathcal{C}^\infty$ regularity will be treated. Replacing in the definition of $\mathcal{G}^\infty$ regularity the sup on ε norm by the corresponding L^p norm, we obtain the corresponding regularity in the new space.

3. Main result

In this short paper the main result is the following.

Proposition 3.1. *If g is "$\mathcal{G}^\infty$" in the sense of the corresponding new Colombeau extension (or even in the usual sense but with representatives measurable on ε), and is strongly average associated to a distribution T then T is $\mathcal{C}^\infty$.*

Note that if g is with continuous dependence on the parameter and $\mathcal{G}^\infty$ in the usual sense, it defines an element of the new space which has also the corresponding $\mathcal{G}^\infty$ regularity.

The main idea is to replace using an "averaging" operator the generalized function g by a new generalized function h belonging to the usual Colombeau algebra, which will have the "same" regularity as g (but in the usual sense), and will be strongly associated to T. Thus the result will follow from the previous well known theorem of comparison of regularities (the same strategy is also used for the cases of other kinds of regularities). Hence we will only have to prove the following two lemmas:

Lemma 3.1. *If g represented by (g_ε) is strongly average associated to the distribution T and $\gamma > 1$, then the generalized function h represented by the net*

$$h_\varepsilon = \frac{1}{\varepsilon}\int_{\varepsilon^\gamma}^{\varepsilon} g_\eta d\eta$$

is strongly associated to T.

The proof is obtained by introducing in the action of $h-T$ on a test function the average on $[0,\varepsilon]$ of the action of $g_\eta - T$ in the following way: Let K be a compact subset of the open set Ω. By hypothesis, there exists a positive number b such that for any test function φ with support in K we have:

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\langle g_\eta - T, \phi\rangle d\eta = o(\varepsilon^b).$$

Now using the definition of h we can easily see that

$$\langle h_\varepsilon - T, \varphi\rangle = \frac{1}{\varepsilon}\int_0^{\varepsilon}\langle g_\eta - T, \phi\rangle d\eta - \frac{1}{\varepsilon}\int_0^{\varepsilon^\gamma}\langle g_\eta - T, \varphi\rangle d\eta - \frac{1}{\varepsilon}\int_0^{\varepsilon^\gamma}\langle T, \varphi\rangle d\eta.$$

Now using the hypothesis of average strong association it is straightforward to conclude that there exists $c > 0$ such that the previous quantity is $o(\varepsilon^c)$.

This lemma being proved we now only have to prove the following second lemma.

Lemma 3.2. *If g is $\mathcal{G}^\infty$ in the new sense and h is as in the previous lemma, then h is an element of Colombeau generalized function space with continuous dependence on the parameter and is $\mathcal{G}^\infty$ in the usual sense.*

Sketch of the proof: For any given compact subset K there exists M such that $\forall\alpha \in \mathbb{N}^n$ we have by our "new" regularity hypothesis the following:

$$\|\eta^M \sup_K |\partial^\alpha g_\eta|\|_p < \infty.$$

We only have to prove that there exists $l \in \mathbb{R}$ depending only on K such that

$$sup_K|\partial^\alpha g_\varepsilon| = o(\varepsilon^l).$$

Note that by the properties of integration and Hölder inequality we have

$$sup_K|\partial^\alpha h_\varepsilon| \leq \frac{1}{\varepsilon}\int_{\varepsilon^\gamma}^{\varepsilon} sup_K|\partial^\alpha g_\eta|d\eta = \frac{1}{\varepsilon}\int_0^1 (\eta^{-M}\chi_{[\varepsilon^\gamma,\varepsilon]})(\eta^M sup_K|\partial^\alpha g_\eta|)d\eta$$

$$\leq \|\eta^M sup_K|\partial^\alpha g_\eta|\|_p \frac{1}{\varepsilon}\|\eta^{-M}\chi_{[\varepsilon^\gamma,\varepsilon]}\|_q,$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Since the first term of the product is bounded, it is easy to conclude the claim.

Now to finish the proof of the theorem we only have to use the old theorem of comparison of regularities under strong association.

The same strategy is used for real analytic regularities, while for the Besov type regularities the methods are much more complicated, but this is outside the scope of this paper.

As a conclusion we can remark that by the use of this passage from g to h we obtain a new generalized function much easier to handle and obtain nice results on regularities of the distribution T even when we have apparently wild oscillation situations.

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