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ON A CLASS OF ULTRADIFFERENTIABLE FUNCTIONS[1]

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Dedicated to Professor Stanković on the occasion of his 90^{th} birthday.

Abstract. We introduce a class of ultradifferentiable functions which contains Gevrey functions and study its basic properties. In particular, we investigate the continuity properties of certain (ultra)differentiable operators. Finally, we discuss microlocal properties in appropriate dual spaces.

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1. Introduction

Since their introduction in the context of regularity properties of fundamental solution of the heat operator in [1], Gevrey classes were used in many situations related to the general theory of linear partial differential operators such as hypoellipticity, local solvability and propagation of singularities. We refer to [6] for the definition and detailed exposition of Gevrey classes and their applications to the theory of linear partial differential operators. It is known that intersection (projective limit) of Gevrey classes contains the space of analytic functions, while its union (inductive limit) is contained in the class of smooth functions. However, there is a gap between the Gevrey classes and the space of smooth functions, so that in certain situations a more refined description of regularity might be useful. The purpose of this paper is to introduce a family of smooth functions which are less regular than the Gevrey functions, and to study its basic properties. The main motivation for our approach is that it can be used in the study of intermediate singularities between the classical C^∞ and the Gevrey type singularities, see Section 4.

We recall Komatsu's approach [4] and introduce a family of sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{Z}_+$, for some $\tau > 0$ and $\sigma > 1$, so that the corresponding space of ultradifferentiable functions contains Gevrey classes. Such sequences

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do not satisfy conditions $\utilde{(M.2)}$ and $\utilde{(M.2)}'$ (cf. Subsection 1.2), which are in our analysis replaced by $\overline{(M.2)}'$ and $\overline{(M.2)}$, see Lemma 2.2.

In Section 3 we define the corresponding space of ultradifferentiable functions and study its basic properties. For example, we show that there exists a nontrivial smooth compactly supported function in our class which does not belong to Gevrey classes.

Recall the condition $(M.2)$ provides the stability under the action of appropriate ultradifferentiable operators. Since $M_p^{\tau,\sigma}$ does not satisfy $(M.2)$ we can not expect that our space is closed under the action of appropriate ultradifferentiable operators. However, the continuity holds if we observe an inductive limit with respect to one of the parameters, Theorems 3.2 and 3.3.

In Section 4 we discuss a new approach to the study of microlocal properties of ultradistributions in the context of the new class of ultradifferentiable functions.

1.1. Notation

Throughout the paper, we use the standard notation: nonnegative integers, integers, positive integers, real numbers, positive real numbers and complex numbers are denoted by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Z}_+$, $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{C}$, respectively. The integer part (the floor function) of $x \in \mathbf{R}_+$ is denoted by $\lfloor x \rfloor := \max\{m \in \mathbf{N} : m \leq x\}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$ we write $\partial^\alpha = \partial^{\alpha_1} \dots \partial^{\alpha_d}$ and $|\alpha| = |\alpha_1| + \dots |\alpha_d|$. We will also use the Stirling formula: $N! = N^N e^{-N} \sqrt{2\pi N} e^{\frac{\theta_N}{12N}}$, for some $0 < \theta_N < 1$, $N \in \mathbf{Z}_+$. By $C^m(K)$, $m \in \mathbf{N}$, we denote the Banach space of m-times continuously differentiable functions on a regular compact set $K \subset\subset U$, where $U \subseteq \mathbf{R}^d$ is an open set, and $C^\infty(K)$ is the corresponding set of smooth functions on K, see [4]. Convolution is denoted with $f * g(x) = \int_{\mathbf{R}^d} f(x-y)g(y)dy$, whenever the integral make sense.

For locally convex topological spaces X and Y we write $X \hookrightarrow Y$ when $X \subseteq Y$ and the identity mapping from X to Y is continuous. If, in addition, $X \neq Y$ then the embedding is strict. By X' we denote the strong dual of X and by $\langle \cdot, \cdot \rangle_X$ the dual pairing between X and X'. The set of continuous linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$.

A linear map $B \in \mathcal{L}(X, Y)$, X, Y are Banach spaces, is *quasi-nuclear* if there exists a sequence $\{x'_j\}$ in X' such that $\sum\limits_{j=1}^{\infty} \|x'_j\|_{X'} < \infty$ and $\|Bx\|_Y \leq \sum\limits_{j=1}^{\infty} |\langle x, x'_j \rangle_X|$. In particular, a quasi-nuclear map $A \in \mathcal{L}(X, Y)$ is *nuclear* if there exists bounded sequences $x'_j \in X'$ (with respect to the strong topology) and $y_j \in Y$, $j \in \mathbf{Z}_+$, and a sequence $\lambda_j \in \mathbf{C}$, $j \in \mathbf{Z}_+$, such that $\sum\limits_{j=1}^{\infty} |\lambda_j| < \infty$ and $Ax = \sum\limits_{j=1}^{\infty} \lambda_j \langle x, x'_j \rangle_X y_j$. We refer to [11, Section III.7] and [8] for an extension of nuclear and quasi-nuclear mappings to arbitrary locally convex topological spaces.

1.2. Classical spaces of ultradifferentiable functions

We use Komatsu's approach to the theory of ultradistributions as follows, see [4].

By $M_p = (M_p)_{p\in\mathbf{N}}$ we denote a sequence of positive numbers such that:

$$
\begin{array}{llr}
(M.0) & M_0 = 1; & \\
(M.1) & M_p^2 \le M_{p-1}M_{p+1}, & p \in \mathbf{Z}_+; \\
(M.2) & (\exists A, B > 0) \;\; M_{p+q} \le AB^p M_p M_q, & p, q \in \mathbf{N}; \\
(M.3)' & \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty. &
\end{array}
$$

Then M_p also satisfies weaker conditions: $(M.1)'$ $M_p M_q \le M_{p+q}$ and $(M.2)'$ $M_{p+q} \le AB_q^p M_p$ for some $A, B > 0$, $p, q \in \mathbf{N}$.

Let there be given sequence M_p which satisfies $(M.0) - (M.3)$ and let $U \subseteq \mathbf{R}^d$ be an open set. A function $\phi \in C^\infty(U)$ is an *ultradifferentiable function* if on each compact subset $K \subset\subset U$ there exist positive constants C and h such that

$$
\sup_{x\in K} |\partial^\alpha \phi(x)| \le C h^{|\alpha|} M_{|\alpha|}, \quad \alpha \in \mathbf{N}^d. \tag{1.1}
$$

If K is a fixed compact set in $\mathbf{R}^d$ and if $h > 0$ is given, then $\phi \in \mathcal{E}^{\{M_p\},h}(K)$ if $\phi \in C^\infty(K)$ and if (1.1) holds for some $C > 0$. If $\phi \in C^\infty(\mathbf{R}^d)$ and $\operatorname{supp}\phi \subset K$, then $\phi \in D_K^{\{M_p\},h}$.

The space of ultradifferentiable functions of class $\{M_p\}$ is given by

$$
\mathcal{E}^{\{M_p\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}(K) = \bigcap_{K\subset\subset U} \bigcup_{h\to\infty} \mathcal{E}^{\{M_p\},h}(K),
$$

and its strong dual is the space of ultradistributions of Roumieu type of class M_p. The space of ultradifferentiable functions of class $\{M_p\}$ with support in K is given by

$$
\mathcal{D}^{\{M_p\}}(U) = \varinjlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{D}_K^{\{M_p\},h} = \bigcup_{K\subset\subset U} \bigcup_{h\to\infty} \mathcal{D}_K^{\{M_p\},h}.
$$

and its strong dual is the space of compactly supported ultradistributions of Roumieu type of class M_p.

In particular, if M_p is *Gevrey sequence*, $M_p = p!^t$, $t > 1$, then $\mathcal{E}^{\{p!^t\}}(U)$ is the *Gevrey class* of ultradifferentiable functions. Note that $p!^t$, $t > 1$, satisfies $(M.0) - (M.3)$. We refer to [4] for a detailed study of different classes of ultradifferentiable functions and their duals.

Let there be given $t \ge 1$, $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then the *Gevrey wave front set* $WF_t(u)$ of $u \in \mathcal{D}'^{\{p!^t\}}(U)$ can be defined as follows: $(x_0, \xi_0) \notin WF_t(u)$

if and only if there exists an open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $u_N \in \mathcal{E}'^{\{p!^t\}}(U)$, such that $u_N = u$ on Ω and

$$|\widehat{u}_N(\xi)| \leq A\,\frac{h^N N!^t}{|\xi|^N}, \quad N \in \mathbf{Z}_+,\, \xi \in \Gamma,$$

for some $A, h > 0$. Here, and in what follows, the Fourier transform $\widehat{u}$ of a distribution u is normalized to be $\widehat{u}(\xi) = \int_{\mathbf{R}^d} u(x)e^{-2\pi i x\xi}dx$, $\xi \in \mathbf{R}^d$, whenever the integral is well defined. If $t = 1$, then the Gevrey wave front set is sometimes called the *analytic wave front set* and denoted by $WF_A(u)$, $u \in \mathcal{D}'^{\{p!\}}(U)$. We refer to [5, 6] for details.

2. New classes of ultradifferentiable functions

In this section we introduce new spaces of ultradifferentiable functions in an analogy to $\mathcal{E}^{\{M_p\}}(U)$ and $\mathcal{D}^{\{M_p\}}(U)$. We introduce more general sequences than the Gevrey type sequences $p!^t$, $t > 1$, and begin our investigations by studying their basic properties. We start with a simple but useful Lemma.

Lemma 2.1. *Let $\tau > 0$, $\sigma > 1$ and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{Z}_+$, $M_0^{\tau,\sigma} = 1$. Then there exist $A, B, C > 0$ such that*

$$M_p^{\tau,\sigma} \leq AC^{p^\sigma}\lfloor p^\sigma\rfloor!^{\tau/\sigma} \quad \text{and} \quad \lfloor p^\sigma\rfloor!^{\tau/\sigma} \leq BM_p^{\tau,\sigma}. \tag{2.1}$$

Proof. By $p^\sigma \leq \lfloor p^\sigma\rfloor + 1$ and $p^\sigma \leq 2\lfloor p^\sigma\rfloor$, $p \in \mathbf{Z}_+$, we have

$$p^{\tau p^\sigma} \leq p^{\tau(\lfloor p^\sigma\rfloor+1)} \leq p^\tau\Big(2\lfloor p^\sigma\rfloor\Big)^{\tau\lfloor p^\sigma\rfloor/\sigma} \leq e^{\tau p^\sigma}2^{\tau\lfloor p^\sigma\rfloor/\sigma}\lfloor p^\sigma\rfloor^{\tau\lfloor p^\sigma\rfloor/\sigma},$$

and the left hand side inequality in (2.1) follows from the Stirling formula.

The right hand side inequality in (2.1) follows directly from the Stirling formula:

$$\lfloor p^\sigma\rfloor!^{\tau/\sigma} \leq \Big(e^{-\lfloor p^\sigma\rfloor}\sqrt{2\pi\lfloor p^\sigma\rfloor}\lfloor p^\sigma\rfloor^{\lfloor p^\sigma\rfloor}\Big)^{\tau/\sigma} \leq B\lfloor p^\sigma\rfloor^{\tau\lfloor p^\sigma\rfloor/\sigma} \leq Bp^{\tau p^\sigma},$$

for some $B > 0$. □

Next we study properties of the sequence $M_p^{\tau,\sigma}$, $\tau > 0$, $\sigma > 1$ with respect to the conditions $(M.0) - (M.3)'$.

Lemma 2.2. *Let $\tau > 0$, $\sigma > 1$ and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{Z}_+$, $M_0^{\tau,\sigma} = 1$. Then the following properties hold:*

$(M.1)$ $(M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma}M_{p+1}^{\tau,\sigma}$, $p \in \mathbf{Z}_+$,

$\widetilde{(M.2)}'$ $M_{p+q}^{\tau,\sigma} \leq C_q^{p^\sigma}M_p^{\tau,\sigma}$, *for some sequence* $C_q \geq 1$, $p, q \in \mathbf{N}$,

$\widetilde{(M.2)}$ $M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma+q^\sigma}M_p^{\tau 2^{\sigma-1},\sigma}M_q^{\tau 2^{\sigma-1},\sigma}$, $p, q \in \mathbf{N}$, *for some constant* $C > 1$.

Proof. Note that $(M.0)$ holds by the assumption and $(M.3)'$ will be proved in Lemma 3.1.

We may assume that $\tau = 1$ without loos of generality.

The condition $(M.1)$ obviously holds when $p = 1$. If $p - 1 \in \mathbf{Z}_+$ then the sequence $\ln M_p$ is convex since the second derivative of $f(t) = t^\sigma \ln t$, $t > 0$, is positive when $t > e^{\frac{1-2\sigma}{\sigma(\sigma-1)}}$. This implies $(M.1)$.

The conditions $\widetilde{(M.2)'}$ and $\widetilde{(M.2)}$ trivially hold when $p = 0$ (or $q = 0$). Let $p, q \in \mathbf{Z}_+$.

To prove $\widetilde{(M.2)'}$ we put $\sigma = n + \delta$ where $n \in \mathbf{Z}_+$ and $0 < \delta \leq 1$. If $\sigma \notin \mathbf{Z}_+$ then $n = \lfloor \sigma \rfloor$, $0 < \delta < 1$, while $n = \sigma - 1$, $\delta = 1$, if $\sigma \in \mathbf{Z}_+$. By the binomial formula we have:

$$\begin{aligned}(p+q)^\sigma &\leq (p+q)^n (p^\delta + q^\delta) = p^\sigma + \sum_{k=1}^{n} \binom{n}{k} p^{\sigma-k} q^k \\ &+ \sum_{k=0}^{n} \binom{n}{k} p^{n-k} q^{k+\delta} \leq p^\sigma + 2^n (p^{\sigma-1} q^n + p^n q^\sigma) \\ &\leq p^\sigma + 2^{n+1} q^\sigma p^{\sigma-\delta},\end{aligned}$$

wherefrom

$$(p+q)^\sigma \ln(p+q) \leq p^\sigma \ln(p+q) + 2^{n+1} q^\sigma p^{\sigma-\delta} \ln(p+q). \tag{2.2}$$

We will use the fact that for any $\alpha > 0$ there exists $A > 0$ such that $\ln x \leq A x^\alpha$, $x \geq 1$. Therefore $p \leq C^{p^\delta}$, for some $C > 1$ when $p \in \mathbf{Z}_+$ and $0 < \delta \leq 1$.

The first term on the right hand side of the inequality (2.2) can be estimated by

$$\begin{aligned}p^\sigma \ln(p+q) &= p^\sigma \Big(\ln p + \ln \Big(1 + \frac{q}{p}\Big)\Big) \leq p^\sigma \ln p + p^{\sigma-1} q \\ &\leq p^\sigma \ln p + q p^\sigma,\end{aligned} \tag{2.3}$$

while for the second term we use

$$\begin{aligned}2^{n+1} q^\sigma p^{\sigma-\delta} \ln(p+q) &= 2^{n+1} q^\sigma p^{\sigma-\delta} \Big(\ln p + \ln \Big(1 + \frac{q}{p}\Big)\Big) \\ &\leq 2^{n+1} q^\sigma p^\sigma \ln C + 2^{n+1} q^\sigma p^\sigma \ln(1+q)\,.\end{aligned} \tag{2.4}$$

Now (2.3) and (2.4) imply $\widetilde{(M.2)'}$ by taking the exponentials in (2.2).

It remains to prove $\widetilde{(M.2)}$. From $(p+q)^\sigma \leq 2^{\sigma-1}(p^\sigma + q^\sigma)$ it follows that

$$(p+q)^{(p+q)^\sigma} \leq (p+q)^{2^{\sigma-1} p^\sigma} (p+q)^{2^{\sigma-1} q^\sigma}.$$

Since

$$\begin{aligned}2^{\sigma-1} p^\sigma \ln(p+q) &= 2^{\sigma-1} p^\sigma \Big(\ln p + \ln \Big(1 + \frac{q}{p}\Big)\Big) \\ &\leq 2^{\sigma-1} p^\sigma \ln p + 2^{\sigma-1} q p^{\sigma-1} \\ &\leq 2^{\sigma-1} p^\sigma \ln p + 2^{\sigma-1} (p+q)^\sigma,\end{aligned}$$

by taking the exponentials we obtain $(p+q)^{2^{\sigma-1}p^\sigma} \leq p^{2^{\sigma-1}p^\sigma} e^{2^{\sigma-1}(p+q)^\sigma}$. Similarly, $(p+q)^{2^{\sigma-1}q^\sigma} \leq q^{2^{\sigma-1}q^\sigma} e^{2^{\sigma-1}(p+q)^\sigma}$. Therefore

$$(p+q)^{(p+q)^\sigma} \leq p^{2^{\sigma-1}p^\sigma} q^{2^{\sigma-1}q^\sigma} e^{2^\sigma (p+q)^\sigma}$$

and $\widetilde{(M.2)}$ is proved.

□

Remark 2.1. From the proof of $\widetilde{(M.2)}'$ it follows that $M_p^{\tau,\sigma}$ does not satisfy the Komatsu condition $(M.2)'$. As a first guess one might assume that the sequence $M_p^{\tau,\sigma}$ satisfies

$$M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma+q^\sigma} M_p^{\tau,\sigma} M_q^{\tau,\sigma}, \quad C>0, p,q \in \mathbf{N}, \tag{2.5}$$

instead. Assume that (2.5) holds for $M_p^{\tau,\sigma}$, and that $\tau=1$. Then, for $p=q\neq 0$, (2.5) gives

$$p^{(2p)^\sigma} \leq (C_1 p)^{2p^\sigma}, \quad p \in \mathbf{Z}_+, \tag{2.6}$$

with $C_1 = \frac{C}{2^{2^{\sigma-1}}}$. By taking the logarithm we obtain $2^{\sigma-1}\ln p \leq \ln C_1 p$, $p\in \mathbf{Z}_+$, but this holds only for finitely many $p \in \mathbf{Z}_+$. This contradiction explains why $\widetilde{(M.2)}'$ is an appropriate substitution of $(M.2)'$ when considering $M_p^{\tau,\sigma}$.

The next simple Lemma will be used later on.

Lemma 2.3. *Let $\tau>0$ and $\sigma>1$ be fixed. Then*

$$\tilde{T}_{\tau,\sigma}(h) := \sup_{\rho>0} \frac{h^{\rho^\sigma}}{\rho^{\tau\rho^\sigma}} = e^{\frac{\tau}{\sigma e} h^{\frac{\sigma}{\tau}}} \quad, h>0. \tag{2.7}$$

Proof. Put $f(\rho) = \frac{h^{\rho^\sigma}}{\rho^{\tau\rho^\sigma}}$, $\rho>0$. Since $(\ln f(\rho))' = \rho^{\sigma-1}(\sigma \ln h - \tau\sigma\ln\rho - \tau)$, $\rho>0$, for $\rho_0 := h^{\frac{1}{\tau}} e^{-\sigma}$ we have $\max_{\rho>0} \ln f(\rho) = f(\rho_0) = \frac{\tau}{e\sigma} h^{\frac{\sigma}{\tau}}$, and Lemma is proved. □

Remark 2.2. In the theory of ultradifferentiable functions, for a given sequence M_p, $p\in\mathbf{N}$, the function T given by $T(h) = \sup_{p>0} \frac{h^p M_0}{M_p}$, $h>0$, is called *the associated function* of the sequence M_p, $p\in\mathbf{N}$ (in [4] the function $\sup_{p>0} \ln \frac{h^p M_0}{M_p}$ is considered instead of $T(h)$). It plays an important role in the study of the spaces of ultradifferentiable functions and their dual spaces. Notice that $\tilde{T}_{\tau,\sigma}$ given by (2.7) is not the associated function of the sequence $M_p^{\tau,\sigma}$. It is known that the associated function $T_\tau(h)$ of the sequence $p!^\tau$, $\tau>0$, satisfies the estimate of the form $C_1 e^{\frac{\tau}{e}h^{\frac{1}{\tau}}} \leq T_\tau(h) \leq C_2 e^{\frac{\tau}{e}h^{\frac{1}{\tau}}}$, for some $C_1, C_2>0$, and for every $h>0$, cf. [2, Chapter IV.2]. This implies that

$$C'(T_\tau(h^\sigma))^{1/\sigma} \leq \tilde{T}_{\tau,\sigma}(h) \leq C''(T_\tau(h^\sigma))^{1/\sigma}$$

for some $C', C''>0$, $h>0$ and for any given $\tau>0$, $\sigma>1$.

Next we introduce a family of spaces of ultradifferentiable functions in an analogy to the spaces $\mathcal{E}^{\{M_p\}}(U)$ and $\mathcal{D}^{\{M_p\}}(U)$ from the Introduction.

Let $U \subseteq \mathbf{R}^d$ be an open set, $K \subset\subset U$ and $h > 0$. Then $\phi \in C^\infty(U)$ belongs to the space $\mathcal{E}_{\tau,\sigma,h}(K)$ if there exists $A > 0$ such that

$$|\partial^\alpha \phi(x)| \leq A h^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}, \quad \alpha \in \mathbf{N}^d. \tag{2.8}$$

Then $\mathcal{E}_{\tau,\sigma,h}(K)$ is a Banach space with the norm given by

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma,h}(K)} = \sup_{\alpha\in\mathbf{N}^d} \sup_{x\in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}}. \tag{2.9}$$

Let $\mathcal{D}^K_{\tau,\sigma,h}$ be the set of $\phi \in C^\infty(\mathbf{R}^d)$ with support in K such that (2.8) holds for some $A > 0$.

Then we define the spaces $\mathcal{E}_{\tau,\sigma}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$of ultradifferentiable functions as follows:

(2.10)

$$\mathcal{E}_{\tau,\sigma}(U) := \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}_{\tau,\sigma,h}(K), \quad \text{and} \quad \mathcal{D}_{\tau,\sigma}(U) := \varinjlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{D}^K_{\tau,\sigma,h}.$$

Remark 2.3. The spaces $\mathcal{E}_{\tau,\sigma}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$ can be represented as projective and inductive limits of a countable family of spaces as follows, cf. [4]. Let $\{K_i\}_{i\in\mathbf{N}}$ be a sequence of compact sets with smooth boundary such that $K_i \subset K_{i+1}$, $i \in \mathbf{N}$, and $\cup_{i\in\mathbf{N}} K_i = U$. Then, for $j \in \mathbf{N}$,

$$\mathcal{E}_{\tau,\sigma}(U) = \varprojlim_{i\to\infty} \varinjlim_{j\to\infty} \mathcal{E}_{\tau,\sigma,j}(K_i), \quad \text{and} \quad \mathcal{D}_{\tau,\sigma}(U) = \varinjlim_{i\to\infty} \varinjlim_{j\to\infty} \mathcal{D}^{K_i}_{\tau,\sigma,j}.$$

Remark 2.4. From Lemma 2.1 it follows that the norms (2.9) and

$$\|\phi\|^{\sim}_{\mathcal{E}_{\tau,\sigma,h}(K)} = \sup_{\alpha\in\mathbf{N}^d} \sup_{x\in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|^\sigma} \lfloor |\alpha|^\sigma \rfloor !^{\tau/\sigma}} < \infty, \quad h > 0. \tag{2.11}$$

are equivalent in $\mathcal{E}_{\tau,\sigma,h}(K)$.

Obviously,

$$\mathcal{E}_{\tau_1,\sigma_1,h_1}(K) \hookrightarrow \mathcal{E}_{\tau_2,\sigma_2,h_2}(K)\,, \quad h_1 < h_2,\, 0 < \tau_1 < \tau_2,\, 1 < \sigma_1 < \sigma_2\,. \tag{2.12}$$

Moreover, the following proposition holds.

Proposition 2.1. *Let $\sigma_1 \geq 1$. Then for every $\sigma_2 > \sigma_1$ we have the strict embedding*

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma_1}(U) \hookrightarrow \varprojlim_{\tau\to 0^+} \mathcal{E}_{\tau,\sigma_2}(U). \tag{2.13}$$

Proof. Let $\tau > 0$ be given and let $\phi \in \mathcal{E}_{\tau_0,\sigma_1,h}(K)$ for some $h, \tau_0 > 0$ and $K \subset\subset U$. Then

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma_2,h}(K)} \leq \sup_{\alpha\in\mathbf{N}^d} \frac{h^{|\alpha|^{\sigma_1}} |\alpha|^{\tau_0|\alpha|^{\sigma_1}}}{h^{|\alpha|^{\sigma_2}} |\alpha|^{\tau|\alpha|^{\sigma_2}}} \|\phi\|_{\mathcal{E}_{\tau_0,\sigma_1,h}(K)}. \tag{2.14}$$

Put $\varepsilon := \sigma_2 - \sigma_1$. Then there is a constant $C_\varepsilon > 0$ so that

$$\tau_0 p^{\sigma_1} \ln p \leq C_\varepsilon \tau_0 p^{\sigma_1+\varepsilon} = C_\varepsilon \tau_0 p^{\sigma_2} \quad p \in \mathbf{N},$$

wherefrom $p^{\tau_0 p^{\sigma_1}} \leq e^{C_\varepsilon \tau_0 p^{\sigma_2}}$ (note that C_ε blows up if $\varepsilon \to 0^+$). If $C := e^{C_\varepsilon \tau_0}$ and $c_h := \max\{1/h, 1\}$, we have

$$\text{(2.15)} \quad \sup_{\alpha \in \mathbf{N}^d} \frac{h^{|\alpha|^{\sigma_1}} |\alpha|^{\tau_0 |\alpha|^{\sigma_1}}}{h^{|\alpha|^{\sigma_2}} |\alpha|^{\tau |\alpha|^{\sigma_2}}} \leq \sup_{\alpha \in \mathbf{N}^d} \frac{(C\, c_h)^{|\alpha|^{\sigma_2}}}{|\alpha|^{\tau |\alpha|^{\sigma_2}}} \leq \tilde{T}_{\tau,\sigma_2}(Cc_h) = e^{\frac{\tau}{e\sigma_2}(C\, c_h)^{\frac{\sigma_2}{\tau}}},$$

where $\tilde{T}_{\tau,\sigma}$ is given in (2.7). Now by (2.14) and (2.15) it follows that $\mathcal{E}_{\tau_0,\sigma_1,h}(K) \hookrightarrow \mathcal{E}_{\tau,\sigma_2,h}(K)$, for arbitrary $\tau, \tau_0 > 0$, which implies (2.13), the embedding obviously being strict. □

As an immediate consequence we obtain that

$$\text{(2.16)} \qquad \mathcal{E}_{\tau_0,\sigma_1}(U) \hookrightarrow \bigcap_{\tau > \tau_0} \mathcal{E}_{\tau,\sigma_1}(U) \hookrightarrow \mathcal{E}_{\tau_0,\sigma_2}(U),$$

for any $\tau_0 > 0$ whenever $\sigma_2 > \sigma_1 \geq 1$. In particular, if $\mathcal{E}^{\{p!^t\}}(U)$, $t > 1$, is the Gevrey space of ultradifferentiable functions on U, then, for every $\tau > 0$ and $\sigma > 1$ we have

$$\text{(2.17)} \qquad \bigcup_{t>1} \mathcal{E}^{\{p!^t\}}(U) \hookrightarrow \mathcal{E}_{\tau,\sigma}(U).$$

Furthermore, with the notation $\mathcal{E}_{\infty,\sigma}(U) := \varinjlim_{\tau \to \infty} \mathcal{E}_{\tau,\sigma}(U)$, Proposition 2.1 implies that for fixed $\tau_0 > 0$ and $\sigma_0 > 1$, we have

$$\varinjlim_{t \to \infty} \mathcal{E}^{\{p!^t\}}(U) \hookrightarrow \varprojlim_{\sigma \to 1^+} \mathcal{E}_{\infty,\sigma}(U) \hookrightarrow \mathcal{E}_{\infty,\sigma_0}(U) \hookrightarrow \varinjlim_{\sigma \to \infty} \mathcal{E}_{\tau,\sigma}(U) \hookrightarrow C^\infty(U).$$

Note that our classes are larger then Gevrey classes but their inductive limits with respect to τ and σ are continuously embedded in $C^\infty(U)$.

3. Basic properties of the new classes of ultradifferentiable functions

In this section we study the basic properties of $\mathcal{E}_{\tau,\sigma}(U)$. In separate subsections we show that $\mathcal{E}_{\tau,\sigma}(U)$ are non-quasianalytic and nuclear spaces, closed under differentiation and pointwise multiplication. Finally, we study the action of ultradifferentiable operators on $\mathcal{E}_{\tau,\sigma}(U)$.

3.1. Compactly supported test functions

In this subsection we construct a compactly supported function in $\mathcal{E}_{\tau,\sigma}(U)$ following the ideas presented in [5]. We begin by showing that the sequence $M_p^{\tau,\sigma}$ satisfies the non-quasianalyticity condition.

Lemma 3.1. *Let $\tau > 0$, $\sigma > 1$ and let $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$. Then*

$$(M.3)' \quad \sum_{p=1}^{\infty} \frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}} < \infty. \tag{3.1}$$

Proof. Since $p^\sigma \ln(1+\frac{1}{p}) = p^{\sigma-1}\ln(1+\frac{1}{p})^p$, $p \in \mathbf{Z}_+$, we have

$$\tau\, p^{\sigma-1} \ln 2 \le \tau p^\sigma \ln\left(1+\frac{1}{p}\right) \le \tau p^{\sigma-1}, \quad p \in \mathbf{Z}_+, \tag{3.2}$$

so that

$$2^{\tau p^{\sigma-1}} \le \left(1+\frac{1}{p}\right)^{\tau p^\sigma} \le e^{\tau p^{\sigma-1}}, \quad p \in \mathbf{Z}_+. \tag{3.3}$$

The left hand side of (3.3) and $p^\sigma \ge (p-1)^{\sigma-1}p = (p-1)^\sigma + (p-1)^{\sigma-1}$, $p \in \mathbf{Z}_+$, give

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{(p-1)^{\tau(p-1)^\sigma}}{p^{\tau p^\sigma}} &\le \sum_{p=1}^{\infty} \frac{(p-1)^{\tau(p-1)^\sigma}}{p^{\tau((p-1)^\sigma+(p-1)^{\sigma-1})}} \\ &= \sum_{p=1}^{\infty} \left((1-\frac{1}{p})^{\tau(p-1)^\sigma}\right) \frac{1}{p^{\tau(p-1)^{\sigma-1}}} \\ &\le \sum_{p=1}^{\infty} \frac{1}{(2p)^{\tau(p-1)^{\sigma-1}}} < \infty, \end{aligned} \tag{3.4}$$

which implies (3.1). □

Corollary 3.1. *There exists a compactly supported function $\phi \in \mathcal{E}_{\{\tau,\sigma\}}(U)$ such that $0 \le \phi \le 1$ and $\int_{\mathbf{R}^d} \phi\, dx = 1$.*

Proof. Our goal is to construct a function in $\mathcal{D}_{\tau,\sigma}(U)$ which is not in $\mathcal{D}^{\{p!^t\}}(U)$, for any $t > 1$. We follow the ideas from [5, Theorem 1.3.5], and repeat some steps of its proof to make the exposition self contained.

We start with one dimensional case. Let χ be the characteristic function of interval $(0,1)$, and for $c > 0$ let $H_c(x) = \frac{1}{c}\chi(\frac{x}{c})$. Clearly $\int_{\mathbf{R}} H_c dx = 1$ and we recall that

$$(H_c * f)'(x) = \frac{1}{c}\left(\int_{x-c}^{x} f(t)dt\right)' = \frac{f(x)-f(x-c)}{c}, \tag{3.5}$$

for any continuous function f on $\mathbf{R}$.

Further we set

$$a_p := \frac{1}{(2(p+1))^{\tau p^{\sigma-1}}}, \quad p \in \mathbf{N},$$

and note that (3.4) implies

$$\frac{M_p^{\tau,\sigma}}{M_{p+1}^{\tau,\sigma}} \le a_p, \quad p \in \mathbf{N}. \tag{3.6}$$

Put $u_m(x) = H_{a_0} * H_{a_1} \cdots * H_{a_m}$, $m \in \mathbf{N}$. Then, by [5, Theorem 1.3.5] it follows that the sequence $\{u_m\}_{m\in\mathbf{N}}$ has a uniform limit $u \in C^\infty(\mathbf{R})$ supported in $[0,a]$ where $a = \sum_{p=0}^{\infty} a_p < \infty$, and $\int_{\mathbf{R}} u\,dx = 1$.

Next we estimate the derivatives $u_m^{(p)}$, $p \le m-1$. After applying p iterations of (3.5) and by using (3.6) we obtain

$$\begin{aligned}
|u_m^{(p)}(x)| &= \prod_{k=0}^{p-1} \frac{|H_{a_p} * \cdots * H_{a_m}(x) - H_{a_p} * \cdots * H_{a_m}(x-a_k)|}{a_k} \\
&\le 2^p \prod_{k=0}^{p-1} \frac{1}{a_k} \sup_{x\in\mathbf{R}} |H_{a_p} * H_{a_{p+1}} \cdots * H_{a_m}(x)| \\
&\le 2^p \Big(\prod_{k=0}^{p-1} \frac{1}{a_k}\Big) \sup_{x\in\mathbf{R}} |H_{a_p}(x)| \Big(\prod_{k=p+1}^{m} \int_{\mathbf{R}} H_{a_k}(x)dx\Big) \\
&= 2^p \prod_{k=0}^{p} \frac{1}{a_k} \le 2^p \prod_{k=0}^{p} \frac{M_{k+1}^{\tau,\sigma}}{M_k^{\tau,\sigma}} = 2^p \frac{M_{p+1}^{\tau,\sigma}}{M_0^{\tau,\sigma}} \\
&= 2^p (p+1)^{\tau(p+1)^\sigma} \le C^{p^\sigma} p^{\tau p^\sigma}, \qquad (3.7)
\end{aligned}$$

where $\widetilde{(M.2)}'$ is used in the last inequality.

From the uniform convergence it follows that the derivatives of u also satisfy (3.7), so that $u \in \mathcal{D}^{[0,a]}_{\tau,\sigma,C}$.

Next, we extend this to higher dimensions by putting $\psi(x) = u(x+a/2)$ and $\phi(x) = \prod_{k=1}^d \psi(x_k)$ for $x = (x_1, x_2, \ldots, x_d)$. Since the sequence $M_p^{\tau,\sigma}$ fulfills the $(M.1)$ property, we obtain

$$|\partial^\alpha \phi(x)| = \prod_{k=1}^{d} |\partial^{\alpha_k} \psi(x_k)| \le \prod_{k=1}^{d} C^{{\alpha_k}^\sigma} {\alpha_k}^{\tau {\alpha_k}^\sigma} \le C^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma},$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d),$$

wherefrom $\phi \in \mathcal{D}^K_{\tau,\sigma,C}$ with $K = [-a/2, a/2]^d$.

Although K does not have a smooth boundary, by [5, Lemma 1.4.3] one can find an appropriate open set U and conclude that $\phi \in \mathcal{D}_{\tau,\sigma}(U)$.

At the same time, $\phi \notin \mathcal{D}^{\{p!^t\}}(U)$, for any $t > 1$. Otherwise the derivatives $u_m^{(p)}$ in (3.7) should be bounded by $C^p p!^t = \prod_{k=0}^{p-1} C(k+1)^t$, for some $C > 0$, $t > 1$, and for arbitrary large $m \in \mathbf{N}$. In that case, the estimates in (3.7) would imply $(k+1)^t > C(2(k+1))^{\tau k^\sigma}$, which is obviously not true for k large enough.

We refer to [5, Lemma 1.3.6] for a discussion about the precision of the presented construction. □

3.2. Nuclearity

In this subsection we show that the spaces in (2.10) are nuclear. This is in agreement with Komatsu's result about the nuclearity of $\mathcal{E}^{\{M_p\}}(U)$ when M_p satisfies $(M.2)'$ (see [4, Theorem 2.6]).

Let us show that for every $h > 0$ there exists $k > h$ such that identity mapping $X \to Y$ is quasi-nuclear, where $X = \mathcal{E}_{\tau,\sigma,h}(K)$ and $Y = \mathcal{E}_{\tau,\sigma,k}(K)$ (resp. $X = \mathcal{D}^K_{\tau,\sigma,h}$ and $Y = \mathcal{D}^K_{\tau,\sigma,k}$). This means that seminorms on $\mathcal{E}_{\tau,\sigma}(K) := \varinjlim_{h\to\infty} \mathcal{E}_{\tau,\sigma,h}(K)$ (resp. $\mathcal{D}^K_{\tau,\sigma} := \varinjlim_{h\to\infty} \mathcal{D}^K_{\tau,\sigma,h}$) are prenuclear, cf. [11, page 177]. By [11, Theorem IV 10.2] this implies that $\mathcal{E}_{\tau,\sigma}(K)$ (resp. $\mathcal{D}^K_{\tau,\sigma}$) is a nuclear space.

The classes under consideration can be represented as projective and inductive limits of a countable family of spaces, cf. Remark 2.3. The nuclearity of $\mathcal{E}_{\tau,\sigma}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$ then follows from [11, Theorem III 7.4].

Theorem 3.1. *The spaces $\mathcal{E}_{\tau,\sigma}(U)$, $\mathcal{D}^K_{\tau,\sigma}$ and $\mathcal{D}_{\tau,\sigma}(U)$ are nuclear.*

Proof. We follow the idea presented in [4]. Let $\phi \in \mathcal{E}_{\tau,\sigma,h}(K)$ and let $u_{\alpha,j}$, $\alpha \in \mathbf{N}^d$, $j \in \mathbf{Z}^d$, be the sequence of linear functionals on $\mathcal{E}_{\tau,\sigma,h}(K)$ given by

$$\langle \phi, u_{\alpha,j}\rangle = \frac{\langle \partial^\alpha \phi, v_j\rangle_{C^{d+1}(K)}}{k^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}\,, \tag{3.8}$$

where $v_j \in (C^{d+1}(K))'$ is defined by the following procedure:

Choose $l > 0$ such that K is contained in the interior of $L = [-l, l]^d$ and let $C^{d+1}_L(\pi L)$ be the space of all $d + 1$ times differentiable functions on πL with support in L. Let $B \in \mathcal{L}(C^{d+1}(K), C^{d+1}_L(\pi L))$ be the (Whitney's) extension operator such that $Bf|_K = f$ and let $t_j \in (C^{d+1}_L(\pi L))'$ be given by

$$\langle f, t_j\rangle := \int_{\pi L} f(y)e^{-iyj/l}\,dy, \quad j \in \mathbf{Z}^d.$$

From [4, Lemma 2.3] it follows that the identity operator from $C^{d+1}(K)$ to $C(K)$, given by

$$f(x) = \frac{1}{(2\pi l)^d}\sum_{j\in\mathbf{Z}^d} e^{-ixj/l}\langle Bf, t_j\rangle_{C^{d+1}_L(\pi L)}, \quad x \in K,$$

is quasi-nuclear. In particular, if we put $v_j = t_j \circ B$, $j \in \mathbf{Z}^d$, it follows that

$$\sum_{j\in\mathbf{Z}^d}^{\infty} ||v_j||_{(C^{d+1}(K))'} < \infty, \quad \text{and} \quad ||f||_{C(K)} \le \sum_{j\in\mathbf{Z}^d}^{\infty} |\langle f, v_j\rangle_{C^{d+1}(K)}|\,. \tag{3.9}$$

By (2.9), (3.8) and the righthand side of (3.9) we obtain

$$\|\phi\|_Y = \sup_{\alpha\in\mathbf{N}^d}\sup_{x\in K}\frac{|\partial^\alpha\phi(x)|}{k^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}} \le \sum_{\alpha\in\mathbf{N}^d}\sum_{j\in\mathbf{Z}^d}|\langle\phi, u_{\alpha,j}\rangle|\,.$$

It remains to show that $\sum\limits_{\alpha,j}||u_{\alpha,j}||_{X'}<\infty$.

Note that for $|\alpha|\geq 1$, $\alpha\in\mathbf{N}^d$ and $h\geq 1$, by $\widetilde{(M.2)'}$ we obtain

$$\begin{aligned}|\langle\phi,u_{\alpha,j}\rangle| &\leq \sup_{|\beta|\leq d+1}\frac{h^{|\alpha+\beta|^\sigma}|\alpha+\beta|^{\tau|\alpha+\beta|^\sigma}}{k^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}||\phi||_X||v_j||_{(C^{d+1}(K))'}\\ &= \sup_{|\beta|\leq d+1}\frac{h^{(1+\frac{|\beta|}{|\alpha|})^\sigma|\alpha|^\sigma}|\alpha+\beta|^{\tau|\alpha+\beta|^\sigma}}{k^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}||\phi||_X||v_j||_{(C^{d+1}(K))'}\\ &\leq \Big(\frac{h^{(d+2)^\sigma}}{k}\Big)^{|\alpha|^\sigma}C_d^{|\alpha|^\sigma}||\phi||_X||v_j||_{(C^{d+1}(K))'},\end{aligned}\tag{3.10}$$

for some $C_d>1$.

For $0<h<1$, note that $h^{|\alpha+\beta|^\sigma}\leq h^{|\alpha|^\sigma}$ and thus by $\widetilde{(M.2)'}$ it follows that

$$\sup_{|\beta|\leq d+1}\frac{h^{|\alpha+\beta|^\sigma}|\alpha+\beta|^{\tau|\alpha+\beta|^\sigma}}{k^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}}\leq\Big(\frac{h}{k}\Big)^{|\alpha|^\sigma}C_d^{|\alpha|^\sigma}.\tag{3.11}$$

Now we choose $k>0$ such that $k>\max\{2C_dh,2C_dh^{(d+2)^\sigma}\}$, so the estimates (3.10), and (3.11) imply

$$\sum_{\alpha\in\mathbf{N}^d}\sum_{j\in\mathbf{Z}^d}||u_{\alpha,j}||_{X'}<\sum_{\alpha\in\mathbf{N}^d}\sum_{j\in\mathbf{Z}^d}\Big(\frac{1}{2}\Big)^{|\alpha|^\sigma}||v_j||_{(C^{d+1}(K))'}<\infty.$$

We conclude that for every $h>0$ there exists $k>h$ such that the identity mapping $\mathcal{E}_{\tau,\sigma,h}(K)\to\mathcal{E}_{\tau,\sigma,k}(K)$ (resp. $\mathcal{D}^K_{\tau,\sigma,h}\to\mathcal{D}^K_{\tau,\sigma,k}$) is quasi-nuclear, and the theorem is proved. □

3.3. Algebra property

Since $M_p^{\tau,\sigma}$ satisfies properties $(M.1)$ and $\widetilde{(M.2)'}$ we have the following.

Proposition 3.1. *$\mathcal{E}_{\tau,\sigma}(U)$ is closed under the pointwise multiplication of functions and under the (finite order) differentiation.*

Proof. Let $\phi\in\mathcal{E}_{\tau,\sigma,h}(K)$ and $\psi\in\mathcal{E}_{\tau,\sigma,k}(K)$ for some $h,k>1$, and, for simplicity, assume that $\tau=1$.

We first show that

$$\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}h^{|\alpha-\beta|^\sigma}k^{|\beta|^\sigma}\leq(h+k)^{|\alpha|^\sigma},\quad h,k>1.\tag{3.12}$$

In fact,

$$\begin{aligned}\Big(\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}h^{|\alpha-\beta|^\sigma}k^{|\beta|^\sigma}\Big)^{\frac{1}{|\alpha|^{\sigma-1}}} &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}h^{|\alpha|(1-\frac{|\beta|}{|\alpha|})^\sigma}k^{|\beta|(\frac{|\beta|}{|\alpha|})^{\sigma-1}}\\ &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}h^{|\alpha-\beta|}k^{|\beta|}=(h+k)^{|\alpha|}.\end{aligned}\tag{3.13}$$

where the last inequality follows from $|\beta| \leq |\alpha|$ and

$$|\alpha|(1 - \frac{|\beta|}{|\alpha|})^{\sigma} \leq |\alpha|(1 - \frac{|\beta|}{|\alpha|}) = |\alpha - \beta|.$$

Now (3.12) follows from (3.13).

By the Leibnitz formula and $(M.1)$ we have

$$\begin{aligned}
|\partial^{\alpha}(\phi\psi)(x)| &\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta}\phi(x)||\partial^{\beta}\psi(x)| \\
&\leq AB \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} h^{|\alpha-\beta|^{\sigma}} |\alpha-\beta|^{|\alpha-\beta|^{\sigma}} k^{|\beta|^{\sigma}} |\beta|^{|\beta|^{\sigma}} \\
&\leq AB|\alpha|^{|\alpha|^{\sigma}} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} h^{|\alpha-\beta|^{\sigma}} k^{|\beta|^{\sigma}}, \quad x \in K. \qquad (3.14)
\end{aligned}$$

Now $\phi\psi \in \mathcal{E}_{\tau,\sigma}(U)$ follows from (3.12).

Next, for any $\beta \in \mathbf{N}^d$, $x \in K$ and $h > 1$ we have

$$\begin{aligned}
|\partial^{\alpha}(\partial^{\beta}\phi(x))| &\leq C h^{|\alpha+\beta|^{\sigma}} |\alpha+\beta|^{|\alpha+\beta|^{\sigma}} \\
&\leq C h^{2^{\sigma-1}|\alpha|^{\sigma}} h^{2^{\sigma-1}|\beta|^{\sigma}} C_{\beta}^{|\alpha|^{\sigma}} |\alpha|^{|\alpha|^{\sigma}},
\end{aligned}$$

where we used $\widetilde{(M.2)'}$ in the last inequality, which proves the Proposition. □

Remark 3.1. Let $P = \sum_{|\alpha|\leq m} a_{\alpha}(x)\partial^{\alpha}$ be the partial differential operator of order m with $a_{\alpha} \in \mathcal{E}_{\tau,\sigma}(U)$. Then, by the proof of Proposition 3.1, it follows that $P : \mathcal{E}_{\tau,\sigma}(U) \to \mathcal{E}_{\tau,\sigma}(U)$ is a continuous and linear map with respect to the topology of $\mathcal{E}_{\tau,\sigma}(U)$. Moreover, if $\phi \in \mathcal{E}_{\tau,\sigma}(U)$ and $\psi \in \mathcal{D}^K_{\tau,\sigma}$, then $\phi\psi \in \mathcal{D}^K_{\tau,\sigma}$. In particular, if $a_{\alpha} \in \mathcal{D}^K_{\tau,\sigma}$, $|\alpha| \leq m$, then $P : \mathcal{E}_{\tau,\sigma}(U) \to \mathcal{D}^K_{\tau,\sigma}$ is also continuous and linear.

3.4. Ultradifferentiable property

In this subsection we study the continuity properties of certain ultradifferentiable operators $P(x,\partial)$ acting on $\mathcal{E}_{\tau,\sigma}(U)$. Recall, if the defining sequence M_p fulfills the condition $(M.2)'$ then the corresponding test function space is closed under the action of ultradifferentiable operators. Since the sequence $M_p^{\tau,\sigma}$ does not satisfy (2.5), the space $\mathcal{E}_{\tau,\sigma}(U)$ can not be closed under the action of $P(x,\partial)$. However, if we consider $\mathcal{E}_{\infty,\sigma}(U) := \varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma}(U)$ then the following results hold true.

Theorem 3.2. *Let $P(x,\partial) = \sum_{|\alpha|=0}^{\infty} a_{\alpha}\partial^{\alpha}$ be a differential operator of infinite order with constant coefficients such that for every $L > 0$ there exists $A > 0$ so that*

$$|a_{\alpha}| \leq A \frac{L^{|\alpha|^{\sigma}}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^{\sigma}}}. \qquad (3.15)$$

Then $\mathcal{E}_{\infty,\sigma}(U)$ is closed under action of $P(x,\partial)$. In particular, the mapping

$$P(x,\partial): \quad \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U)\,, \tag{3.16}$$

is continuous.

Proof. Let $\phi \in \mathcal{E}_{\tau,\sigma,h}(K)$ for some $h>0$. Then using $\widetilde{(M.2)}$ we have

$$\begin{aligned}
&|\partial^\beta(a_\alpha \partial^\alpha \phi(x))| \\
&\leq \; A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}} h^{|\alpha+\beta|^\sigma} (|\alpha+\beta|)^{\tau|\alpha+\beta|^\sigma} \\
&\leq \; A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} \frac{L^{|\alpha|^\sigma} C^{|\alpha|^\sigma+|\beta|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}} h^{|\alpha+\beta|^\sigma} |\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma} |\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma} \\
&\leq \; A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)} C^{|\beta|^\sigma} h^{2^{\sigma-1}|\beta|^\sigma} |\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma} (CLh^{2^{\sigma-1}})^{|\alpha|^\sigma},
\end{aligned}$$

where $x\in K$ and C is the constant in $\widetilde{(M.2)}$. Now we choose $L>0$ such that

$$CLh^{2^{\sigma-1}} \leq 1/2$$

and the theorem is proved. □

By the use of $(M.1)$ we are able to extend Theorem 3.2 to a class of operators with non-constant coefficients.

Definition 3.1. A differential operator of infinite order

$$P(x,\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x)\partial^\alpha \tag{3.17}$$

is an ultradifferential operator of class $\{\tau,\sigma\}$ on $U\subseteq \mathbf{R}^d$ if $a_\alpha \in \mathcal{E}_{\tau,\sigma}(U)$, $\alpha\in\mathbf{N}^d$, and for every $K\subset\subset U$ there exists $h>0$ such that for any $L>0$ there exists $A>0$ such that

$$\sup_{x\in K} |\partial^\beta a_\alpha(x)| \leq A h^{|\beta|^\sigma} |\beta|^{\tau|\beta|^\sigma} \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}, \quad \alpha,\beta\in\mathbf{N}^d. \tag{3.18}$$

Theorem 3.3. *Let $P(x,\partial)$ be an ultradifferential operator of class $\{\tau,\sigma\}$. Then $\mathcal{E}_{\infty,\sigma}(U)$ is closed under action of $P(x,\partial)$. In particular, the mapping*

$$P(x,\partial): \quad \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U)\,, \tag{3.19}$$

is continuous.

Proof. We are following the idea given in the proof of Theorem 2.4 in [3]. Let $a_\alpha \in \mathcal{E}_{\tau,\sigma,h}(K)$, $\alpha\in\mathbf{N}^d$, so that (3.18) holds, and let $\phi\in\mathcal{E}_{\tau,\sigma,k}(K)$, for some $k>0$ which will be determined later on. Then, by $(M.1)'$ and $\widetilde{(M.2)}$ we obtain

$$
\begin{aligned}
&|\partial^\beta(a_\alpha(x)\partial^\alpha\phi(x))| \\
&\quad \leq \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}|\partial^{\beta-\gamma}a_\alpha(x)||\partial^{\alpha+\gamma}\phi(x)| \\
&\quad \leq A||\phi||_{\mathcal{E}_{\tau,\sigma,k}(K)}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma}(|\beta-\gamma|)^{\tau|\beta-\gamma|^\sigma} \\
&\quad \cdot \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}k^{|\alpha+\gamma|^\sigma}(|\alpha+\gamma|)^{\tau|\alpha+\gamma|^\sigma} \\
&\quad \leq A||\phi||_{\mathcal{E}_{\tau,\sigma,k}(K)}\frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}(|\alpha+\beta|)^{\tau|\alpha+\beta|^\sigma} \\
&\quad \cdot \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma}k^{|\alpha+\gamma|^\sigma} \\
&\quad \leq A||\phi||_{\mathcal{E}_{\tau,\sigma,k}(K)}(CL)^{|\alpha|^\sigma}C^{|\beta|^\sigma}|\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma}C_{h,k,\beta}, \quad x\in K, \qquad (3.20)
\end{aligned}
$$

for some constant $C>0$ which does not depend on α,β and K and we put

$$C_{h,k,\beta}=\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma}k^{|\alpha+\gamma|^\sigma}.$$

It remains to estimate $C_{h,k,\beta}$. Without loss of generality we may assume that $h>1$ and $k\geq h$. Since

$$|\beta-\gamma|^\sigma+|\alpha+\gamma|^\sigma\leq(|\beta|-|\gamma|+|\alpha|+|\gamma|)^\sigma=(|\alpha|+|\beta|)^\sigma\leq 2^{\sigma-1}(|\alpha|^\sigma+|\beta|^\sigma),$$

it follows that $C_{h,k,\beta}\leq 2^{|\beta|}k^{2^{\sigma-1}(|\alpha|^\sigma+|\beta|^\sigma)}$. Now, (3.20) implies

$$|\partial^\beta(a_\alpha(x)\partial^\alpha\phi(x))|\leq B||\phi||_{\mathcal{E}_{\tau,\sigma,k}(K)}(k^{2^{\sigma-1}}CL)^{|\alpha|^\sigma}(2Ck^{2^{\sigma-1}})^{|\beta|^\sigma}|\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma}.$$

By choosing $L>0$ such that $LCk^{2^{\sigma-1}}<1/2$ holds and then taking the sum with respect to α and the supremum with respect to β and $x\in K$, we obtain

$$||P(\partial)\phi||_{\mathcal{E}_{\tau 2^{\sigma-1},\sigma,l}(K)}\leq\tilde{C}||\phi||_{\mathcal{E}_{\tau,\sigma,k}(K)},$$

for some $\tilde{C}>0$, where $l=2Ck^{2^{\sigma-1}}$. This proves the theorem. □

4. Motivation

To conclude this paper, we present a motivation for studying $\mathcal{E}_{\tau,\sigma}(U)$ which comes from microlocal analysis. It is known that for a Schwartz distribution u we have

$$\mathrm{WF}(u)\subseteq\mathrm{WF}_t(u)\subseteq\mathrm{WF}_A(u)\,, \qquad (4.1)$$

where WF_t, $t > 1$ is the Gevrey wave front set, WF_A is the analytic wave front set (see Introduction), and WF is the standard C^∞ wave front (see [5, 7]). In [6], the inclusion on the righthand side of (4.1) is extended to Gevrey type ultradistributions.

In the existing literature there are no wave front sets which detect singularities that are "heavier" than classical C^∞ singularities and "lighter" than Gevrey type singularities. The "heavier" singularities related to $t < 1$ are considered in [9]. The "lighter" singularities can be studied within the framework of $\mathcal{E}_{\tau,\sigma}(U)$ and its dual space of Roumier ultradistributions. Let us explain the main idea of our approach and leave more details to the forthcoming paper [10]

In the study of regularity properties (as opposed to the singularity properties) of a function (or of a distribution), one is interested in the points (x_0, ξ_0) in which the decrease of $|\widehat{u\phi}(\xi)|$ ($\phi = 1$ in a neighborhood if x_0) is faster than $|\xi|^{-N}$, $|\xi| \to \infty$, for any $N \in \mathbf{N}$, but, at the same time, slower than $e^{-|\xi|^{1/t}}$, $|\xi| \to \infty$, for any $t > 1$. In other words, u is locally more than being C^∞, but less than being Gevrey ultradifferentiable.

We therefore start with the following decay properties on the Fourier transform side. Let $u \in \mathcal{D}'^{\{p!^t\}}(U)$ and let $\{u_N\}_{N\in\mathbf{N}}$ be a sequence of compactly supported smooth functions such that $u_N = u$ on Ω, and such that some of the following regularity conditions hold:

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^{N^t}\lfloor N^t\rfloor!}{|\xi|^{\lfloor N^t\rfloor}}, \quad N \in \mathbf{N},\, t > 0 \tag{4.2}$$

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^N N!^t}{|\xi|^N}, \quad N \in \mathbf{N},\, t > 1 \tag{4.3}$$

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^N N!^t}{|\xi|^{\lfloor N^t\rfloor}}, \quad N \in \mathbf{N},\, 0 < t < 1. \tag{4.4}$$

for some $A, h > 0$ and $\xi \in \mathbf{R}^d \backslash \{0\}$.

We note that the condition (4.3) is related to the Gevrey wave front WF_t, that is to the Gevrey type regularity for $t > 1$ (see Introduction).

Next, note that if in (4.2) we put $N^{1/t}$ instead of N we get

$$|\widehat{u_{N^{1/t}}}(\xi)| \le A\,\frac{h^N N!}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\}. \tag{4.5}$$

We call this process *enumeration.* We may also say that after *the left hand side enumeration* $N \to N^{1/t}$, (4.2) is equivalent to local analyticity.

If we now apply the same enumeration $N \to N^{1/t}$ in (4.4) then we obtain

$$|\widehat{u_{N^{1/t}}}(\xi)| \le A\,\frac{h^{N^{1/t}}\lfloor N^{1/t}\rfloor!^t}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\}, \tag{4.6}$$

that is, after the enumeration and with $\sigma = 1/t > 1$, from Lemma 2.1 and Remark 2.4, it follows that (4.4) is equivalent to

$$|\widehat{u_{N^\sigma}}(\xi)| \leq B\,\frac{k^{N^\sigma} N^{N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\}. \tag{4.7}$$

Now, if in (4.5) and (4.7) we write u_N instead of u_{N^σ} we conclude that

$$(4.2) \Rightarrow (4.3) \Rightarrow (4.4)\,.$$

Moreover, when $\tau > 0$ and $\sigma > 1$, we propose a new regularity condition:

$$|\widehat{u}_N(\xi)| \leq A\,\frac{h^N N!^{1/\sigma}}{|\xi|^{\lfloor (N/\tau)^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\}. \tag{4.8}$$

By similar arguments as above we note that after the enumeration $N \to \tau N^\sigma$, the condition (4.8) is equivalent to

$$|\widehat{u}_{\tau N^\sigma}(\xi)| \leq A\,\frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\}. \tag{4.9}$$

for some $A, h > 0$, and thus, if we write u_N instead of $u_{\tau N^\sigma}$ in (4.9), we obtain (4.2) $\Rightarrow$ (4.3) $\Rightarrow$ (4.8). Note that for $\sigma = 1/t$, (4.8) $\Leftrightarrow$ (4.4) when $\tau = 1$, while (4.8) $\Rightarrow$ (4.4) when $\tau \in (0, 1)$.

Thus, the regularity condition (4.8) can be used to define a new type of wave front sets in $\mathcal{D}'^{\{p!^t\}}(U)$. We recall that the idea behind the condition (4.4) is to construct a (bounded) sequence of cutoff functions in $\mathcal{D}^{\{p!^t\}}(U)$ similar to the one constructed in [5] for WF_A (see the proof of [5, Proposition 8.4.2.]).

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