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ON BILIPSCHICITY OF QUASICONFORMAL HARMONIC MAPPINGS[1]

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In honor of Professor Bogoljub Stanković on the occasion of his 90th birthday

Abstract. We show that quasiconformal harmonic mappings on domains in $\mathbb{R}^2$ are bilipschitz with respect to euclidean metric on those parts of the domain where the boundary is flat.

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1. Introduction

Continuity properties of harmonic quasiconformal mappings $f : D \longrightarrow D'$, where D and D' are domains in plane, with respect to various natural metrics have been studied extensively in [5], [6], [7], [8], [9], [10] and [11]. Since the inverse of K-quasiconformal mapping is also K-quasiconformal mapping, such results apply at the same time to f and f^{-1}. Note that if f is harmonic then f^{-1} is not in general harmonic.

We will consider a method to achieve local bilipschitz behaviour when part of the boundary is flat. This is local generalization of the work of Kalaj and Pavlović [7]. Our philosophy is to use the boundary Harnack inequality for this problem.

The following theorem will be important for proving our main results.

Theorem 1.1. *[9] Let $f : \Omega \longrightarrow \mathbb{C}$ be a harmonic map whose Jacobian determinant $J = |f_z|^2 - |f_{\bar z}|^2$ is positive everywhere in Ω. Then $\log J$ is a superharmonic function.*

This theorem has many applications. One of these is to prove that quasiconformal harmonic mappings on proper domains in $\mathbb{R}^2$ are bi-Lipschitz with respect to the quasihyperbolic metric [9, Theorem 1].

Another application is in establishing the minimum principle for the Jacobian determinant which is the novelty for the new analytic proof of celebrated Radó–Kneser–Choquet theorem [4].

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It is also used for studying higher dimensional counterparts [2] to the well-known theorem of Pavlovic [11], that every harmonic quasiconformal mapping of the disk is bi-Lipschitz.

We next recall definition from [1, Definition 1.5]

$$\alpha_f(z) = \exp\left(\frac{1}{n}(\log J_f)_{B_z}\right),$$

where

$$\log(J_f)_{B_z} = \frac{1}{m(B_z)}\int_{B_z} \log J_f \, dm, \quad B_z = B(z, d(z, \partial D)).$$

In the case $n = 2$ we have

$$\frac{1}{\alpha_f(z)} = \exp\left(\frac{1}{2}\frac{1}{m(B_z)}\int_{B_z} \log\frac{1}{J_f(w)}\, dm(w)\right).$$

For our main results we also need the counterpart of Koebe theorem established by Astala and Gehring.

Theorem 1.2. *[1, Theorem 1.8] Suppose that D and D' are domains in $\mathbb{R}^n$ if $f : D \longrightarrow D'$ is K-qc, then*

$$\frac{1}{c}\frac{d(f(z), \partial D')}{d(z, \partial D)} \leq \alpha_f(z) \leq c\,\frac{d(f(z), \partial D')}{d(z, \partial D)}$$

for $z \in D$, where c is a constant which depends only on K and n.

2. Main Results

We are going to need the following boundary Harnack inequality ([3], exercise 6, p. 28):

Theorem 2.1. *Let u and v be positive harmonic functions on unit disk $\mathbb{D}$ in $\mathbb{R}^2$ with $u(0) = v(0)$, let $I \subset \partial\mathbb{D}$ be an open arc and assume*

$$\lim_{z\to\zeta} u(z) = \lim_{z\to\zeta} v(z) = 0$$

for all $\zeta \in I$. Then for every compact $A \subset \mathbb{D} \cup I$ there is a constant $C(A)$ independent of u and v such that on $A \cap \mathbb{D}$

$$\frac{1}{C(A)} \leq \frac{u(z)}{v(z)} \leq C(A).$$

Proof. We are going to consider the case $I = \partial\mathbb{D} \cap \mathbb{H}_-$, $\mathbb{H}_- = \{z : Im(z) < 0\}$.

Since u is positive and harmonic, we have $u(z) = \int_{S^1} P_z(t)d\mu(t)$, where μ is positive measure, with $u(0) = \int_{S^1} d\mu = \mu(S^1)$, and similarly v is defined via positive measure ν.

Suppose $v, u \geq 0$ are harmonic in $\mathbb{D}$ and $u\Big|_I = v\Big|_I = 0$, $u(0) = v(0) = 1$, i.e. $\mu(S^1) = \nu(S^1) = 1$. Since u is harmonic and μ is supported on $\{z : \mathrm{Im}(z) \geq 0, |z| = 1\} = S^1_+$, we have

$$u(z) = \int_{S^1} \frac{1-|z|^2}{|\xi - z|^2} d\mu(\xi) = \int_{S^1_+} \frac{1-|z|^2}{|\xi - z|^2} d\mu(\xi)$$

For $\delta_0 = dist(A, supp(\mu))$ and $z \in A$ we have $dist(z, S^1_+) \geq \delta_0 > 0$ and

$$u(z) \leq (1-|z|^2) \int_{S^1_+} \frac{1}{|\xi - z|^2} \, d\mu(\xi) \leq \frac{2(1-|z|)}{\delta_0^2}$$

Since $|\xi - z| \leq 2$, we have

$$v(z) \geq (1-|z|) \int_{S^1_+} \frac{1}{|\xi - z|^2} \, d\nu(\xi) \geq \frac{1-|z|}{4}$$

and we conclude that for $z \in A$ we have $u(z)/v(z) \leq \frac{8}{\delta_0^2}$ and analogously $v(z)/u(z) \leq \frac{8}{\delta_0^2}$, and hence

$$\frac{\delta_0^2}{8} \leq \frac{u(z)}{v(z)} \leq \frac{8}{\delta_0^2}.$$

□

To illustrate the use of the boundary Harnack inequality, we will first prove the following special case:

Theorem 2.2. *Suppose that $\mathbb{D}$ is the unit disc and $\mathbb{H}_+$ is upper-half plane in $\mathbb{R}^2$. If $f : \mathbb{D} \longrightarrow \mathbb{H}_+$ is hqc homeomorphism then $f\Big|_{\mathbb{D}_-}$ is bi-Lipschitz with respect to Euclidean metric, where $\mathbb{D}_- = \{z : z \in \mathbb{D}, Im(z) < 0\}$.*

Proof. Without loss of generality we will assume that $f(0) = i$. Consider the Möbius transformation

$$M(z) = \frac{1 - iz}{z - i}$$

such that $M(\pm 1) = \pm 1$, $M(0) = i$, $M(-i) = 0$ and choose

$$u = Im(f), \quad v = Im(M(z)) = \frac{1-|z|^2}{|z-i|^2}.$$

It holds that $u(0) = v(0) = 1$, and for any ξ such that $Im(\xi) < 0$ $|\xi| = 1$,

$$\lim_{z \to \xi} u(z) = \lim_{z \to \xi} v(z) = 0.$$

Since in our setting $Im(f(z)) \equiv d(f(z), \partial\mathbb{H}_+)$, from 2.1 we now have

$$\frac{1}{C(A)} \leq \frac{d(f(z), \partial\mathbb{H}_+)}{\frac{1-|z|^2}{|z-i|^2}} \leq C(A)$$

on $A \cap \mathbb{D}$ for some constant $C(A)$, for every compact $A \subset \mathbb{D} \cup I$, where $I = \partial\mathbb{D} \cap \mathbb{H}_-$.

Because $|z-i|^2 \leq 4$, $d(z,\partial\mathbb{D}) = 1-|z|$ it follows that

$$\frac{1}{2C(A)} \leq \frac{d(f(z),\partial\mathbb{H}_+)}{d(z,\partial\mathbb{D})} \leq 4C(A).$$

Using Theorem 1.2 we conclude that

$$\frac{1}{c} \leq \alpha_f(x) \leq c,$$

where c is constant which depends only on A.

Finally, from the proof of Theorem 1.1, ([9]) it follows that

$$\alpha_f(x) \asymp \|f'(x)\|,$$

and since f is qc, it follows that it is bi-Lipschitz.

□

By developing the ideas above we can consider local questions of bilipschicity phenomena when only part of the boundary is flat. Here we need to use quasiconformal geometry.

Definition 2.3. $\partial\Omega$ is flat at some $x_0 \in \partial\Omega$ if, up to rotations,

$$\partial\Omega \cap B(x_0,\rho) = [x_0-\rho, x_0+\rho]$$

for some $\rho > 0$.

Theorem 2.4. *Suppose that $\mathbb{D}$ is unit disc, Ω is simply connected and $f : D \longrightarrow \Omega$ is harmonic and quasiconformal mapping such that $f(\mathbb{D}) = \Omega$. Suppose also that $\partial\Omega$ is flat at x_0, and that f is normalised so that $f(\pm 1) = x_0 \pm \rho$ with $f(-i) = x_0$.*

If $\Omega_1 = f^{-1}[B(x_0,\rho/2) \cap \Omega]$, then $f : \Omega_1 \longrightarrow B(x_0,\rho/2) \cap \Omega$ is bi-Lipschitz. Indeed,

$$\frac{1}{L_0} \leq \frac{|f(x)-f(y)|}{\rho|x-y|} \leq L_0$$

for some L_0 depending only on $K(f)$.

For the proof we need a local version of Theorem 2.2.

Lemma 2.5. *Let $\mathbb{D}_+ = \mathbb{D} \cap \mathbb{H}_+$, and $g : \mathbb{D} \to \mathbb{D}_+$ a harmonic K-quasiconformal mapping with*

$$g(\pm 1) = \pm 1, \qquad g(-i) = 0.$$

If $A \subset \overline{D}$ is a compact subset with $\delta_0 := \operatorname{dist}(A, S^1 \cap \mathbb{H}_+) > 0$, then

$$\frac{1}{c(K,\delta_0)} \leq \frac{\operatorname{dist}(g(z),\partial\mathbb{D}_+)}{1-|z|} \leq c(K,\delta_0), \qquad z \in A.$$

The constant $c(K,\delta_0) < \infty$ depends only on K and δ_0.

Proof. First, the map $g : \mathbb{D} \to \mathbb{D}_+$ is η-quasisymmetric, where η depends only on K. Indeed, every K-quasiconformal mapping of the unit disk $\mathbb{D}$ fixing ± 1 and $-i$ is η-quasisymmetric, and the case of our mapping is quickly reduced to this fact, e.g. by using a suitable bilipschitz mapping from $\mathbb{D}_+$ to $\mathbb{D}$.

It follows that if $u(z) = Im(g(z)), z \in A$, then firstly $c(K) \leq u(0) \leq 1$, and secondly, that

$$\frac{1}{c(K,\delta_0)} \leq \frac{u(z)}{\operatorname{dist}(g(z), \partial\mathbb{D}_+)} \leq c(K,\delta_0)$$

for some constant $c(K) < \infty$ depending only on K and δ_0. Therefore we can argue similarly as in Theorem 2.2 to prove the claim.

□

The proof of Theorem 2.4 is reduced to Lemma 2.5, via the conformal mapping $\phi : \mathbb{D} \to \tilde{\Omega} = f^{-1}[B(x_0,\rho) \cap \Omega]$, where $\phi(\pm 1) = \pm 1$ and $\phi(-i) = -i$. One mainly needs to notice that ϕ is bilipschitz on $A = f^{-1}[B(x_0,\rho/2) \cap \Omega]$.

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