Novi Sad J. Math.
Vol. 44, No. 2, 2014, 77-88

ON LORENTZIAN α–SASAKIAN MANIFOLDS ADMITTING A TYPE OF SEMI-SYMMETRIC METRIC CONNECTION

Ajit Barman[1]

Abstract. The object of the present paper is to study a Lorentzian α-Sasakian manifold admitting a semi-symmetric metric connection.

AMS Mathematics Subject Classification (2010): 53C15, 53C25

Key words and phrases: Lorentzian α-Sasakian manifold, semi-symmetric metric connection, locally ϕ-symmetric, ξ-projectively flat

1. Introduction

In 1969, Tanno [21] classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. The sectional curvature of the manifolds of plain sections containing ξ is a constant, say c. The sectional curvature of plain sections can be divided into three classes:

(1.1) homogeneous normal contact Riemannian manifolds with $c > 0$,

(1.2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and

(1.3) a warped product space $\mathbb{R} \times_f \mathbb{C}$ if $c < 0$.

The manifolds of class (1.1) are characterized by admitting a Sasakian structure. Kenmotsu [16] characterized the differential geometric properties of the manifolds of class (1.3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [16].

In 1980, Gray and Hervella [12], classification of almost Hermitian manifolds there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehlerian manifolds [10]. An almost contact metric structure on the manifold M is called a trans-Sasakian structure [17] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ ([15], [16]) coincides with the class of trans-Sasakian structure of type (α, β). We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [3], β-Kenmotsu [14] and α-Sasakian [14] respectively.

In 2005, Yildiz and Murathan [25] studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α-Sasakian manifolds are locally isometric with a sphere.

[1]Department of Mathematics, Assistant Professor, Kabi-Nazrul Mahavidyalaya, P.O.-Sonamura-799181, P.S.-Sonamura, Dist.- Sepahijala, Tripura, India, e-mail: ajitbarmanaw@yahoo.in

In 2012, Yadav and Suthar [23] studied Lorentzian α-Sasakian manifolds.

Hayden [13] introduced semi-symmetric linear connection on a Riemannian manifold. Let M be an n-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and ∇ be the Levi-Civita connection on (M^n, g).

A linear connection $\bar{\nabla}$ defined on (M^n, g) is said to be semi-symmetric [11] if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

where η is a 1-form and ξ is a vector field given by

$$\eta(X) = g(X, \xi), \tag{1.2}$$

for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on M^n.

A semi-symmetric connection $\bar{\nabla}$ is called a semi-symmetric metric connection [13] if it further satisfies

$$\bar{\nabla} g = 0. \tag{1.3}$$

A relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on (M^n, g) has been obtained by Yano [24] which is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \tag{1.4}$$

We also have

$$(\bar{\nabla}_X \eta)(Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y) + \eta(\xi)g(X, Y). \tag{1.5}$$

Further, a relation between the curvature tensor $\bar{R}$ of the semi-symmetric metric connection $\bar{\nabla}$ and the curvature tensor R of the Levi-Civita connection ∇ is given by

$$\begin{aligned}\bar{R}(X, Y)Z = R(X, Y)Z + \gamma(X, Z)Y - \gamma(Y, Z)X +& \\ g(X, Z)QY - g(Y, Z)QX,&\end{aligned} \tag{1.6}$$

where γ is a tensor field of type (0,2) and a tensor field Q of type (1,1) is given by

$$\gamma(Y, Z) = g(QY, Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y, Z). \tag{1.7}$$

From (1.6) and (1.7), we obtain

$$\begin{aligned}\tilde{\bar{R}}(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) - \gamma(Y, Z)g(X, W) +& \\ \gamma(X, Z)g(Y, W) - g(Y, Z)\gamma(X, W) +& \\ g(X, Z)\gamma(Y, W),&\end{aligned} \tag{1.8}$$

where

$$\tilde{\bar{R}}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$$
$$\text{and} \quad \tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W). \tag{1.9}$$

The study of semi-symmetric metric connection was further developed by Amur and Pujar [1], Binh [2], Chaki and Konar [4], De ([5], [6]), De and Biswas [7], De and De [8], De and De [9], Prvanović [18], Sharfuddin and Hussain [19], Yano [24] and many others.

The Projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor vanishes. Here the projective curvature tensor $\bar{P}$ with respect to the semi-symmetric metric connection is defined by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y], \tag{1.10}$$

for X, Y, $Z \in \chi(M)$, where $\bar{S}$ is the Ricci tensor with respect to the semi-symmetric metric connection. In fact M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

The present paper is organized as follows: The section 2 is equipped with some prerequisites about Lorentzian α-Sasakian manifolds. In section 3, we establish the relation of the curvature tensor between the Levi-Civita connection and the semi-symmetric metric connection of a Lorentzian α-Sasakian manifold. Locally ϕ-symmetric Lorentzian α-Sasakian manifolds with respect to the semi-symmetric metric connection have been studied in section 4. In the next section we consider ξ-projectively flat Lorentzian α-Sasakian manifolds. Finally, we construct an example of a 3-dimensional Lorentzian α-Sasakian manifold with respect to the semi-symmetric metric connection which support the result obtained in section 4 and section 5.

2. Lorentzian α-Sasakian manifolds

A $(2n+1)$-dimensional differentiable manifold M is called a Lorentzian α-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy [25]

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = -1, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M.

Also Lorentzian α-Sasakian manifolds satisfy [25],

$$\nabla_X \xi = -\alpha \phi X, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y), \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and $\alpha \in \mathbb{R}$.

Further on a Lorentzian α-Sasakian manifold M the following relations hold ([25], [23]):

$$\eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.6}$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.7}$$

$$R(\xi, X)\xi = \alpha^2[\eta(X)\xi + X], \tag{2.8}$$

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$S(X, \xi) = 2n\alpha^2\eta(X), \tag{2.10}$$

$$(\nabla_X \phi)(Y) = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.11}$$

where S is the Ricci tensor of the Levi-Civita connection.

3. Curvature tensor of a Lorentzian α-Sasakian manifold with respect to the semi-symmetric metric connection

Using (2.1) and (2.5) in (1.7), we get

$$\gamma(X, Y) = g(QX, Y) = -\alpha g(\phi X, Y) - \eta(X)\eta(Y) - \frac{1}{2}g(X, Y). \tag{3.1}$$

From (3.1), it follows that

$$QX = -\alpha\phi X - \eta(X)\xi - \frac{1}{2}X. \tag{3.2}$$

Again using (3.1) and (3.2) in (1.6), we have

$$\bar{R}(X, Y)Z$$

$$
\begin{aligned}
&= R(X,Y)Z - \alpha g(X,\phi Z)Y - \eta(X)\eta(Z)Y \\
&\quad +\alpha g(Y,\phi Z)X + \eta(Y)\eta(Z)X - g(X,Z)Y \\
&\quad +g(Y,Z)X - \alpha g(X,Z)\phi Y + \alpha g(Y,Z)\phi X \\
&\quad -g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi. \qquad (3.3)
\end{aligned}
$$

Taking the inner product of (3.3) with W, it follows that

$$
\begin{aligned}
&\tilde{\bar{R}}(X,Y,Z,W) \\
&= \tilde{R}(X,Y,Z,W) - \alpha g(X,\phi Z)g(Y,W) - \eta(X)\eta(Z)g(Y,W) \\
&\quad +\alpha g(Y,\phi Z)g(X,W) + \eta(Y)\eta(Z)g(X,W) - g(X,Z)g(Y,W) \\
&\quad +g(Y,Z)g(X,W) - \alpha g(X,Z)g(\phi Y,W) + \alpha g(Y,Z)g(\phi X,W) \\
&\quad -g(X,Z)\eta(Y)\eta(W) + g(Y,Z)\eta(X)\eta(W). \qquad (3.4)
\end{aligned}
$$

Taking a frame field from (3.3), we obtain

$$
\begin{aligned}
\bar{S}(Y,Z) &= S(Y,Z) + (2n-1)\alpha g(Y,\phi Z) + (2n-1)\eta(Y)\eta(Z) \\
&\quad +[2n-1+\alpha trace\phi]g(Y,Z). \qquad (3.5)
\end{aligned}
$$

Putting $Z=\xi$ in (3.5) and using (2.1) and (2.10), we get

$$\bar{S}(Y,\xi) = [2n\alpha^2 + \alpha trace\phi]\eta(Y). \qquad (3.6)$$

From the above discussions we can state the following theorem:

Theorem 3.1. *For a Lorentzian α-Sasakian manifold M with respect to the semi-symmetric metric connection* $\bar{\nabla}$

(i) The curvature tensor $\bar{R}$ *is given by*

$$
\begin{aligned}
\bar{R}(X,Y)Z &= R(X,Y)Z - \alpha g(X,\phi Z)Y - \eta(X)\eta(Z)Y \\
&\quad +\alpha g(Y,\phi Z)X + \eta(Y)\eta(Z)X - g(X,Z)Y \\
&\quad +g(Y,Z)X - \alpha g(X,Z)\phi Y + \alpha g(Y,Z)\phi X \\
&\quad -g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi,
\end{aligned}
$$

(ii) The Ricci tensor $\bar{S}$ *is given by*

$$
\begin{aligned}
\bar{S}(Y,Z) &= S(Y,Z) + (2n-1)\alpha g(Y,\phi Z) + (2n-1)\eta(Y)\eta(Z) \\
&\quad +[2n-1+\alpha trace\phi]g(Y,Z),
\end{aligned}
$$

(iii) The Ricci tensor $\bar{S}$ *is symmetric.*

(iv) $\bar{S}(Y,\xi) = [2n\alpha^2 + \alpha trace\phi]\eta(Y)$.

4. Locally ϕ-symmetric Lorentzian α-Sasakian manifolds with respect to the semi-symmetric metric connection

Definition 4.1. *A Lorentzian α-Sasakian manifold M with respect to the semi-symmetric metric connection is called to be locally ϕ-symmetric if*

$$\phi^2((\bar{\nabla}_W \bar{R})(X,Y)Z) = 0, \tag{4.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ on M. This notion was introduced by Takahashi [20], for a Sasakian manifold.

Taking covariant differentiation of (3.3) with respect to W and using (1.3), (2.1), (2.4), (2.5) and (2.11), we have

$$\begin{aligned}
&(\bar{\nabla}_W \bar{R})(X,Y)Z \\
&= (\nabla_W R)(X,Y)Z - \eta(X)R(W,Y)Z \\
&\quad -\eta(Y)R(X,W)Z - \eta(Z)R(X,Y)W - \acute{R}(X,Y,Z,W)\xi \\
&\quad +2\eta(X)\eta(Z)\eta(W)Y - 2\eta(Y)\eta(Z)\eta(W)X + 2\eta(Y)\eta(W)g(X,Z)\xi \\
&\quad -2\eta(X)\eta(W)g(Y,Z)\xi - (\alpha^3+\alpha^2-1)g(X,Z)g(Y,W)\xi \\
&\quad +(\alpha^3+\alpha^2-1)g(X,W)g(Y,Z)\xi + (\alpha^3-\alpha^2+1)\eta(Y)g(X,Z)W \\
&\quad -(\alpha^3-\alpha^2+1)\eta(X)g(Y,Z)W - \alpha^2\eta(Z)g(X,W)Y \\
&\quad +\alpha^2\eta(Z)g(Y,W)X + \alpha^2\eta(Y)g(Z,W)X \\
&\quad -\alpha^2\eta(X)g(Z,W)Y + \alpha\eta(Z)g(X,\phi W)Y + \alpha\eta(X)g(Z,\phi W)Y \\
&\quad +\eta(X)g(Z,W)Y - \alpha\eta(Z)g(Y,\phi W)X - \alpha\eta(Y)g(Z,\phi W)X \\
&\quad -\eta(Z)g(Y,W)X - \eta(Y)g(Z,W)X + 2\alpha g(X,Z)g(Y,\phi W)\xi \\
&\quad -2\alpha g(Y,Z)g(X,\phi W)\xi + 2\alpha\eta(Y)g(X,Z)\phi W \\
&\quad -2\alpha\eta(X)g(Y,Z)\phi W + \eta(X)g(Z,W)Y.
\end{aligned} \tag{4.2}$$

Now applying ϕ^2 on both sides of (4.2) and using (2.1) and (2.2), it follows that

$$\begin{aligned}
&\phi^2((\bar{\nabla}_W \bar{R})(X,Y)Z) \\
&= \phi^2((\nabla_W R)(X,Y)Z) - \eta(X)R(W,Y)Z - \eta(X)\eta(R(W,Y)Z)\xi \\
&- \eta(Y)R(X,W)Z - \eta(Y)\eta(R(X,W)Z)\xi - \eta(Z)R(X,Y)W \\
&- \eta(Z)\eta(R(X,Y)W)\xi + 2\eta(X)\eta(Z)\eta(W)Y - 2\eta(Y)\eta(Z)\eta(W)X \\
&+ (\alpha^3-\alpha^2+1)\eta(Y)g(X,Z)W + (\alpha^3-\alpha^2+1)\eta(Y)\eta(W)g(X,Z)\xi \\
&- (\alpha^3-\alpha^2+1)\eta(X)g(Y,Z)W - (\alpha^3-\alpha^2+1)\eta(X)\eta(W)g(Y,Z)\xi \\
&- \alpha^2\eta(Z)g(X,W)Y - \alpha^2\eta(Z)\eta(Y)g(X,W)\xi \\
&+ \alpha^2\eta(Z)g(Y,W)X + \alpha^2\eta(X)\eta(Z)g(Y,W)\xi + (\alpha^2-1)\eta(Y)g(Z,W)X \\
&- (\alpha^2-1)\eta(X)g(Z,W)Y + \alpha\eta(Z)g(X,\phi W)Y
\end{aligned}$$

$$+ \alpha\eta(Z)\eta(Y)g(X,\phi W)\xi + \eta(Z)g(X,W)Y$$
$$+ \eta(Z)\eta(Y)g(X,W)\xi + \alpha\eta(X)g(Z,\phi W)Y$$
$$- \alpha\eta(Z)g(Y,\phi W)X - \alpha\eta(Z)\eta(X)g(Y,\phi W)\xi - \eta(Z)g(Y,W)X$$
$$- \eta(Z)\eta(X)g(Y,W)\xi - \alpha\eta(Y)g(Z,\phi W)X$$
$$+ 2\alpha\eta(Y)g(X,Z)\phi W - 2\alpha\eta(X)g(Y,Z)\phi W. \tag{4.3}$$

Now taking X, Y, Z, W orthogonal to ξ, the equation (4.3) gives

$$\phi^2((\bar{\nabla}_W\bar{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{4.4}$$

Hence we state the following theorem:

Theorem 4.1. *A $(2n+1)$-dimensional Lorentzian α-Sasakian manifold is locally ϕ-symmetric with respect to the semi-symmetric metric connection if and only if the manifold is also locally ϕ-symmetric with respect to the Levi-Civita connection.*

5. ξ-projectively flat Lorentzian α-Sasakian manifolds with respect to the semi-symmetric metric connection

Definition 5.1. *A Lorentzian α-Sasakian manifold M with respect to the semi-symmetric metric connection is said to be ξ-projectively flat if*

$$\bar{P}(X,Y)\xi = 0, \tag{5.1}$$

for all vector fields X, Y on M. This notion was first defined by Tripathi and Dwivedi [22]. If equation (5.1) just holds for X, Y orthogonal to ξ, we called such a manifold a horizontal ξ-projectively flat manifold.

Using (3.3) in (1.10), we get

$$\begin{aligned}\bar{P}(X,Y)Z \;=\;& R(X,Y)Z - \alpha g(X,\phi Z)Y - \eta(X)\eta(Z)Y\\ &+\alpha g(Y,\phi Z)X + \eta(Y)\eta(Z)X - g(X,Z)Y\\ &+g(Y,Z)X - \alpha g(X,Z)\phi Y + \alpha g(Y,Z)\phi X\\ &-g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi\\ &-\frac{1}{2n}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y].\end{aligned} \tag{5.2}$$

Putting $Z = \xi$ and using (2.1), (2.9) and (3.6) in (5.2), we get

$$\begin{aligned}\bar{P}(X,Y)\xi \;=\;& [\alpha^2 - \frac{2n-1+\alpha trace\phi}{2n}][\eta(Y)X - \eta(X)Y]\\ &-\alpha[\eta(X)\phi Y - \eta(Y)\phi X].\end{aligned} \tag{5.3}$$

From (5.3), implies that

$$\bar{P}(X,Y)\xi = 0; \forall X, Y \text{ orthogonal to } \xi, \tag{5.4}$$

we called such a manifold a horizontal ξ-projectively flat manifold.

Hence we state the following theorem:

Theorem 5.1. *A* $(2n+1)$*-dimensional Lorentzian* α*-Sasakian manifold is horizontal* ξ*-projectively flat with respect to the semi-symmetric metric connection.*

Again using (3.5) in (5.2), we have

$$\begin{aligned}\bar{P}(X,Y)Z &= P(X,Y)Z - \frac{1}{2n}\alpha g(X,\phi Z)Y - \frac{1}{2n}\eta(X)\eta(Z)Y \\ &+\frac{1}{2n}\alpha g(Y,\phi Z)X + \frac{1}{2n}\eta(Y)\eta(Z)X + [\frac{\alpha trace\phi - 1}{2n}]g(X,Z)Y \\ &-[\frac{\alpha trace\phi - 1}{2n}]g(Y,Z)X - \alpha g(X,Z)\phi Y + \alpha g(Y,Z)\phi X \\ &-g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi,\end{aligned} \tag{5.5}$$

where P be the projective curvature tensor with respect to the Levi-Civita connection.

Putting $Z = \xi$ in (5.5) and using (2.1), it follows that

$$\begin{aligned}\bar{P}(X,Y)\xi &= P(X,Y)\xi + \eta(X)[\frac{\alpha trace\phi}{2n}Y - \alpha\phi Y] \\ &-\eta(Y)[\frac{\alpha trace\phi}{2n}X - \alpha\phi X].\end{aligned} \tag{5.6}$$

From (5.6), implies that

$$\bar{P}(X,Y)\xi = P(X,Y)\xi;\ \forall\, X, Y \text{ orthogonal to } \xi. \tag{5.7}$$

In view of above discussions we state the following theorem:

Theorem 5.2. *A* $(2n+1)$*-dimensional Lorentzian* α*-Sasakian manifold is horizontal* ξ*-projectively flat with respect to the semi-symmetric metric connection if and only if the manifold is* ξ*-projectively flat with respect to the Levi-Civita connection.*

6. Example

In this section we construct an example of locally ϕ- symmetric and ξ-projectively flat on a Lorentzian α-Sasakian manifold with respect to the semi-symmetric metric connection which verifies the result of section 4 and section 5.

We consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinate in R^3. We choose the vector fields

$$e_1 = e^z\frac{\partial}{\partial y},\ \ e_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}),\ \ e_3 = \alpha\frac{\partial}{\partial z}$$

which are linearly independent at each point of M and α is constant.

Let g be the Lorentzian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$$

and

$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1,$$

that is, the form of the metric becomes

$$g = \frac{1}{(e^z)^2}(dy)^2 - \frac{1}{\alpha^2}(dz)^2,$$

which is a Lorentzian metric.

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_3)$$

for any $Z \in \chi(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_1,\ \phi e_2 = -e_2,\ \phi e_3 = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_3) = -1$$

$$\phi^2(Z) = Z + \eta(Z)e_3$$

and

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W)$$

for any $U, W \in \chi(M)$.

Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = -\alpha e_1,\ [e_2, e_3] = -\alpha e_2.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, W) &= Xg(Y, W) + Yg(X, W) - Wg(X, Y) - g(X, [Y, W]) \\ &\quad -g(Y, [X, W]) + g(W, [X, Y]). \end{aligned} \tag{6.1}$$

Using Koszul's formula we get the following

$$\nabla_{e_1} e_1 = -\alpha e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = -\alpha e_1,$$

$$\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -\alpha e_3,\ \nabla_{e_2} e_3 = -\alpha e_2,$$

$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0.$$

Using (1.4) in above equation, we obtain

$$\bar{\nabla}_{e_1} e_1 = -(1+\alpha)e_3,\ \bar{\nabla}_{e_1} e_2 = 0,\ \bar{\nabla}_{e_1} e_3 = -(1+\alpha)e_1,$$

$$\bar{\nabla}_{e_2}e_1 = 0,\ \bar{\nabla}_{e_2}e_2 = -(1+\alpha)e_3,\ \bar{\nabla}_{e_2}e_3 = -(1+\alpha)e_2,$$

$$\bar{\nabla}_{e_3}e_1 = 0,\ \bar{\nabla}_{e_3}e_2 = 0,\ \bar{\nabla}_{e_3}e_3 = 0.$$

Therefore the manifold is a Lorentzian α-Sasakian manifold with respect to the semi-symmetric metric connection.

By using the above results, we can easily obtain the components of the curvature tensor as follows:

$$R(e_1,e_2)e_2 = -\alpha^2 e_2,\ \ R(e_1,e_3)e_3 = -\alpha^2 e_1,\ \ R(e_2,e_1)e_1 = \alpha^2 e_2,$$

$$R(e_2,e_3)e_3 = -\alpha^2 e_2,\ \ R(e_3,e_1)e_1 = \alpha^2 e_3,\ \ R(e_3,e_2)e_2 = \alpha^2 e_3,$$

$$R(e_1,e_2)e_3 = 0,\ \ R(e_2,e_3)e_2 = -\alpha^2 e_3,\ \ R(e_1,e_2)e_2 = \alpha^2 e_1,$$

and

$$\bar{R}(e_1,e_2)e_2 = (1+\alpha)^2 e_1,\ \bar{R}(e_3,e_1)e_1 = \alpha(1+\alpha)e_3,$$

$$\bar{R}(e_3,e_2)e_2 = \alpha(1+\alpha)e_3,\ \ \bar{R}(e_2,e_1)e_1 = (1+\alpha)^2 e_2,$$

$$\bar{R}(e_1,e_2)e_3 = 0,\ \bar{R}(e_1,e_3)e_3 = -\alpha(1+\alpha)e_2,$$

$$\bar{R}(e_2,e_3)e_2 = -\alpha(1+\alpha)e_3,\ \ \bar{R}(e_1,e_2)e_1 = -(1+\alpha)^2 e_2,$$

$$\bar{R}(e_2,e_3)e_3 = -\alpha(1+\alpha)e_2.$$

From the above expression of the curvature tensor which it follows that

$$\phi^2((\bar{\nabla}_W \bar{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z) = 0.$$

Therefore, this example supports Theorem 4.1.

Using the expressions of the curvature tensors with respect to the semi-symmetric metric connection we find the values of the Ricci tensors as follows:

$$\bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = 1+\alpha, \bar{S}(e_3,e_3) = -\alpha(1+\alpha),$$

$$\bar{S}(e_1,e_2) = \bar{S}(e_1,e_3) = \bar{S}(e_2,e_3) = 0.$$

Let X and Y are any two vector fields given by
$X = a_1e_1 + a_2e_2 + a_3e_3$ and $Y = b_1e_1 + b_2e_2 + b_3e_3$.
Using (1.10) and above relations, we get

$$\begin{aligned} \bar{P}(X,Y)e_3 &= \alpha(\alpha+1)[\frac{1}{2n}(a_1b_3 - a_3b_1)e_1 + (\frac{1}{2n}a_2b_3 + a_3b_2 - a_1b_3 \\ &\quad + a_3b_1 - a_2b_3 - \frac{1}{2n}a_3b_2)e_2]. \end{aligned} \tag{6.2}$$

Therefore, the manifold will be ξ-projectively flat on a Lorentzian α-Sasakian manifold with respect to the semi-symmetric metric connection if $\alpha = -1$ which verifies the Theorem 5.1.

Acknowledgement

The author is thankful to the referee for his valuable suggestions in the improvement of the paper.

References

[1] Amur, K., Pujar, S.S., On submanifolds of a Riemannian manifold admitting a metric semi-symmetric connection. Tensor, N.S. 32 (1978), 35-38.

[2] Binh, T.Q., On semi-symmetric connections. Periodica Math. Hungerica 21 (1990), 101-107.

[3] Blair D.E., Contact manifolds in Riemannian geometry. Lecture Note in Mathematics, 509, Springer-Verlag Berlin 1976.

[4] Chaki, M.C., Konar, A., On a type of semi-symmetric connection on a Riemannian manifold. J. Pure Math., Calcutta University (1981), 77-80.

[5] De, U.C., On a type of semi-symmetric connection on a Riemannian manifold, Indian J. Pure Appl. Math. 21 (1990), 334-338.

[6] De, U.C., On a type of semi-symmetric metric connection on a Riemannian manifold. Analele Stiintifice Ale Universitatii, " AL. I. CUZA " DIN IASI, Math., Tomul XXXVIII, Fasc. 38 (1991), 105-108.

[7] De, U.C., Biswas, S.C., On a type of semi-symmetric metric connection on a Riemannian manifold. Pub. De L' Institut Math., N. S. 61 (1997), 90-96.

[8] De, U.C., De, B.K., On a type of semi-symmetric connection on a Riemannian manifold. Ganita 47 (1996), 11-14.

[9] De, U.C., De, K., On trans-Sasakian manifold admitting a semi-symmetric metric connection. Bull. Pure. Appl. Math. 5 (2011),360-366.

[10] Dragomir, S., Ornea, L., Locally conformal Keahler geometry. In: Progress in Mathematics, 155, Birkhauser Boston, Inc. M.A. 1998.

[11] Friedmann A, Schouten J.A., *Ü*ber die Geometric der halbsymmetrischen Übertragung. Math. Zeitschr. 21 (1924), 211-223.

[12] Gray A., Hervella L.M., The sixteen classes of almost Hermitian manifolds and their linear invariants. Ann. Mat. Pure Appl. 123 (1980), 35-58.

[13] Hayden H.A., Subspaces of space with torsion. Proc. London Math. Soc. 34 (1932), 27-50.

[14] Kenmotsu K., A class of almost contact Riemannian manifolds. Tohoku Math. J. 24 (1972), 93-103.

[15] Marrero J.C., The local structure of trans-Sasakian manifolds. Ann. Mat. Pure Appl. 162 (1992), 77-86.

[16] Marrero J.C., Chinea D., On trans-Sasakian manifolds. Proceeding of the XIV^{th} Spanish-Portuguese Conference on Mathematics. I-III, (Spanish) (Puerto de la Cruz, 1989), Univ. La Laguna, La Laguna, (1990), 655-659.

[17] Oubina, J.A., New classes of contact metric structures. Publ. Math. Debrecen (32) (1985), 187-193.

[18] Prvanović, M., On pseudo metric semi-symmetric connections, Pub. De L' Institut Math., Nouvelle serie 32 (1975), 157-164.

[19] Sharfuddin, A., Hussain, S.I., Semi-symmetric metric connexions in almost contact manifolds. Tensor, N. S. 30 (1976), 133-139.

[20] Takahashi, T., Sasakian ϕ-symmetric spaces. *Tôhoku* Math. J. 29 (1977), 91-113.

[21] Tanno S., The automorphisam groups of almost contact Riemannian manifolds. Tohoku Math. J. 21 (1969), 21-38.

[22] Tripathi, M.M., Dwivedi, M.K., The structure of some classes of K-contact manifolds. Proc. Indian Acad. Sci. Math. Sci. 118 (2008), 371-379.

[23] Yadav, S., Suthar, D.L., Certain derivation on Lorentzian α-Sasakian manifolds. Mathematics and Decision Science 12 (2012).

[24] Yano K., On semi-symmetric connection. Revue Roumaine de Math. Pure et Appliques 15 (1970), 1570-1586.

[25] Yildiz A, Murathan C., On Lorentzian α-Sasakian manifolds. Kyungpook Math. J. 45 (2005), 95-103.

Received by the editors July 13, 2013