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AN ASYMPTOTIC NUMERICAL METHOD FOR SINGULARLY PERTURBED WEAKLY COUPLED SYSTEM OF CONVECTION-DIFFUSION TYPE DIFFERENTIAL DIFFERENCE EQUATIONS

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Abstract. In this paper an asymptotic numerical method is suggested to solve singularly perturbed weakly coupled system of convection-diffusion type second order ordinary differential equations with delay (negative shift) terms. An error estimate is derived in the supremum norm and it is found to be of order $O(N^{-1} \ln N)$ provided that $\varepsilon \leq CN^{-1}$, where ε is small perturbation parameter and N is the discretization parameter. Numerical results are provided to illustrate the theoretical results.

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1. Introduction

In many applications, one assumes the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. However, under closer scrutiny, it becomes apparent that the principle of causality is often a first order approximation to the true situation and more realistic model would involve some of the past states of the system. This kind of systems are governed by differential equations with delay arguments.

A subclass of these equations consists of singularly perturbed ordinary differential equations with a delay, that is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay term. Such type of equations arise frequently in the mathematical modeling of various practical phenomena, for example, in the modeling of the human pupil-light reflex [17], the mathematical model of the determination of expected time for generation of action potentials in nerve cell by random synaptic inputs in the dendrites [14] and variational problems in control theory [8], etc. For a fixed $\varepsilon > 0$, the existence and uniqueness of solutions of the boundary value problems for delay differential equations of second

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order ordinary differential equations have been well studied in the literature. For more details one may refer to [3, 4].

It is well known that standard discretization methods for solving singularly perturbed non delay differential equations are sometimes unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to solve this type of equations, whose accuracy does not depend on the parameter ε, that is the methods are uniformly convergent with respect to the parameter. For more details of this type of numerical methods one may refer to [2, 5, 6, 7, 19, 23, 15, 16].

In [9, 20] and the references therein the authors presented various numerical methods for the following Singularly Perturbed Delay Differential Equations (SPDDEs)

$$\varepsilon y'' + a(x)y'(x-\delta) + b(x)y'(x) + c(x)y(x-\delta) + e(x)y(x) = f(x), \tag{1.1}$$
$$y(x) = \phi(x),\ x \in [-\delta, 0],\ y(1) = \gamma,\ \delta = o(\varepsilon). \tag{1.2}$$

In fact, for various combinations of conditions on the coefficients $a,\ b,\ c$ and e (for an example $a \neq 0,\ b = c = 0,\ e \neq 0$ [9]) they suggested suitable numerical methods. In all these methods first they applied Taylor's expansion for $y'(x-\delta)$ or $y(x-\delta)$ and reduced the DDEs to non DDEs. For the resulting non DDEs they applied standard numerical methods available in the literature. Following this procedure some authors [10, 18] suggested numerical methods to the following Boundary Value Problem (BVP):

$$\varepsilon y'' + c(x)y(x-\delta) + d(x)y(x+\eta) + e(x)y(x) = f(x), \tag{1.3}$$
$$y(x) = \phi(x),\ x \in [-\delta, 0],\ y(x) = \gamma(x),\ x \in [1, 1+\eta], \tag{1.4}$$

where $\delta = o(\varepsilon),\ \eta = o(\varepsilon)$. Some authors [11] considered the above DDE (1.1)-(1.2) with $a \neq 0,\ b = 0,\ c = 0$ and suggested numerical methods without applying Taylor's expansion, that is, reducing the DDE to non DDE. Using the Taylor's series expansion procedure as mentioned above and Newton's quasi linearization process some authors [12] solved nonlinear problems numerically. Subburayan and Ramanujam [21, 22] suggested a numerical method namely initial value technique for the following BVP

$$\begin{cases} -\varepsilon u'' + a(x)u'(x) + b(x)u(x) + c(x)u(x-1) = f(x), x \in (0,1) \cup (1,2), \\ u(x) = \phi(x),\ [-1, 0],\ u(2) = l, \end{cases} \tag{1.5}$$

where a can be either continuous throughout the domain $[0, 2]$ or continuous except at $x = 1$. In [21, 22], the authors considered single second order delay differential equation and applied initial value technique, whereas the present paper considers system and applied asymptotic numerical method which is different from the initial value technique. The motivation for the consideration of the above SPDDE (1.5) and below (2.1) has come from the paper of Lange and Miura [13].

In the present paper, as mentioned in the abstract, we consider the following weakly coupled system of singularly perturbed boundary value problem (2.1) for second order ordinary differential equations of convection-diffusion type with a negative shifts and suggest an asymptotic numerical method. It is proved that this method is convergent of order $O(N^{-1}\ln N)$.

The present paper is organized as follows. In Section 2, the problem under study is stated. A maximum principle for the DDE is established in Section 3. Further a stability result is derived. Some analytical results are derived in Section 4. In Section 5, a mesh selection strategy is explained. Further the fourth order Runge Kutta method with piecewise cubic Hermite interpolation on this mesh for a system of first order delay differential equations and an upwind finite difference scheme for weakly coupled system of singularly perturbed second order ordinary differential equations are described. Also their error estimates are given. Section 6 presents the asymptotic numerical method and its error analysis. Section 7 presents numerical results.

2. Statement of the problem

Throughout the paper, it is assumed that $\varepsilon \leq CN^{-1}$ and $C_1,\ C$ denote generic positive constants independent of the singular perturbation parameter ε and the discretization parameter N of the discrete problem. The supremum norm is used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem:

$$\|\overline{w}\|_D = \max\{\|w_1\|_D, \|w_2\|_D\},$$

where $\overline{w} = (w_1, w_2),\ \|w_i\|_D = \sup\limits_{x\in D}\{|\ w_i(x)\ |\},\ i = 1, 2.$

We consider the following Boundary Value Problem (BVP).
Find $\overline{u} = (u_1,\ u_2),\ u_1,\ u_2 \in Y = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$ such that

$$\begin{cases} -\varepsilon u_1''(x) + a_1(x)u_1'(x) + \sum_{k=1}^{2} b_{1k}(x)u_k(x) \\ \qquad\qquad + \sum_{k=1}^{2} c_{1k}(x)u_k(x-1) = f_1(x),\ x \in \Omega^*, \\ -\varepsilon u_2''(x) + a_2(x)u_2'(x) + \sum_{k=1}^{2} b_{2k}(x)u_k(x) \\ \qquad\qquad + \sum_{k=1}^{2} c_{2k}(x)u_k(x-1) = f_2(x),\ x \in \Omega^*, \\ u_1(x) = \phi_1(x),\ x \in [-1, 0],\ u_1(2) = l_1, \\ u_2(x) = \phi_2(x),\ x \in [-1, 0],\ u_2(2) = l_2, \end{cases} \tag{2.1}$$

where $0 < \varepsilon \ll 1,\ a_i(x) \geq \alpha_i > 0,\ i = 1, 2,\ 0 < \alpha < \min\{\alpha_1,\ \alpha_2\},\ b_{11}(x) \geq 0,$ $b_{12}(x) \leq 0,\ b_{21}(x) \leq 0,\ b_{22}(x) \geq 0,\ c_{ij}(x) \leq 0,\ i, j = 1, 2,\ b_{i1}(x) + b_{i2} \geq \beta_i \geq 0,\ i = 1, 2,\ c_{i1}(x) + c_{i2}(x) \geq \gamma_i,\ i = 1, 2,\ 2\alpha_i + 5\beta_i + 5\gamma_i \geq \eta_i > 0,\ i = 1, 2,$ and $a_i,\ b_{ij},\ c_{ij},\ f_i,\ i = 1, 2,\ j = 1,\ 2$ are sufficiently smooth functions on $\overline{\Omega}$, $\Omega = (0, 2)$, $\overline{\Omega} = [0, 2]$, $\Omega^* = \Omega^- \cup \Omega^+$, $\Omega^- = (0, 1)$, $\Omega^+ = (1, 2)$ and $\phi_i,\ i = 1, 2$ are smooth on $[-1, 0]$.

The above problem is equivalent to

$$(2.2)\quad P_1\overline{u}(x)\colon = \begin{cases} -\varepsilon u_1''(x) + a_1(x)u_1'(x) + \sum_{k=1}^{2} b_{1k}(x)u_k(x) \\ \qquad\qquad = f_1(x) - \sum_{k=1}^{2} c_{1k}(x)\phi_k(x-1),\ x \in \Omega^-, \\ -\varepsilon u_1''(x) + a_1(x)u_1'(x) + \sum_{k=1}^{2} b_{1k}(x)u_k(x) \\ \qquad\qquad + \sum_{k=1}^{2} c_{1k}(x)u_k(x-1) = f_1(x),\ x \in \Omega^+, \end{cases}$$

$$(2.3)\quad P_2\overline{u}(x)\colon = \begin{cases} -\varepsilon u_2''(x) + a_2(x)u_2'(x) + \sum_{k=1}^{2} b_{2k}(x)u_k(x) \\ \qquad\qquad = f_2(x) - \sum_{k=1}^{2} c_{2k}(x)\phi_k(x-1),\ x \in \Omega^-, \\ -\varepsilon u_2''(x) + a_2(x)u_2'(x) + \sum_{k=1}^{2} b_{2k}(x)u_k(x) \\ \qquad\qquad + \sum_{k=1}^{2} c_{2k}(x)u_k(x-1) = f_2(x),\ x \in \Omega^+, \end{cases}$$

$$u_1(0) = \phi_1(0),\ u_1(1-) = u_1(1+),\ u_1'(1-) = u_1'(1+),\ u_1(2) = l_1,$$
$$u_2(0) = \phi_2(0),\ u_2(1-) = u_2(1+),\ u_2'(1-) = u_2'(1+),\ u_2(2) = l_2,$$

where $u_1(1-)$ and $u_1(1+)$ denote the left and right limits of u_1 at $x = 1$ and the similar expressions are true for other functions. The above problem (2.1) has a solution [4].

3. Stability Result

In the following, the function $\bar{s}$ defined by $\bar{s}(x) = (s_1(x), s_2(x))$, where

$$(3.1)\qquad s_1(x) = s_2(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, & x \in [0,1], \\ \frac{3}{8} + \frac{x}{4}, & x \in [1,2] \end{cases}$$

is used. The following theorem states a maximum principle for the BVP (2.1).

Theorem 3.1. *(Maximum principle) Let* $\overline{w} = (w_1,\ w_2),\ w_1,\ w_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$ *be any function satisfying* $w_i(0) \geq 0,\ w_i(2) \geq 0,\ P_i\overline{w}(x) \geq 0,\ \forall x \in \Omega^*$, $i = 1, 2$ *and* $w_i'(1+) - w_i'(1-) = [w_i'](1) \leq 0,\ i = 1, 2.$ *Then* $w_i(x) \geq 0$, $\forall x \in \overline{\Omega},\ i = 1, 2.$

Proof. Let $\bar{s}$ be a test function defined by (3.1). It is easy to see that, $P_i\bar{s}(x) > 0,\ \forall x \in \Omega^*,\ i = 1, 2,\ [s_i'](1) < 0,\ i = 1, 2,\ s_i(x) > 0, \forall x \in \overline{\Omega},\ i = 1, 2.$ Let

$$\mu = \max\left\{ \max_{x\in\overline{\Omega}}\{\frac{-w_1(x)}{s_1(x)}\},\ \max_{x\in\overline{\Omega}}\{\frac{-w_2(x)}{s_2(x)}\} \right\}.$$

Then there exists at least one point $x_0 \in \overline{\Omega}$ such that $w_1(x_0) + \mu s_1(x_0) = 0$ or $w_2(x_0) + \mu s_2(x_0) = 0$ or both and $w_i(x) + \mu s_i(x) \geq 0,\ \forall x \in \overline{\Omega},\ i = 1, 2.$ Without the loss of generality we assume that, $w_1(x_0) + \mu s_1(x_0) = 0$. Therefore the function $(w_1 + \mu s_1)$ attains its minimum at $x = x_0$. Suppose the theorem

does not hold true then $\mu > 0$.
Let $x_0 \in \Omega^-$.

$$0 < P_1(\overline{w} + \mu\overline{s})(x_0) = -\varepsilon(w_1 + \mu s_1)''(x_0) + a_1(x_0)(w_1 + \mu s_1)'(x_0) \\ + b_{11}(x_0)(w_1 + \mu s_1)(x_0) + b_{12}(x_0)(w_2 + \mu s_2)(x_0) \leq 0.$$

It is a contradiction.
Similarly one can consider the case $x_0 \in \Omega^+$ and get a contradiction.
Let $x_0 = 1$.

$$0 \leq [(w_1 + \mu s_1)'](1) = [w_1'](1) + \mu[s_1'](1) < 0.$$

It is a contradiction. Hence the proof of the theorem. □

Theorem 3.2. *(Stability Result) Let $\overline{u} = (u_1,\ u_2),\ u_1,\ u_2 \in Y$ be any function. Then*

$$| u_i(x) | \leq C\ \max\left\{\max_{j=1,2}\{| u_j(0) |\},\ \max_{j=1,2}\{| u_j(2) |\}, \max_{j=1,2}\{\sup_{\zeta\in\Omega^*} | P_j\overline{u}(\zeta) |\}\right\}, \\ \forall\ x \in \overline{\Omega},\ i = 1, 2.$$

Proof. Define $\overline{\psi}^{\pm}(x) = (\psi_1^{\pm}(x),\ \psi_2^{\pm}(x))$ as $\psi_i^{\pm}(x) = C\ C_1 s_i(x) \pm u_i(x),\ x \in \overline{\Omega},\ i = 1, 2,$ where $C_1 = \max\{\max_{j=1,2}\{| u_j(0) |\},\ \max_{j=1,2}\{| u_j(2) |\},\ \max_{j=1,2}\{\sup_{\zeta\in\Omega^*} | P_j\overline{u}(\zeta) |\}\}$ and $\bar{s}$ defined above. Further, we have $\psi_i^{\pm}(0) = C\ C_1 s_i(0) \pm u_i(0) > 0,\ i = 1, 2$ and $\psi_i^{\pm}(2) = C\ C_1 s_i(2) \pm u_i(2) > 0,\ i = 1, 2$ by a proper choice of C.
Let $x \in \Omega^-$.

$$P_1\overline{\psi}^{\pm}(x) = C\ C_1 P_1\overline{s}(x) \pm P_1\overline{u}(x) \geq \frac{C\ C_1}{8}(4\alpha_1 + \beta_1) \pm P_1\overline{u}(x) \geq 0,$$

by a proper choice of C.
It is easy to show that $P_1\overline{\psi}^{\pm}(x) \geq 0$ in Ω^+. Therefore $P_1\overline{\psi}^{\pm}(x) \geq 0$ in Ω^*. Similarly one can prove that $P_2\overline{\psi}^{\pm}(x) \geq 0$ in Ω^*.

Let $x = 1$.

$$[\psi_i^{\pm'}](1) = C\ C_1[s_i'](1) + [u_i'](1) < 0,\ i = 1, 2.$$

Then by Theorem 3.1 we have $\psi_i^{\pm}(x) \geq 0,\ x \in \overline{\Omega},\ i = 1, 2.$ Therefore

$$| u_i(x) | \leq C\ \max\left\{\max_{j=1,2}\{| u_j(0) |\},\ \max_{j=1,2}\{| u_j(2) |\},\ \max_{j=1,2}\{\sup_{\zeta\in\Omega^*} | P_j\overline{u}(\zeta) |\}\right\}, \\ \forall\ x \in \overline{\Omega},\ i = 1, 2.$$

Hence the proof. □

An immediate application of the above Theorem 3.2 is that, the solution of the BVP (2.1) is unique.

4. Analytical Results

Let $\overline{u}_0(x) = (u_{01}(x), u_{02}(x)),\ u_{0i} \in C^0(\overline{\Omega}) \cap C^1(\Omega \cup \{2\}),\ i = 1, 2$, be the solution of the reduced problem of (2.1) given by

$$(4.1)\qquad \begin{cases} a_1(x)u'_{01}(x) + \sum_{k=1}^{2} b_{1k}(x)u_{0k}(x) \\ \qquad\qquad + \sum_{k=1}^{2} c_{1k}(x)u_{0k}(x-1) = f_1(x),\ x \in \Omega \cup \{2\}, \\ a_2(x)u'_{02}(x) + \sum_{k=1}^{2} b_{2k}(x)u_{0k}(x) \\ \qquad\qquad + \sum_{k=1}^{2} c_{2k}(x)u_{0k}(x-1) = f_2(x),\ x \in \Omega \cup \{2\}, \\ u_{01}(x) = \phi_1(x),\ x \in [-1, 0], \\ u_{02}(x) = \phi_2(x),\ x \in [-1, 0]. \end{cases}$$

Further, we assume that $\| u''_{0i} \|_{\Omega^*} \le C,\ i = 1, 2$.

Theorem 4.1. Let $\overline{u}$ be the solution of (2.1) and $\overline{u}_0$ be its reduced problem solution defined by (4.1). Then, $| u_i(x) - u_{0i}(x) | \le C\varepsilon + C \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right),\ x \in \overline{\Omega},\ i = 1, 2$.

Proof. Consider the barrier function $\overline{\varphi}^{\pm}(x) = (\varphi_1^{\pm}(x), \varphi_2^{\pm}(x))$, where

$$\varphi_i^{\pm}(x) = C_1\varepsilon s_i(x) + C_1 \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) \pm (u_i(x) - u_{0i}(x)),\ x \in \overline{\Omega},\ i = 1, 2.$$

It is easy to see that $\varphi_i^{\pm} \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$. Further, $\varphi_i^{\pm}(0) \ge 0,\ i = 1, 2$ and $\varphi_i^{\pm}(2) \ge 0$, $i = 1, 2$ for a suitable choice of $C_1 > 0$.

Case (i): $(x \in \Omega^-)$

$$\begin{aligned} P_1\overline{\varphi}^{\pm}(x) &= C_1\varepsilon[a_1(x)s'_1(x) + b_{11}(x)s_1(x) + b_{12}(x)s_2(x)] \\ &\quad + C_1\left[\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + (b_{11}(x) + b_{12}(x))\right] \exp(-\alpha(2-x)/\varepsilon) \\ &\quad \pm P_1(\overline{u}(x) - \overline{u}_0(x)) \\ &\ge C_1\varepsilon[\alpha_1/2 + \beta_1/8] + C_1\left[\frac{\alpha}{\varepsilon}(\alpha_1 - \alpha) + \beta_1\right] \exp(-\alpha(2-x)/\varepsilon) \mp C\varepsilon \\ &\ge 0, \end{aligned}$$

for a suitable choice of $C_1 > 0$.

Case (ii):$(x \in \Omega^+)$

$$\begin{aligned} P_1\overline{\varphi}^{\pm}(x) &= C_1\varepsilon[a_1(x)s'_1(x) + b_{11}(x)s_1(x) + b_{12}(x)s_2(x) + c_{11}(x)s_1(x-1) \\ &\quad + c_{12}(x)s_2(x-1)] + C_1\left[\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + (b_{11}(x) + b_{12}(x))\right] \\ &\quad \times \exp(-\alpha(2-x)/\varepsilon) + \left[(c_{11}(x) + c_{12}(x)) \exp(\frac{-\alpha}{\varepsilon})\right] \\ &\quad \times \exp(-\alpha(2-x)/\varepsilon) \pm P_1(\overline{u}(x) - \overline{u}_0(x)) \\ &\ge C_1\varepsilon[\alpha_1/4 + 5\beta_0/8 + 5\gamma_0/8] + C_1\left[\frac{\alpha}{\varepsilon}(\alpha_1 - \alpha) + \beta_1 + \gamma_1 \exp(\frac{-\alpha}{\varepsilon})\right] \\ &\quad \times \exp(-\alpha(2-x)/\varepsilon) \mp C\varepsilon \ge 0, \end{aligned}$$

for a suitable choice of $C_1 > 0$.
Similarly one can prove that $P_2\overline{\varphi}^{\pm}(x) \geq 0,\ x \in \Omega^*$. Further, $[\varphi_i^{\pm'}](1) < 0$, $i = 1, 2$.

Then by the Theorem 3.1, we have $\varphi_i^{\pm}(x) \geq 0,\ x \in \overline{\Omega},\ i = 1, 2$. That is,

$$\mid u_i(x) - u_{0i}(x) \mid \leq C\varepsilon + C \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right),\ x \in \overline{\Omega},\ i = 1, 2.$$

Hence the proof of the theorem. □

Note 4.2. From the above Theorem 4.1, it is clear that the solution $\overline{u}$ of the BVP (2.1) exhibits a strong boundary layer at $x = 2$ and further, away from the boundary layer and in particular on $[0, 1]$, we have

$$\mid u_i(x) - u_{0i}(x) \mid \leq C\left(\varepsilon + \exp\left(\frac{-\alpha}{\varepsilon}\right)\right) \leq C\varepsilon,\ x \in [0, 1],\ i = 1, 2.$$

Now we define an auxiliary problem to (2.1). Find $\overline{u}^*(x) = (u_1^*(x), u_2^*(x))$, $u_1^*,\ u_2^* \in Y^* = C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$(4.2) \qquad P_j^*\overline{u}^*(x) := -\varepsilon u_j^{*''}(x) + a_j(x)u_j^{*'}(x) + \sum_{k=1}^{2} b_{jk}(x)u_k^*(x) = f_j^*(x),$$
$$x \in \Omega,\ j = 1, 2,$$
$$u_j^*(0) = u_j(0),\ u_j^*(2) = u_j(2),\ j = 1, 2,$$

where

$$f_j^*(x) = \begin{cases} f_j(x) - \sum_{k=1}^{2} c_{jk}(x)\phi_k(x-1),\ x \in \Omega^- \cup \{1\}, \\ f_j(x) - \sum_{k=1}^{2} c_{jk}(x)u_{0k}(x-1),\ x \in \Omega^+,\ j = 1, 2. \end{cases}$$

Theorem 4.3. Let $\overline{w}(x) = (w_1(x), w_1(x)),\ w_1, w_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$ be any function satisfying $w_i(0) \geq 0,\ w_i(2) \geq 0,\ i = 1, 2,\ P_j^*\overline{w}(x) \geq 0,\ x \in \Omega^*$, $j = 1, 2$ and $[w_i'](1) \leq 0,\ i = 1, 2$. Then, $w_i(x) \geq 0,\ x \in \overline{\Omega},\ i = 1, 2$.

Proof. See [23]. □

Theorem 4.4. Let $\overline{u}$ and $\overline{u}^*$ be the solutions of the problems (2.1) and (4.2), respectively. Then, $\mid u_i(x) - u_i^*(x) \mid \leq C\varepsilon,\ x \in \overline{\Omega},\ i = 1, 2$.

Proof. Consider the barrier function $\overline{\varphi}^{\pm}(x) = (\varphi_1^{\pm}(x), \varphi_2^{\pm}(x))$, where

$$\varphi_i^{\pm}(x) = C_1\varepsilon s_i(x) \pm (u_i(x) - u_i^*(x)),\ x \in \overline{\Omega},\ i = 1, 2,$$

C_1 is a positive constant. Note that $\varphi_i^{\pm} \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+),\ i = 1, 2$. Further, $\varphi_i^{\pm}(0) \geq 0,\ i = 1, 2$ and $\varphi_i^{\pm}(2) \geq 0,\ i = 1, 2$ for a suitable choice of $C_1 > 0$.

Case (i): $(x \in \Omega^-)$

$$\begin{aligned} P_1\overline{\varphi}^{\pm}(x) &= C_1\varepsilon[a_1(x)s_1'(x) + b_{11}(x)s_1(x) + b_{12}(x)s_2(x)] \pm 0 \\ &\geq C_1\varepsilon[\alpha_1/2 + \beta_1/8] \pm 0 \ \geq 0, \end{aligned}$$

for a suitable choice of $C_1 > 0$.

Case (ii):$(x \in \Omega^+)$

$$\begin{aligned} P_1\overline{\varphi}^{\pm}(x) &= C_1\varepsilon[a_1(x)s_1'(x) + b_{11}(x)s_1(x) + b_{12}(x)s_2(x) \\ &+ c_{11}(x)s_1(x-1) + c_{12}(x)s_2(x-1)] \\ &\pm c_{11}(x)(u_{01}(x-1) - u_1(x-1)) + c_{12}(x)(u_{02}(x-1) - u_2(x-1)) \\ &\geq C_1\varepsilon[\alpha_1/4 + 5\beta_0/8 + 5\gamma_0/8] \mp C\varepsilon \geq 0, \end{aligned}$$

for a suitable choice of $C_1 > 0$.
Similarly one can prove that $P_2\overline{\varphi}^{\pm}(x) \geq 0,\ x \in \Omega^*$. Further, $[\varphi_i^{\pm'}](1) < 0$, $i = 1, 2$.

Then by the Theorem 3.1, we have $\varphi_i^{\pm}(x) \geq 0,\ x \in \overline{\Omega},\ i = 1, 2$. That is,

$$| u_i(x) - u_i^*(x) | \leq C\varepsilon,\ x \in \overline{\Omega},\ i = 1, 2.$$

Hence the proof of the theorem.

□

5. Discrete Problem

In this section, first a mesh selection strategy is explained. Then the fourth order Runge Kutta method with piecewise cubic Hermite interpolation on this mesh for initial value problem (4.1) and an upwind finite difference scheme for the BVP (4.2) are presented. Further error estimates of these methods are given.

5.1. Mesh Selection Strategy

The BVP (2.1) and the auxiliary problem (4.2) exhibit a strong boundary layer at $x = 2$ and the functions f_1^* and f_2^* are continuous in $[0, 2]$ but not differentiable at $x = 1$. Further, $x = 1$ is a primary discontinuity point [1]. Therefore, we choose a piece-wise uniform Shishkin mesh on $[0, 2]$. In fact, we divide the interval $[0, 2]$ into four subintervals, namely $\Omega_1 = [0, 1-\tau]$, $\Omega_2 = [1-\tau, 1]$, $\Omega_3 = [1, 2-\tau]$, $\Omega_4 = [2-\tau, 2]$, where $\tau = \min\left\{0.5,\ \frac{2\varepsilon \ln N}{\alpha}\right\}$.

Let $h = 2N^{-1}\tau$ & $H = 2N^{-1}(1-\tau)$. The mesh $\overline{\Omega}^{2N} = \{x_0,\ x_1,\ \ldots,\ x_{2N}\}$ is defined by

$$\begin{aligned} &x_0 = 0.0,\ x_i = x_0 + iH,\ i = 1(1)\frac{N}{2},\ x_{i+\frac{N}{2}} = x_{\frac{N}{2}} + ih,\ i = 1(1)\frac{N}{2}, \\ &x_{i+N} = x_N + iH,\ i = 1(1)\frac{N}{2},\ x_{i+\frac{3N}{2}} = x_{\frac{3N}{2}} + ih,\ i = 1(1)\frac{N}{2}. \end{aligned}$$

5.2. Numerical Solution for (4.1)

In order to obtain a numerical solution for the problem (4.1), we apply the fourth order Runge-Kutta method with piecewise cubic Hermite interpolation on $\overline{\Omega}^{2N}$ [1]. In fact, the numerical solution is given by

$$
\begin{aligned}
(5.1)\qquad & \overline{U}_0(x_i) = (U_{01}(x_i),\ U_{02}(x_i)),\ i = 0(1)2N,\\
& U_{0j}(x_0) = \phi_j(x_0),\ j = 1,2,\\
& U_{0j}(x_{i+1}) = U_{0j}(x_i) + \frac{1}{6}(K_{j1} + 2K_{j2} + 2K_{j3} + K_{j4}),\\
& \qquad\qquad i = 0(1)2N-1,\ j = 1,2,
\end{aligned}
$$

where

$$
\begin{cases}
K_{j1} = h^*\Big[\frac{f_j(x_i)}{a_j(x_i)} - \frac{\sum_{l=1}^{2} b_{jl}(x_i)U_{0l}(x_i)}{a_j(x_i)} - \frac{\sum_{l=1}^{2} c_{jl}(x_i)U_{0l}^{h^*}(x_i)}{a_j(x_i)}\Big],\\
K_{j2} = h^*\Big[\frac{f_j(x_i+\frac{h^*}{2})}{a_j(x_i+\frac{h^*}{2})} - \frac{\sum_{l=1}^{2} b_{jl}(x_i+\frac{h^*}{2})(U_{0l}(x_i)+\frac{K_{l1}}{2})}{a_j(x_i+\frac{h^*}{2})}\\
\qquad - \frac{\sum_{l=1}^{2} c_{jl}(x_i+\frac{h^*}{2})U_{0l}^{h^*}(x_i+\frac{h^*}{2})}{a_j(x_i+\frac{h^*}{2})}\Big],\\
K_{j3} = h^*\Big[\frac{f_j(x_i+\frac{h^*}{2})}{a_j(x_i+\frac{h^*}{2})} - \frac{\sum_{l=1}^{2} b_{jl}(x_i+\frac{h^*}{2})(U_{0l}(x_i)+\frac{K_{l2}}{2})}{a_j(x_i+\frac{h^*}{2})}\\
\qquad - \frac{\sum_{l=1}^{2} c_{jl}(x_i+\frac{h^*}{2})U_{0l}^{h^*}(x_i+\frac{h^*}{2})}{a_j(x_i+\frac{h^*}{2})}\Big],\\
K_{j4} = h^*\Big[\frac{f_j(x_i+h^*)}{a_j(x_i+h^*)} - \frac{\sum_{l=1}^{2} b_{jl}(x_i+h^*)(U_{0l}(x_i)+K_{l3})}{a_j(x_i+h^*)}\\
\qquad - \frac{\sum_{l=1}^{2} c_{jl}(x_i+h^*)U_{0l}^{h^*}(x_i+h^*)}{a_j(x_i+h^*)}\Big],
\end{cases}
$$

$$
h^* = \begin{cases} H,\ i = 0(1)\frac{N}{2}-1,\ i = N(1)\frac{3N}{2}-1,\\ h,\ i = \frac{N}{2}(1)N-1,\ i = \frac{3N}{2}(1)2N-1, \end{cases}
$$

$$
U_{0l}^{h^*}(x) = \begin{cases} \phi_l(x-1),\ x \in [x_i, x_{i+1}],\ i = 0(1)N-1,\\ U_{0l}(x_m)A_m(x-1) + U_{0l}(x_{m+1})A_{m+1}(x-1)\\ +B_m(x-1)\tilde{f}_l(x_m) + B_{m+1}(x-1)\tilde{f}_l(x_{m+1}), x \in [x_i, x_{i+1}],\\ \qquad i = N(1)2N-1,\ m = i-N,\ l = 1,2, \end{cases}
$$

$$
\begin{aligned}
A_m(x) &= \Big[1 - \frac{2(x-x_m)}{x_m - x_{m+1}}\Big]\frac{(x-x_{m+1})^2}{(x_m - x_{m+1})^2},\\
A_{m+1}(x) &= \Big[1 - \frac{2(x-x_{m+1})}{x_{m+1} - x_m}\Big]\frac{(x-x_m)^2}{(x_{m+1} - x_m)^2},\\
B_m(x) &= \frac{(x-x_m)(x-x_{m+1})^2}{(x_m - x_{m+1})^2},\ B_{m+1}(x) = \frac{(x-x_{m+1})(x-x_m)^2}{(x_{m+1}-x_m)^2},\\
\tilde{f}_j(x_m) &= \frac{f_j(x_m)}{a_j(x_m)} - \frac{\sum_{l=1}^{2} b_{jl}(x_m)U_{0l}(x_m)}{a_j(x_m)} - \frac{\sum_{l=1}^{2} c_{jl}(x_m)\phi_l(x_m - 1)}{a_j(x_m)},\ j = 1,2.
\end{aligned}
$$

The following theorem gives an error estimate for the above method.

Theorem 5.1. Let $\overline{u}_0(x)$ be the solution of the problem (4.1). Further, let $\overline{U}_0(x_i) = (U_{01}(x_i), U_{02}(x_i))$ be its numerical solution defined by (5.1). Then, $\| \overline{u}_0 - \overline{U}_0 \|_{\overline{\Omega}^{2N}} \leq CH^4$.

Proof. See [1]. □

5.3. A Finite Difference Method for the BVP (4.2)

On $\overline{\Omega}^{2N}$, we define the following scheme for the BVP (4.2):

(5.2)

$$\begin{cases} P_j^{*N}\overline{U}^*(x_i) = -\varepsilon\delta^2 U_j^*(x_i) + a_j(x_i)D^- U_j^*(x_i) + \sum_{k=1}^{2} b_{jk}(x_i)U_k^*(x_i) \\ \qquad\qquad = F_j^*(x_i),\ j = 1,2,\ i = 1(1)N-1, N+1(1)2N-1, \\ D^- U_j^*(x_N) = D^+ U_j^*(x_N),\ j = 1,2, \\ U_j^*(x_0) = u_j^*(0),\ U_j^*(x_{2N}) = u_j^*(2),\ j = 1,2, \end{cases}$$

where

$$\delta^2 U_j^*(x_i) = \frac{2}{x_{i+1} - x_{i-1}} \Big[D^+ U_j^*(x_i) - D^- U_j^*(x_i) \Big],$$
$$D^- U_j^*(x_i) = \frac{U_j^*(x_i) - U_j^*(x_{i-1})}{x_i - x_{i-1}},\ D^+ U_j^*(x_i) = \frac{U_j^*(x_{i+1}) - U_j^*(x_i)}{x_{i+1} - x_i},$$

$$F_j^*(x_i) = f_j^*(x_i),\ x_i \in \overline{\Omega}^{2N} \setminus \{x_0, x_N, x_{2N}\}, \tag{5.3}$$

or

$$F_j^*(x_i) = \begin{cases} f_j(x_i) - \sum_{k=1}^{2} c_{jk}(x_i)\phi_k(x_i - 1),\ x_i \in \Omega^- \cap \overline{\Omega}^{2N}, \\ f_j(x_i) - \sum_{k=1}^{2} c_{jk}(x_i)U_{0k}(x_{i-N}),\ x_i \in \Omega^+ \cap \overline{\Omega}^{2N}. \end{cases} \tag{5.4}$$

Theorem 5.2. *(Discrete maximum principle) Suppose a mesh function* $\overline{Z}(x_i)$ *satisfies* $Z_j(x_0) \geq 0,\ Z_j(x_{2N}) \geq 0,\ j = 1,2,\ P_j^{*N}\overline{Z}(x_i) \geq 0,\ x_i \in \overline{\Omega}^{2N} \setminus \{x_0, x_N, x_{2N}\},\ j = 1,2$ *and* $[D]Z_j(x_N) = D^+ Z_j(x_N) - D^- Z_j(x_N) \leq 0$, $j = 1,2$. *Then,* $Z_j(x_i) \geq 0,\ \forall x_i \in \overline{\Omega}^{2N},\ j = 1,2.$

Proof. See [23]. □

A consequence of this theorem is the following stability result.

Theorem 5.3. Let $\overline{U}^*(x_i)$ be a numerical solution of the problem (4.2) defined by (5.2) either with (5.3) or (5.4). Then,

$$| U_k^*(x_i) |$$
$$\leq C \max \left\{ \max_{j=1,2}\{| U_j^*(x_0) |\},\ \max_{j=1,2}\{| U_j^*(x_{2N}) |\}, \max_{j=1,2}\{\max_{l \in J} | P_j^* \overline{U}^*(x_l) |\} \right\},$$
$$\forall\ x_i \in \overline{\Omega}^{2N},\ k = 1,2,\ J = \{1, 2, \cdots, N-1, N+1, \cdots, 2N-1\}.$$

Theorem 5.4. Let $\overline{u}^*$ be the solution of the auxiliary problem (4.2) and let $\overline{U}^*(x_i)$ be the corresponding numerical solution defined by (5.2) and (5.3). Then,

$$| u_j^*(x_i) - U_j^*(x_i) | \leq CN^{-1} \ln N, \ x_i \in \overline{\Omega}^{2N}, \ j = 1, 2.$$

Proof. See [23]. □

Theorem 5.5. Let $\overline{U}^*(x_i)$ be a numerical solution of (4.2) defined by (5.2) and (5.4). If $\varepsilon \leq CN^{-1}$, then we have

$$| u_j^*(x_i) - U_j^*(x_i) | \leq CN^{-1} \ln N, \ x_i \in \overline{\Omega}^{2N}, \ j = 1, 2.$$

Proof. Using the Theorems 5.1 and 5.3 and the result given in [23] one can derive the desired result. It may be noted that, the result derived in [23] is true only when $\varepsilon \leq CN^{-1}$. □

6. Asymptotic Numerical Method

We now explain how to obtain a numerical solution for the BVP (2.1) by the asymptotic numerical method. First we solve the reduced problem (4.1) either exactly or numerically. Then we solve numerically the auxiliary problem (4.2) by using the scheme (5.2) with either (5.3) or (5.4). This numerical solution is taken as an approximation to the exact solution of the BVP (2.1). An error estimate for this approximation is given in the following theorem.

Theorem 6.1. Let $\overline{u}$ be the solution of the problem (2.1) and let $\overline{U}^*(x_i)$ be a numerical solution defined by (5.2) with either (5.3) or (5.4) . If $\varepsilon \leq CN^{-1}$, then we have, $\| \overline{u} - \overline{U}^* \|_{\overline{\Omega}^{2N}} \leq CN^{-1} \ln N$.

Proof. Then by the Theorems 4.4 and 5.4 or 5.5 we have,

$$\begin{aligned} | u_j(x_i) - U_j^*(x_i) | &\leq | u_j(x_i) - u_j^*(x_i) | + | u_j^*(x_i) - U_j^*(x_i) |, \ x_i \in \overline{\Omega}^{2N}, \\ &\leq C\varepsilon + CN^{-1} \ln N \\ &\leq CN^{-1} + CN^{-1} \ln N \leq CN^{-1} \ln N, \ j = 1, 2 \end{aligned}$$

since $\varepsilon \leq CN^{-1}$. Hence the proof of the theorem. □

7. Numerical Results

In this section, three examples are given to illustrate the numerical technique discussed in this paper. We use the double mesh principle to estimate the error and compute the experiment rate of convergence in our computed solution. For this we put

$$D_{k,\varepsilon}^M = \max_{0 \leq i \leq M} | U_k^M(x_i) - U_k^{2M}(x_{2i}) |, \ k = 1, 2,$$

where $U_k^M(x_i)$ and $U_k^{2M}(x_{2i})$ are the i^{th} and $2i^{th}$ components of the numerical solutions on meshes of M and $2M$ points, respectively. Here $M = 2N$. We compute the error and rate of convergence as

$$D_k^M = \max_{\varepsilon} D_{k,\varepsilon}^M, \; p_k^M = \log_2\left(\frac{D_k^M}{D_k^{2M}}\right), \; k = 1, 2.$$

For the following examples the numerical results are presented using the expression (5.4) and the range of the perturbation parameter is from $\varepsilon = 2^{-4}$ to 2^{-23}.

Example 7.1. (Variable Coefficient Problem)

$$\begin{aligned}
-\varepsilon u_1''(x) + 11u_1'(x) + 6u_1(x) - 2u_2(x) - (x^2+1)u_1(x-1) \\
- (x+1)u_2(x-1) = 0, \\
-\varepsilon u_2''(x) + 16u_2'(x) - 2u_1(x) + 5u_2(x) - xu_1(x-1) - xu_2(x-1) = 0, \\
u_1(x) = 1, \; x \in [-1, 0], \; u_1(2) = 1, \\
u_2(x) = 1, \; x \in [-1, 0], \; u_2(2) = 1.
\end{aligned}$$

Table 1 presents the values of D_k^M, p_k^M, $i = 1, 2$ and Figure 1 represents the numerical solution of this Example 7.1.

Example 7.2. (Constant Coefficient Problem)

$$\begin{aligned}
-\varepsilon u_1''(x) + 11u_1'(x) + 6u_1(x) - 2u_2(x) - u_1(x-1) = 0, \\
-\varepsilon u_2''(x) + 16u_2'(x) - 2u_1(x) + 5u_2(x) - u_2(x-1) = 0, \\
u_1(x) = 1, \; x \in [-1, 0], \; u_1(2) = 1, \\
u_2(x) = 1, \; x \in [-1, 0], \; u_2(2) = 1.
\end{aligned}$$

Table 2 presents the values of D_k^M, p_k^M, $i = 1, 2$ and Figure 2 represents the numerical solution of this Example 7.2.

Example 7.3.

$$\begin{aligned}
-\varepsilon u_1''(x) + 11u_1'(x) + 6u_1(x) - 2u_2(x) - u_1(x-1) = \exp(x), \\
-\varepsilon u_2''(x) + 16u_2'(x) - 2u_1(x) + 5u_2(x) - u_2(x-1) = x^2, \\
u_1(x) = \exp(x), \; x \in [-1, 0], \; u_1(2) = 1, \\
u_2(x) = 1 + x, \; x \in [-1, 0], \; u_2(2) = 1.
\end{aligned}$$

Table 3 presents the values of D_k^M, p_k^M, $i = 1, 2$ and Figure 7 represents the numerical solution of this Example 7.3.

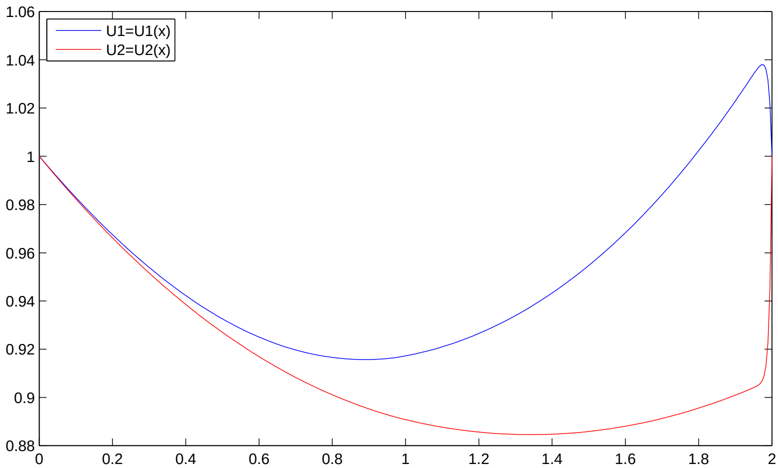

Figure 1: Numerical solution of the above Example 7.1.

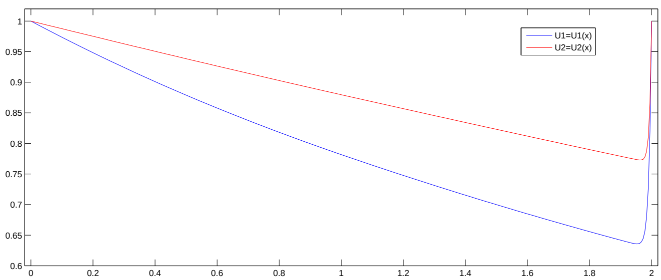

Figure 2: Numerical solution of the above Example 7.2.

Table 1: Numerical Results for the Example 7.1

	N (Number of mesh points)						
	32	64	128	256	512	1024	2048
D_1^M	7.4709e-3	3.8838e-3	1.9843e-3	9.9963e-4	5.0116e-4	2.5084e-4	1.2548e-4
p_1^M	9.4383e-1	9.6884e-1	9.8915e-1	9.9611e-1	9.9850e-1	9.9937e-1	-
D_2^M	1.3064e-2	9.8093e-3	6.9348e-3	4.4954e-3	2.8118e-3	1.6521e-3	9.4091e-4
p_2^M	4.1336e-1	5.0029e-1	6.2540e-1	6.7693e-1	7.6721e-1	8.1218e-1	-

8. Conclusion

A BVP for one type of SPDDEs is considered. To obtain an approximate solution for this type of problems, an asymptotic numerical method is presented. The method is shown to be convergent of order $O(N^{-1}\ln N)$. This is very much reflected in the numerical results given in Tables 1-3. The problem

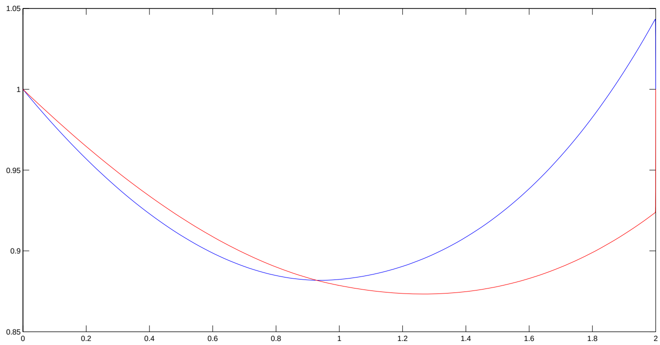

Figure 3: Numerical solution of the above Example 7.3.

Table 2: Numerical Results for the Example 7.2

	N (Number of mesh points)						
	32	64	128	256	512	1024	2048
D_1^M	1.4853e-2	1.0525e-2	6.6142e-3	4.2351e-3	2.5371e-3	1.4828e-3	8.4244e-4
p_1^M	4.9694e-1	6.7014e-1	6.4316e-1	7.3920e-1	7.7484e-1	8.1571e-1	-
D_2^M	9.2969e-3	7.2360e-3	5.3258e-3	3.5093e-3	2.1976e-3	1.2987e-3	7.4206e-4
p_2^M	3.6154e-1	4.4221e-1	6.0182e-1	6.7523e-1	7.5888e-1	8.0745e-1	-

Table 3: Numerical Results for the Example 7.3

	N (Number of mesh points)						
	32	64	128	256	512	1024	2048
D_1^M	7.4709e-3	3.8838e-3	1.9843e-3	9.9963e-4	5.0116e-4	2.5084e-4	1.2548e-4
p_1^M	9.4383e-1	9.6884e-1	9.8915e-1	9.9611e-1	9.9850e-1	9.9937e-1	-
D_2^M	1.3064e-2	9.8093e-3	6.9348e-3	4.4954e-3	2.8118e-3	1.6521e-3	9.4091e-4
p_2^M	4.1336e-1	5.0029e-1	6.2540e-1	6.7693e-1	7.6721e-1	8.1218e-1	-

(4.2) is solved numerically by the scheme (5.2). This gives only almost first order accuracy. It is true that, it is enough to apply Euler finite difference scheme with linear interpolation to get first order accuracy. One can improve this using the higher order schemes for the problem (4.2). Only for this reason, we have used fourth-order Runge-Kutta method and piecewise cubic Hermite interpolation.

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