Novi Sad J. Math.
Vol. 44, No. 2, 2014, 41-51

ON C-BOCHNER CURVATURE TENSOR OF (k, μ)-CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to study the C-Bochner curvature tensor in an n-dimensional $(n \geq 5)$ (k, μ)-contact metric manifold.

AMS Mathematics Subject Classification (2010): 53C15, 53C25

Key words and phrases: C-Bochner semi-symmetric manifold, (k, μ)-contact metric manifold, Einstein manifold, η-Einstein manifold, Sasakian manifold.

1. Introduction

In modern mathematics the study of contact geometry has become a matter of growing interest due to its role in explaining physical phenomena in the context of mathematical physics. An important class of contact manifolds is Sasakian manifolds introduced by S. Sasaki [18]. Among the geometric properties of manifolds symmetry is an important one. A Riemannian manifold M is called locally symmetric if its curvature tensor R is parallel, i.e., $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. As a generalization of locally symmetric spaces, many geometers have considered semisymmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semisymmetric if its curvature tensor R satisfies

$$R(X, Y).R = 0, \qquad X, Y \in T(M),$$

where $R(X, Y)$ acts on R as a derivation. In contact geometry, S. Tanno [19] showed that a semisymmetric K-contact manifold M is locally isometric to the unit sphere $S^n(1)$.

On the other hand, S. Bochner [6] introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor. A geometric meaning of the Bochner curvature tensor was given by D. E. Blair [5]. By using the Boothby-Wang's fibration [8], M. Matsumoto and G. Chuman [17] constructed the C-Bochner curvature tensor from the Bochner curvature tensor. The C-Bochner curvature

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tensor is given by

$$
\begin{aligned}
(1.1)\qquad B(X,Y)Z &= R(X,Y)Z + \frac{1}{n+3}[S(X,Z)Y - S(Y,Z)X \\
&+ g(X,Z)QY - g(Y,Z)QX + S(\phi X,Z)\phi Y \\
&- S(\phi Y,Z)\phi X + g(\phi X,Z)Q\phi Y \\
&- g(\phi Y,Z)Q\phi X + 2S(\phi X,Y)\phi Z \\
&+ 2g(\phi X,Y)Q\phi Z - S(X,Z)\eta(Y)\xi \\
&+ S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY \\
&+ \eta(Y)\eta(Z)QX] - \frac{p+n-1}{n+3}[g(\phi X,Z)\phi Y \\
&- g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\
&- \frac{p-4}{n+3}[g(X,Z)Y - g(Y,Z)X] \\
&+ \frac{p}{n+3}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\
&+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X],
\end{aligned}
$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and $p = \frac{n+r-1}{n+1}$, r being the scalar curvature of the manifold.

H. R. Choi and U. H. Kim [10] studied Sasakian manifolds with constant scalar curvature where the C-Bochner curvature vanishes. Also, Sasakian manifolds with vanishing C-Bochner curvature have been studied in [13]. $N(k)$-contact metric manifods satisfying $B.R = 0$, $R.B = 0$ are studied by J. S . Kim, M. M. Tripathi and J. D. Choi in [16]. In this paper they also considered non-Sasakian (k,μ)-contact manifolds satisfying $B(\xi,X).S = 0$. Beside these, J. T. Cho [9] studied (k,μ)-contact manifold with vanishing C-Bochner curvature tensor. C-Bochner curvature tensor has also been studied by A. De [11] on an $N(k)$-contact metric manifold. Motivated by these studies we consider C-Bochner semisymmetry on a (k,μ)-contact metric manifold which is defined as follows:

Definition 1.1. *An n-dimensional (k,μ)-contact metric manifold is said to be C-Bochner semi-symmetric if*

$$(1.2)\qquad R(X,Y).B = 0,$$

where B is the C-Bochner curvature tensor.

The present paper is organized as follows:

After preliminaries in section 3, we study C-Bochner semisymmetry on a (k,μ)-contact metric manifold and prove that this manifold is η-Einstein. Beside this, some important corollaries are given in this section. In section 4, we deal with (k,μ)-contact metric manifolds satisfying $B(\xi,U).R = 0$. In this section we prove that such a manifold is either a Sasakian or an Einstein manifold provided that $[\frac{4(k-1)}{n+3} + \frac{\mu^2(n+3)}{4}] \neq 0$. Section 5 is devoted to study an Einstein

(k,μ)-contact metric manifold and we prove that the relation $B(\xi, X).S = 0$ holds identically in such a manifold.

2. Preliminaries

By a contact manifold we mean an $n = (2m+1)$-dimensional differentiable manifold M^n which carries a global 1-form η and there exists a unique vector field ξ, called the characteristic vector field, such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$. A Riemannian metric g on M^n is said to be an associated metric if there exists a $(1,1)$ tensor field ϕ such that

$$d\eta(X,Y) = g(X,\phi Y), \;\; \eta(X) = g(X,\xi), \;\; \phi^2 = -I + \eta \otimes \xi. \tag{2.1}$$

From these equations we have

$$\phi\xi = 0, \;\; \eta \circ \phi = 0, \;\; g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

The manifold M^n equipped with the contact structure (ϕ, ξ, η, g) is called a contact metric manifold [1].

Given a contact metric manifold $M^n(\phi, \xi, \eta, g)$ we define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. Thus, if λ is an eigenvalue of h with eigenvector X, $-\lambda$ is also an eigenvalue with eigenvector ϕX. Also we have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi h X. \tag{2.3}$$

A contact metric manifold is said to be Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant and η-Einstein if $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.4}$$

$X, Y \in TM$, where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^n(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian [14]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [2]. On the other hand, on a Sasakian manifold the following holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.5}$$

It is well known that there exists contact metric manifolds for which the curvature tensor R and the direction of the characteristic vector field ξ satisfy $R(X,Y)\xi = 0$ for any vector fields X and Y. For example, the tangent bundle of a flat Riemannian manifold admits such a structure.

As a generalization of both $R(X,Y)\xi = 0$ and the Sasakian case: D. E. Blair, Th. Koufogiorgos and B. J. Papantoniou [3] considered the (k,μ) nullity condition on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ [[3],[15]] of a contact metric manifold is defined by

$$\begin{aligned} N(k,\mu) : p \longrightarrow N_p(k,\mu) &= [W \in T_pM \mid R(X,Y)W \\ &= (kI + \mu h)(g(Y,W)X - g(X,W)Y)], \end{aligned}$$

for all $X,\ Y \in$TM, where $(k,\mu) \in \mathbb{R}^2$. A contact metric manifold M^n with $\xi \in N(k,\mu)$ is called a (k,μ) contact metric manifold. Then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.6}$$

Applying a D-homothetic deformation to a contact metric manifold with $R(X,Y)\xi = 0$, we obtain a contact metric manifold satisfying (2.6). In [3], it is proved that the standard contact metric structure on the tangent sphere bundle T_1(M) satisfies the condition that ξ belongs to the (k,μ)-nullity distribution if and only if the base manifold is the space of constant curvature. There exist examples in all dimensions and the condition that ξ belongs to the (k,μ)-nullity distribution is invariant under D-homothetic deformations; in dimension greater than 5, the condition determines the curvature completely; dimension 3 includes the 3-dimensional unimoduler Lie groups with the left invariant metric.

On a (k,μ)-contact metric manifold one has $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, the (k,μ)-nullity condition completely determines the curvature of M^n [4]. In fact, for a (k,μ)-contact manifold, the conditions of being Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent. Again a (k,μ)-contact manifold reduces to an $N(k)$-contact manifold if and only if $\mu = 0$.

In a (k,μ) contact manifold, the following relations hold [[3],[7]]:

$$h^2 = (k-1)\phi^2, \ \ k \leq 1, \tag{2.7}$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.8}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.9}$$

$$S(X,\xi) = (n-1)k\eta(X), \tag{2.10}$$

$$\begin{aligned} S(X,Y) &= [(n-3) - \frac{n-1}{2}\mu]g(X,Y) \\ &\quad +[(n-3) + \mu]g(hX,Y) \\ &\quad +[(3-n) + \frac{n-1}{2}(2k+\mu)]\eta(X)\eta(Y), \ n \geq 5, \end{aligned} \tag{2.11}$$

$$r = (n-1)(n-3+k-\frac{n-1}{2}\mu), \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X,Y) - (n-1)k\eta(X)\eta(Y) - 2(n-3+\mu)g(hX,Y), \tag{2.13}$$

where S is the Ricci tensor of type $(0,2)$ and r is the scalar curvature of the manifold. From (2.4) it follows that

$$(\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{2.14}$$

Also in a (k,μ)-manifold, the following holds

$$\begin{aligned} \eta(R(X,Y)Z) &= k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &\quad + \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)], \end{aligned} \tag{2.15}$$

for $Z \in N(k,\mu)$.

Especially for the case $\mu = (3-n)$, from (2.11) it follows that the manifold is η-Einstein. For more details we refer to [4].

It is well known that in a Sasakian manifold the Ricci operator Q commutes with ϕ. But in a (k,μ)-contact metric manifold, Q does not commute with ϕ, in general. In a (k,μ)-contact metric manifold D. E. Blair, Th. Koufogiorgos and B. J. Papantoniou [3] proved the following:

Lemma 2.1. *Let M^n be a (k,μ)-contact metric manifold. Then the relation*

$$Q\phi - \phi Q = 2[(n-3)+\mu]h\phi \quad \textit{holds.}$$

From the definition of η-Einstein manifold it follows that $Q\phi = \phi Q$, since $\phi\xi = 0$. Hence from Lemma 2.1 we have either $\mu = -(n-3)$ or the manifold is Sasakian. Using $\mu = -(n-3)$ from (2.11) we get the manifold is an η-Einstein manifold. Therefore we state the following:

Proposition 2.1. *In a non-Sasakian (k,μ)-contact metric manifold the following conditions are equivalent:*
i) η-Einstein manifold,
ii) $Q\phi = \phi Q$.

From (1.1) it can be easily verified that on a (k,μ)-contact manifold the C-Bochner curvature tensor satisfies the following:

$$\begin{aligned} B(X,Y)\xi &= \frac{4(k-1)}{n+3}[\eta(Y)X - \eta(X)Y] \\ &\quad + \mu[\eta(Y)hX - \eta(X)hY], \end{aligned} \tag{2.16}$$

$$\begin{aligned} B(\xi,Y)Z &= \frac{4(k-1)}{n+3}[g(Y,Z)\xi - \eta(Z)Y] \\ &\quad + \mu[g(hY,Z)\xi - \eta(Z)hY], \end{aligned} \tag{2.17}$$

$$B(X,\xi)Z = \frac{4(k-1)}{n+3}[\eta(Z)X - g(X,Z)\xi] + \mu[\eta(Z)hX - g(hX,Z)\xi], \tag{2.18}$$

$$B(\xi,Y)\xi = \frac{4(k-1)}{n+3}[\eta(Y)\xi - Y] - \mu hY, \tag{2.19}$$

$$B(X,\xi)\xi = \frac{4(k-1)}{n+3}[X - \eta(X)\xi] + \mu hX. \tag{2.20}$$

Taking inner product with W we obtain from (1.1)

$$\begin{aligned}\tilde{B}(X,Y,Z,W) &= g(R(X,Y)Z,W)\\ &+\frac{1}{n+3}[S(X,Z)g(Y,W)\\ &-S(Y,Z)g(X,W) + g(X,Z)S(Y,W) - g(Y,Z)S(X,W)\\ &+S(\phi X,Z)g(\phi Y,W) - S(\phi Y,Z)g(\phi X,W) + g(\phi X,Z)S(\phi Y,W)\\ &-g(\phi Y,Z)S(\phi X,W) + 2S(\phi X,Y)g(\phi Z,W)\\ &+2g(\phi X,Y)S(\phi Z,W) - S(X,Z)\eta(Y)\eta(W)\\ &+S(Y,Z)\eta(X)\eta(W) - \eta(X)\eta(Z)S(Y,W)\\ &+\eta(Y)\eta(Z)S(X,W)] - \frac{p+n-1}{n+3}[g(\phi X,Z)g(\phi Y,W)\\ &-g(\phi Y,Z)g(\phi X,W) + 2g(\phi X,Y)g(\phi Z,W)]\\ &-\frac{p-4}{n+3}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]\\ &+\frac{p}{n+3}[g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\\ &+\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)],\end{aligned} \tag{2.21}$$

where $\tilde{B}(X,Y,Z,W) = g(B(X,Y)Z,W)$. Let

$$\{e_1, e_2, ..., e_m, e_{m+1} = \phi e_1, ..., e_{2m} = \phi e_m, e_{2m+1} = \xi\}$$

be a ϕ-basis of the manifold. Putting $X = W = e_i$ in (2.21) and taking summation over $i = 1$ to n we obtain by virtue of (2.13)

$$\sum_{i=1}^{n} \tilde{B}(e_i,Y,Z,e_i) = \frac{(n-1)k + p(2-n) + r}{n+3}\eta(Y)\eta(Z) + \frac{6(n-3+\mu)}{n+3}g(hY,Z). \tag{2.22}$$

Replacing Z by hZ in (2.22) and using (2.7), (2.1) we get

$$\sum_{i=1}^{n} \tilde{B}(e_i,Y,hZ,e_i) = \frac{6(k-1)(n-3+\mu)}{n+3}\eta(Y)\eta(Z) - \frac{6(k-1)(n-3+\mu)}{n+3}g(Y,Z). \tag{2.23}$$

Again from (2.11) we obtain

$$\sum_{i=1}^{n} g(he_i,e_i) = \frac{1}{n-3+\mu}[r - \frac{n-1}{2}\{2n-6-2k-(n-1)\mu\}]. \tag{2.24}$$

3. C-Bochner semisymmetric (k,μ)-contact manifolds

We devote this section to the study of C-Bochner semisymmetric (k,μ)-contact metric manifolds. Putting $Y=\xi$ in (1.2) we obtain

$$\begin{aligned} &R(X,\xi).B(U,V)W - B(R(X,\xi)U,V)W \\ &-B(U,R(X,\xi)V)W - B(U,V)R(X,\xi)W = 0. \end{aligned} \tag{3.1}$$

Using (2.9) in (3.1), we get

$$\begin{aligned} &k[\eta(B(U,V)W)X - g(X,B(U,V)W)\xi - \eta(U)B(X,V)W \\ &+g(X,U)B(\xi,V)W - \eta(V)B(U,X)W + g(X,V)B(U,\xi)W \\ &-\eta(W)B(U,V)X + g(X,W)B(U,V)\xi] + \mu[\eta(B(U,V)W)hX \\ &-g(hX,B(U,V)W)\xi - \eta(U)B(hX,V)W + g(hX,U)B(\xi,V)W \\ &-\eta(V)B(U,hX)W + g(hX,V)B(U,\xi)W - \eta(W)B(U,V)hX \\ &+g(hX,W)B(U,V)\xi] = 0. \end{aligned} \tag{3.2}$$

Putting $W=\xi$ in (3.2) and using (2.16), (2.19) and (2.20) we have

$$\begin{aligned} &\frac{4k(k-1)}{n+3}[g(X,V)U - g(X,U)V] + \frac{4\mu(k-1)}{n+3}[g(hX,V)U \\ &-g(hX,U)V] + \mu k[g(X,hV)\eta(U)\xi - g(X,hU)\eta(V)\xi \\ &-g(X,U)hV + g(X,V)hU] + \mu^2[g(hX,hV)\eta(U)\xi \\ &-g(hX,hU)\eta(V)\xi + g(hX,V)hU \\ &-g(hX,U)hV] - kB(U,V)X - \mu B(U,V)hX = 0. \end{aligned} \tag{3.3}$$

Taking inner product of (3.3) with Z we obtain

$$\begin{aligned} &\frac{4k(k-1)}{n+3}[g(X,V)g(U,Z) - g(X,U)g(V,Z)] \\ &+\frac{4\mu(k-1)}{n+3}[g(hX,V)g(U,Z) - g(hX,U)g(V,Z)] \\ &+\mu k[g(X,hV)\eta(U)\eta(Z) - g(X,hU)\eta(V)\eta(Z) \\ &-g(X,U)g(hV,Z) + g(X,V)g(hU,Z)] + \mu^2[g(hX,hV)\eta(U)\eta(Z) \\ &-g(hX,hU)\eta(V)\eta(Z) + g(hX,V)g(hU,Z) \\ &-g(hX,U)g(hV,Z)] - k\tilde{B}(U,V,X,Z) - \mu\tilde{B}(U,V,hX,Z) = 0. \end{aligned} \tag{3.4}$$

Putting $U=Z=e_i$ in (3.4) and summing up over 1 to n we obtain by using (2.22), (2.22), (2.22) and (2.11),

$$S(X,V) = ag(X,V) + b\eta(X)\eta(V), \tag{3.5}$$

where a and b are given by

$$a = \frac{2n-6-(n-1)\mu}{2} - \frac{[2(k-1)\{2k(n-1)+3\mu(n-3+\mu)\}+(n+3)\mu kt](n-3+\mu)}{2(k-1)\{2\mu(n-1)-3k(n-3+\mu)\}+\mu^2 t(n+3)} \tag{3.6}$$

and

$$b = \frac{1}{2}\{(6-2n)+2(n-1)k+(n-1)\mu\} + \frac{[k^2(n-1)+pk(2-n)+rk+6\mu(n-3+\mu)(k-1)](n-3+\mu)}{2(k-1)\{2\mu(n-1)-3k(n-3+\mu)\}+\mu^2 t(n+3)}, \tag{3.7}$$

$t = \sum_{i=1}^{n} g(he_i, e_i) = \frac{1}{n-3+\mu}[r - \frac{n-1}{2}\{2n-6-2k-(n-1)\mu\}]$.

In view of (3.5) we conclude the following:

Theorem 3.1. *Let M be an n-dimensional ($n \geq 5$) C-Bochner semisymmetric (k,μ)-contact metric manifold. Then the manifold is an η-Einstein manifold.*

Again by virtue of (3.4) we have the following:

Corollary 3.1. *A C-Bochner semisymmetric Sasakian manifold M^n ($n \geq 5$), is C-Bochner flat.*

The above Corollary has already been proved in [12].

In view of the Proposition 2.1 we state the following:

Corollary 3.2. *Let M be an n-dimensional ($n \geq 5$) C-Bochner semisymmetric non-Sasakian (k,μ)-contact metric manifold. Then the Ricci operator Q commutes with ϕ.*

4. (k,μ)-contact metric manifold satisfying $B(\xi,U).R = 0$

This section deals with an n-dimensional (k,μ)-contact metric manifolds satisfying $B(\xi,U).R(X,Y)Z = 0$. The relation $B(\xi,U).R(X,Y)Z = 0$ gives

$$B(\xi,U)R(X,Y)Z - R(B(\xi,U)X,Y)Z - R(X,B(\xi,U)Y)Z - R(X,Y)B(\xi,U)Z = 0. \tag{4.1}$$

Using (2.17), by (4.1) we get

$$\begin{aligned}&\frac{4(k-1)}{n+3}[g(U,R(X,Y)Z)\xi - \eta(R(X,Y)Z)U - g(U,X)R(\xi,Y)Z \\ &+\eta(X)R(U,Y)Z - g(U,Y)R(X,\xi)Z + \eta(Y)R(X,U)Z \\ &-g(U,Z)R(X,Y)\xi + \eta(Z)R(X,Y)U] + \mu[g(hU,R(X,Y)Z)\xi \\ &-\eta(R(X,Y)Z)hU - g(hU,X)R(\xi,Y)Z + \eta(X)R(hU,Y)Z \\ &-g(hU,Y)R(X,\xi)Z + \eta(Y)R(X,hU)Z - g(hU,Z)R(X,Y)\xi \\ &+\eta(Z)R(X,Y)hU] = 0.\end{aligned} \tag{4.2}$$

Taking the inner product with ξ in (4.2) and using $h\xi = 0$, $g(R(X,Y)\xi,\xi) = 0$, we obtain

$$\begin{aligned}(4.3)\qquad &\frac{4(k-1)}{n+3}[g(U,R(X,Y)Z) - \eta(R(X,Y)Z)\eta(U)\\ &-g(U,X)g(R(\xi,Y)Z,\xi) + \eta(X)g(R(U,Y)Z,\xi)\\ &-g(U,Y)g(R(X,\xi)Z,\xi) + \eta(Y)g(R(X,U)Z,\xi)\\ &+\eta(Z)g(R(X,Y)U,\xi)] + \mu[g(hU,R(X,Y)Z)\\ &-g(hU,X)g(R(\xi,Y)Z,\xi) + \eta(X)g(R(hU,Y)Z,\xi)\\ &-g(hU,Y)g(R(X,\xi)Z,\xi) + \eta(Y)g(R(X,hU)Z,\xi)\\ &+\eta(Z)g(R(X,Y)hU,\xi)] = 0.\end{aligned}$$

Let $\{e_i\}, i = 1,2,...,n$ be an orthonormal basis of the tangent space. Putting $Y = Z = e_i$ in (4.3) and summing up over 1 to n we obtain

$$\begin{aligned}(4.4)\qquad &\frac{4(k-1)}{n+3}[S(X,U) - (n-1)kg(X,U)]\\ &+\mu[S(X,hU) - (n-1)kg(X,hU)] = 0.\end{aligned}$$

Replacing U by hU in (4.1) and using (2.7), (2.1), we get

$$\begin{aligned}(4.5)\qquad &(k-1)[\frac{4}{n+3}\{S(X,hU) - (n-1)kg(X,hU)\}\\ &-\mu\{S(X,U) - (n-1)kg(X,U)\}] = 0.\end{aligned}$$

From (4.5) we have either $k = 1$, or

$$(4.6)\quad S(X,hU) - (n-1)kg(X,hU) = \frac{\mu(n+3)}{4}[S(X,U) - (n-1)kg(X,U)].$$

Using (4.6) in (4.4), we obtain

$$(4.7)\qquad [\frac{4(k-1)}{n+3} + \frac{\mu^2(n+3)}{4}][S(X,U) - (n-1)kg(X,U)] = 0.$$

From (4.7) we have

$$(4.8)\qquad S(X,U) = (n-1)kg(X,U),$$

if $[\frac{4(k-1)}{n+3} + \frac{\mu^2(n+3)}{4}] \neq 0$.

In view of the above discussions we state the following:

Theorem 4.1. *An n-dimensional (k,μ)-contact metric manifold satisfying $B(\xi,U).R = 0$ is either a Sasakian manifold or an Einstein manifold, provided $[\frac{4(k-1)}{n+3} + \frac{\mu^2(n+3)}{4}] \neq 0$.*

Remark 4.1. *In a Sasakian manifold it can be easily verified that $B(\xi,X).R = 0$ holds identically.*

5. (k, μ)-contact manifold satisfying $B(\xi, X).S = 0$

Let $M^n (n \geq 5)$ be an Einstein (k, μ)-contact metric manifold. Then we have $S(X, Y) = \lambda g(X, Y)$, where λ is a constant.

Now

$$\begin{aligned}
&B(\xi, X).S(U, V) \\
&= -S(B(\xi, X)U, V) - S(U, B(\xi, X)V) \\
&= -\lambda[g(B(\xi, X)U, V) + g(U, B(\xi, X)V)] \\
&= -\lambda\{g(\frac{4(k-1)}{n+3}[g(X, U)\xi - \eta(U)X] + \mu[g(hX, U)\xi - \eta(U)hX], V) \\
&\quad +g(U, \frac{4(k-1)}{n+3}[g(X, V)\xi - \eta(V)X] + \mu[g(hX, V)\xi - \eta(V)hX])\} \\
&= 0.
\end{aligned}$$

Thus we can state the following:

Theorem 5.1. *Let $M^n (n \geq 5)$ be an $n(= 2m+1)$-dimensional Einstein (k, μ)-contact metric manifold. Then the condition $B(\xi, X).S = 0$ holds on M^n.*

Acknowledgement: The authors are thankful to the referee for his valuable suggestions towards the improvement of this paper.

References

[1] Blair, D. E., Contact manifolds in Riemannian geometry. Lecture notes in math., 509, Springer-Verlag, Berlin, 1976.

[2] Blair, D. E., Two remarks on contact metric structures. Tohoku Math. J., 29 (1977), 319-324.

[3] Blair, D. E., Koufogiorgos, Th. and Papantoniou, B. J., Contact metric manifolds satisfying a nullity condition. Israel J. Math. 91(1995), 189-214.

[4] Blair, D. E., Riemannian geometry of contact and symplectic manifolds. Progress in Mathematics, 203 (Boston, MA: Birkhauser Boston Inc.) 2002.

[5] Blair, D. E., On the geometric meaning of the Bochner tensor. Geom. Dedicata, 4 (1975), 33-38.

[6] Bochner, S., Curvature and Betti numbers. Ann. of Math., 50 (1949), 77-93.

[7] Boeckx, E., A full classification of contact metric (k, μ)-spaces. Illinois J. Math. 44 (2000), 212-219.

[8] Boothby, W. M. and Wang, H. C., On contact manifolds. Annals of Math., 68 (1958), 721-734.

[9] Cho, J. T., A conformally flat contact Riemannian (k, μ)-space. Indian J. of Pure and Appl. Math., 32 (2001), 501-508.

[10] Choi, H. R. and Ki, U. H., On Sasakian manifolds with constant scalar curvature whose C-Bochner curvature tensor vanishes. Nihonkai Math. J., 5 (1994), 101 - 107.

[11] De, A., C-Bochner curvature tensor on $N(k)$-contact metric manifolds. Lobachevskii J. Math., 31(2010), 208-213.

[12] De, U. C. and Shaikh, A. A., Sasakian manifolds with C-Bochner curvature tensor. Indian J. Math., 41 (1999), 131-137.

[13] Hasegawa, K. and Nakane, T., On Sasakian manifolds with vanishing contact Bochner curvature tensor. Hokkaido Math. J., 9 (1980), 184-189.

[14] Jun. J-B. and Kim, U. K., On 3-dimensional almost contact metric manifolds. Kyungpook Math. J., 34 (1994), 293-301.

[15] Jun, J-B., Yildiz, A. and De, U. C., On ϕ-recurrent (k, μ)-contact metric manifolds. Bull. Korean Math. Soc. 45 (2008), 689-700.

[16] Kim. J. S, Tripathi, M. M. and Choi. J. D., On C-Bochner curvature tensor of a contact metric manifold. Bull. Korean Math. Soc. 42 (2005), 713-724.

[17] Matsumoto, M. and Chuman, G., On the C-Bochner curvature tensor. TRU Math. 5,21 (1969), 21-30.

[18] Sasaki, S., Lecture notes on almost contact manifolds. Part-1. Tohoku University. 1965.

[19] Tanno, S., Isometric immersions of Sasakian manifolds in spheres. Kodai Math. Sem. Rep. 21 (1969), 448-458.

Received by the editors December 3, 2012