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A DEFINABLE CONDENSATION OF LINEAR ORDERINGS

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Abstract. We introduce an $\mathcal{L}_{\omega,\omega}$-definable condensation suitable for studying a class of elementary equivalent linear orderings.

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1. Introduction

The main idea in studying isomorphism types of linear orderings is that of condensations; the domain is partitioned in a suitable way into convex pieces, so that the isomorphism type of the original order is determined (in a very weak sense) by the isomorphism types of pieces and the order type of pieces (ordered by the inherited order). If we view a linear ordering as an algebra with a single binary operation $\min$, then condensations correspond to congruence relations, hence the order is studied by studying the quotient algebra and the congruence classes. If we are interested in a class of elementary equivalent linear orderings, for example in the class of all models of a complete theory of linear orderings, then the congruence should be definable; except in Section 2, by definable we mean $\mathcal{L}_{\omega,\omega}$-definable (unless otherwise stated). The reason is that elementary equivalent structures have elementary equivalent definable quotients. The ideal environment for studying a first order structure and its definable quotients is Shelah's multi sorted ^{eq}-universe where we have all the definable quotients as separate sorts.

In this article we first clarify and motivate the use of model-theoretic methods in studying first order properties of linear orderings, especially the elementary equivalence. Then we introduce a definable discrete/dense condensation $\mathbf{c}_\delta$ and, as an application, show that any complete theory of linear orderings with countably many unary predicates added is interpretable in a complete theory of (pure) linear orderings. In the first section preliminary notions and facts are reviewed. The second section contains a brief exposition of composition and ordinal iteration of condensations from a model theorist's point of view. In the third section we prove the main result.

2. Preliminaries

Let $\mathbf{A} = (A, <)$ be a linear ordering. By closed intervals we mean subsets of the form $[a, b] = \{x \mid a \leqslant x \leqslant b\}$, where $a, b \in A$ are its *endpoints* and all other point are

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inner; similarly for open intervals. A subset $C \subset A$ is *convex* if whenever $a, b \in C$ then $[a, b] \subseteq C$. We say that $b \in A$ is an *immediate successor* of $a \in A$ if $a < b$ and the interval $[a, b]$ has no inner points; similarly *immediate predecessor* of a is defined, and they are called *neighbours* of A. $\mathbf{A}$ is *dense* if it has no endpoints and no pair of neighbours; this slightly differs from the standard notation, but the motivation is model-theoretical because the theory of dense orders, as we defined it, is complete. $\mathbf{A}$ is *scattered* if the ordered rationals cannot be embedded into $\mathbf{A}$. Isomorphism classes of linear orderings are called *order types* (or linear order types) and some of them are denoted by characteristic members. For example:

- $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ are the order types of $(\alpha, <)$ and $(\alpha, >)$ respectively, for any ordinal $\alpha > 0$;
- $\boldsymbol{\zeta}$ is the order type of $(\mathbb{Z}, <)$;
- $\boldsymbol{\eta}$ is the order type of $(\mathbb{Q}, <)$.

Basic operations with linear orders and order types are sums and (lexicographical) products. Let $\mathbf{I} = (I, <_I)$ and $\mathbf{A}_i = (A_i, <_i)$ for $i \in I$ be linear orderings. The sum $\oplus_{\mathbf{I}} \mathbf{A}_i$, is the ordering $(A, <)$ defined as follows:
1. $A = \bigcup_{i \in I} \{i\} \times A_i$
2. $(i, a) < (j, b)$ if and only if either $i < j$ or $i = j$ and $a < b$.

In other words, $(A, <)$ is a linear ordering obtained from $\mathbb{I}$ by replacing each $i \in I$ by (a copy of) $\mathbf{A}_i$. Sums of linear order types are defined adequately. The lexicographical product $\mathbf{I} \times \mathbf{A}$ is $\oplus_{\mathbf{I}} \mathbf{A}_i$ where $\mathbf{A}_i = \mathbf{A}$ for all i.

By a linearly ordered structure we mean a first order structure $\mathbf{A} = (A, <, \dots)$ which is linearly ordered by $<$ and may have additional relations and functions defined. Of particular interest for us are relational linearly ordered structures in which all the additional relations are unary.

A linear ordering $(A, <)$ is called *discrete* if every non-minimal element has an immediate predecessor and every non-maximal element has an immediate successor. The motivation for using word 'discrete' here is that an equivalent definition is that the order topology is discrete. Examples of discrete order types are $\boldsymbol{\omega}, \boldsymbol{\omega}^*$ and $\boldsymbol{\zeta}$, while $\boldsymbol{\omega} + \mathbf{1}$ is not discrete because the maximum does not have an immediate predecessor; similarly $1 + \boldsymbol{\omega}^*$ is not discrete. The fact that a linear ordering is discrete is easily expressible by first order sentences, hence the class of discrete linear orderings is finitely axiomatizable; we will call that theory dLO. It is not a complete theory but completions are easy to describe. Since any finite linear ordering is discrete, for each n the theory of a linear ordering with n elements is a completion. Among infinite discrete linear orderings there are precisely four classes up to elementary equivalence. The division line is the existence of endpoints and details can be found in [2]. For example, the theory $\mathrm{Th}(\omega, <, s, 0)$ (where s is the successor function) is denoted by dLO^+. It is proved that dLO^+ admits elimination of quantifiers; its models are precisely the unbounded, discrete linear orderings having the minimal element and the successor function named. In particular, the order type of any model of dLO^+ is $\boldsymbol{\omega} + \mathbf{L} \times \boldsymbol{\zeta}$ for a unique order type $\mathbf{L}$ (or $\mathbf{L} = \emptyset$). We have the following classes of order types of discrete orders:

(W)	$\boldsymbol{\omega} + \mathbf{L} \times \boldsymbol{\zeta}$
(W^*)	$\mathbf{L} \times \boldsymbol{\zeta} + \boldsymbol{\omega}^*$
(Z)	$\mathbf{L} \times \boldsymbol{\zeta}$
(C_∞)	$\boldsymbol{\omega} + \mathbf{L} \times \boldsymbol{\zeta} + \boldsymbol{\omega}^*$
(C_n)	$\boldsymbol{n}.$

Let $dLO_{(i,j)}$ for $i, j \in \{0, 1\}$ be the theory dLO plus two axioms: one saying that the minimal element exists iff $i = 1$, and the other saying that the maximum exists iff $j = 1$. In particular, each $dLO_{(i,j)}$ has finite set of axioms.

Remark 1. (1) $dLO_{(1,0)}$ is complete and axiomatizes orders of type (W).

(2) $dLO_{(0,1)}$ is complete and axiomatizes orders of type (W^*).

(3) $dLO_{(0,0)}$ is complete and axiomatizes orders of type (Z).

(4) $dLO_{(1,1)}$ is an incomplete theory and axiomatizes linear orderings whose order type is in some (C_ξ) for $\xi \in \mathbb{N} \cup \{\infty\}$.

Let $\mathbf{A} = (A, <)$ be a linear ordering. We say that an equivalence relation on A is *convex* if its classes are convex subsets of A. For each convex equivalence relation e on A let $e(A) = \{[a]_e \mid a \in A\}$ be the quotient set where we define: $[a]_e \leqslant_e [b]_e$ if and only if $a' \leqslant b'$ for some $a' \in [a]_e$ i $b' \in [b]_e$. Then $e(\mathbf{A}) = (e(A), <_e)$ is a linear ordering called the condensations of $\mathbf{A}$ by e. We will also say that e is a condensing relation on A. The notation $e(A)$ is used intentionally, because we will use e to denote the canonical mapping A onto $e(A)$. Hence by e we will denote both the equivalence relation and the mapping; whenever the meaning of e is not clear from the context, we will explicitly refer to either a relation or a mapping e. The mapping e is an epimorphism (surjective homomorphism) of linear orderings; recall that $f : A \longrightarrow B$ is a homomorphism from $\mathbf{A}$ into a linear ordering $\mathbf{B} = (B, <')$ if $a \leqslant a'$ implies $f(a) \leqslant' f(a')$ for all $a, a' \in A$. On the other hand, any homomorphism $g : A \longrightarrow B$ induces a convex equivalence relation e on A defined by $g(x) = g(y)$.

As we mentioned in the introduction, the ideal environment for any analysis of a first order structure in which definable homomorphisms are involved is Shelah's *eq*-expansion of the original structure, where we have copies of all homomorphic images. For any first order $\mathcal{L}$-structure $\mathbf{M}$, $\mathbf{M}^{eq}$ is a multi sorted structure expanding $\mathbf{M}$.

The language has sort names S_E for every definable equivalence relation E on some M^n (recall that by definable we mean $\mathcal{L}_{\omega,\omega}$-definable). In a multi sorted language any variable has a sort name specified, as well as any place in a relation or function symbol; no variable is interpreted outside the sort whose name it carries, similarly for functions and relations. For each such E there is a function symbol π_E such that the variables $x_1, \ldots, x_n$ in $\pi_E(x_1, \ldots, x_n) = Y$ are of sort $S_=$ and Y is of sort S_E. All other language symbols are those interpreted in $\mathbf{M}$ and refer only to $S_=$.

$\mathbf{M}^{eq}$ is defined as follows. The sort $S_=(\mathbf{M}^{eq})$, called the home sort, is $\mathbf{M}$. The domain of $S_E(\mathbf{M})$ is M^n/E and has no language symbols interpreted (however it has all the structure from $\mathbf{M}$ projected). Functions π_E are canonical projections from M^n onto M^n/E. It turns out that any definable set D of the sort S_E ($D \subseteq (M^n/E)^m$) is the π_E-projection of a definable $\subseteq M^m$; in particular, there are no new definable sets in the $S_=$-sort $\mathbf{M}$. In the case when $\mathbf{M}$ is an algebra and E is a definable congruence on $\mathbf{M}$ one of the sorts represents the domain of the quotient algebra; its quotient structure is definable in the multi-sorted language. Details on $\mathbf{M}^{eq}$ the reader can find in [1] and [3]. We list some basic facts.

Fact 2. *(a) Every definable in $\mathbf{M}^{eq}$ subset of M^n is definable in $\mathbf{M}$.*

(b) Every elementary embedding from $\mathbf{M}$ into $\mathbf{N}$ extends uniquely to an elementary embedding of $\mathbf{M}^{eq}$ into $\mathbf{N}^{eq}$.

(c) $\mathbf{M} \equiv \mathbf{N}$ implies $\mathbf{M}^{eq} \equiv \mathbf{N}^{eq}$.

(d) Any complete first order theory T has a natural multi sorted extension T^{eq} such that $\mathbf{M} \models T$ if and only if $\mathbf{M}^{eq} \models T^{eq}$.

(e) T^{eq} eliminates imaginaries: if $\epsilon(x_E, y_E)$ defines an equivalence relation on S_E then for some E' there is an $\mathbf{M}^{eq}$-definable identification of $S_{E'}$ and S_E/ϵ.

Let $\mathbf{M} = (M, \dots)$ be an $\mathcal{L}$-structure and $\mathbf{M}_1 = (M_1, R, \dots, f \dots)$ an $\mathcal{L}_1$-structure. We say that $\mathbf{M}_1$ is *interpretable* in $\mathbf{M}$ if there is n, a definable equivalence relation E on M, and a bijection $F : M^n/E \longrightarrow M_1$ such that: for any basic relation $R \subseteq M_1^m$ the set $F^{-1}(R) \subset M^{m \cdot n}$ is definable in $\mathbf{M}$, and similar condition for basic functions holds. For example, whenever we name definable subsets of a fixed sort in $\mathbf{M}^{eq}$, then the obtained structure (whose domain is the sort) is interpretable in $\mathbf{M}$.

3. Compositions and iterations of $\mathcal{L}_{\infty,\infty}$-definable condensations

Recall that an $\mathcal{L}_{\kappa,\lambda}$-formula is one whose conjunctions are all of size $< \kappa$ while quantifications apply only to sequences with $< \lambda$ variables; $\mathcal{L}_{\infty,\infty}$ means being $\mathcal{L}_{\kappa,\lambda}$ for some κ, λ. $\mathcal{L}_{\infty,\infty}$-sentences are preserved under isomorphism.

Usually in linear orderings the interesting convex equivalence relations are defined by an $\mathcal{L}_{\infty,\infty}$-formula in the language $\{<\}$ such that the defining formula defines an equivalence relation with convex classes on any linear ordering. In this case the isomorphism type of $e(\mathbf{A})$ depends only on the isomorphism type of $\mathbf{A}$, hence such formulas may be considered as defining condensing relations of linear order types, as well. We will say that an $\mathcal{L}_{\kappa,\lambda}$-formula in the language $\{<\}$ is a *condensing formula* if it defines a condensing relation on any linear ordering. Condensing relations defined by them will be called *uniformly definable*; if the defining formula is $\mathcal{L}_{\kappa,\lambda}$, then we say that the relation is uniformly $\mathcal{L}_{\kappa,\lambda}$-definable. For $\epsilon(x, y)$ a condensing formula by $\mathbf{c}_\epsilon$ we denote the corresponding condensation as a mapping of linear order types; by $\mathbf{c}_\epsilon(\mathbf{A})$ we will denote the condensation of linear ordering $\mathbf{A} = (A, <)$ defined by ϵ, by $\mathbf{c}_\epsilon(A)$ its domain, and by $\mathbf{c}_\epsilon$ the condensing relation on $\mathbf{A}$ (there is no danger of confusion for using simply $\mathbf{c}_\epsilon$ instead of $\mathbf{c}_\epsilon^{\mathbf{A}}$). The most common examples of uniformly definable condensing relations are:

- $\mathbf{c_F}$ where $F(x, y)$ is '$[x, y]$ is finite' ;
- $\mathbf{c_W}$ where $W(x, y)$ is '$[x, y]$ is well-ordered';
- $\mathbf{c}_S$ where $S(x, y)$ is '$[x, y]$ is scattered'.

Here W and S are $\mathcal{L}_{\omega_1,\omega_1}$-formulas, while F is an $\mathcal{L}_{\omega_1,\omega}$-formula. For the application of these condensations in analysing linear orderings the reader should consult Rosenstein's book ([4]). Here we only give an interesting example of iterated condensations.

Example 3. Consider $(\omega^\omega, <)$ and the condensation $\mathbf{c_F}$. Each equivalence class contains a single limit ordinal, so $\mathbf{c}_F(\omega^\omega)$ may be identified with the set of all limit ordinals below ω^ω. In the second iteration we get limit-limit ordinals ... At level ω we have a single class:

$$\boldsymbol{\omega^\omega} = \mathbf{c_F}(\boldsymbol{\omega^\omega}) = \mathbf{c}^2_\mathbf{F}(\boldsymbol{\omega^\omega}) = \ldots \text{ and } \mathbf{c}^\omega_\mathbf{F}(\boldsymbol{\omega^\omega}) = \mathbf{1}.$$

Suppose that $\mathbf{c}$ is a uniformly definable condensing relation of order types. As a mapping it is naturally iterated n times, i.e $\mathbf{c}^n$ is defined as the adequate composition. The idea of defining $\mathbf{c}^\omega$ is that it should be the inverse limit of the ω-sequence. Instead of using category theory language, in our setting the adequate definition is in terms of equivalence relations. For $\mathcal{L}_{\omega,\omega}$-definable relations, any finite iteration may be identified with a sort in the ^{eq}-universe.

Let $\epsilon(x, y)$ be a condensing $\mathcal{L}_{\kappa,\lambda}$-formula and let ϕ be *any* $\mathcal{L}_{\kappa,\lambda}$-formula (possibly having many free variables). In Lemma 4 we will prove that '$\phi([x]_\epsilon, [y]_\epsilon)$ holds in the quotient structure' is expressed by a formula $\phi \circ \epsilon(x, y)$ defined in the following way: $\phi \circ \epsilon(x, y)$ is the formula obtained from $\phi(x, y)$ by:

- replacing each occurrence of $u < v$ by $\forall u'\forall v'(\epsilon(u, u') \wedge \epsilon(v, v') \Rightarrow u' < v')$ (where u', v' do not occur in ϵ, ϕ); in this way we turn $x < y$ into $[x]_\epsilon < [y]_\epsilon$.
- replacing each occurrence of $u = v$ by $\epsilon(u, v)$.

Notice that $\phi \circ \epsilon(x, y)$ is an $\mathcal{L}_{\kappa,\lambda}$-formula.

Lemma 4. *If* $\mathbf{A} = (A, <)$ *is a linear ordering and* $\epsilon(x, y)$ *a condensing* $\mathcal{L}_{\infty,\infty}$*-formula, then for any* $\mathcal{L}_{\infty,\infty}$*-formula* $\phi(\overrightarrow{x})$ *and any sequence* $\overrightarrow{b}$ *of elements of* A *we have:*

$$\mathbf{A} \models \phi \circ \epsilon(\overrightarrow{b}) \text{ if and only if } \mathbf{c}_\epsilon(\mathbf{A}) \models \phi(\overrightarrow{[b]}_\epsilon).$$

Proof. By induction on the ordinal complexity of ϕ. If ϕ is atomic the conclusion follows from the definition; it is crucial there that $\mathbf{c}_\epsilon$ is convex. Induction steps are straightforward, we prove only the one for existential quantification. Suppose first that $\mathbf{A} \models \exists\overrightarrow{y}\phi\circ\epsilon(\overrightarrow{y}, \overrightarrow{b})$, and choose $\overrightarrow{c}$ in A such that $\mathbf{A} \models \phi\circ\epsilon(\overrightarrow{c}, \overrightarrow{b})$. The induction hypothesis implies $\mathbf{c}_\epsilon(\mathbf{A}) \models \phi(\overrightarrow{[c]}_\epsilon, \overrightarrow{[b]}_\epsilon)$ and hence $\mathbf{c}_\epsilon(\mathbf{A}) \models \exists\overrightarrow{x}\phi(\overrightarrow{x}, \overrightarrow{[b]}_\epsilon)$. For the other direction assume $\mathbf{c}_\epsilon(\mathbf{A}) \models \exists\overrightarrow{x}\phi(\overrightarrow{x}, \overrightarrow{[b]}_\epsilon)$ and choose $\overrightarrow{c'}$ such that $\mathbf{c}_\epsilon(\mathbf{A}) \models \phi(\overrightarrow{[c']}_\epsilon, \overrightarrow{[b]}_\epsilon)$. The induction hypothesis implies $\mathbf{A} \models \phi \circ \epsilon(\overrightarrow{c}', \overrightarrow{b})$ and $\mathbf{A} \models \exists\overrightarrow{x}\phi \circ \epsilon(\overrightarrow{x}, \overrightarrow{b})$ follows. □

Fact 5. *If* $\epsilon(x, y)$ *and* $\delta(x, y)$ *are condensing* $\mathcal{L}_{\kappa,\lambda}$*-formulas, then* $\delta \circ \epsilon(x, y)$ *is a condensing* $\mathcal{L}_{\kappa,\lambda}$*-formula, too. Moreover, if* $\mathbf{L}$ *is an order type, then* $\mathbf{c}_{\delta\circ\epsilon}(\mathbf{L}) = \mathbf{c}_\delta(\mathbf{c}_\epsilon(\mathbf{L}))$. □

Let $\epsilon(x, y)$ be any $\mathcal{L}_{\kappa,\lambda}$-formula. The formula $\epsilon^\alpha(x, y)$ is defined recursively for any ordinal α in the following way.

(1) $\epsilon^0(x, y)$ is $x = y$, $\epsilon^1(x, y)$ is $\epsilon(x, y)$.
(2) $\epsilon^{\beta+1}(x, y) = \epsilon \circ \epsilon^\beta$
(3) For limit α define: $\epsilon^\alpha(x, y) = \bigvee_{\xi<\alpha} \epsilon^\xi(x, y)$.

Clearly, each $\epsilon^\alpha(x, y)$ is an $\mathcal{L}_{\kappa+|\alpha|,\lambda+|\alpha|}$-formula. The following fact collects basic facts about iterations of a condensing formula.

Fact 6. *Suppose that $\epsilon(x, y)$ is a condensing $\mathcal{L}_{\kappa,\lambda}$-formula and that $\mathbf{A} = (A, <)$ is a linear ordering.*

(a) Each $\mathbf{c}_\epsilon^\alpha$ is a convex equivalence relation on A.

(b) $\mathbf{c}_\epsilon^\alpha \subseteq \mathbf{c}_\epsilon^{\alpha+1}$ and $\mathbf{c}_\epsilon(\mathbf{c}_\epsilon^\alpha(\mathbf{A})) \cong \mathbf{c}_\epsilon^{\alpha+1}(\mathbf{A})$.

(c) If α is a limit ordinal, then $\mathbf{c}_\epsilon^\alpha = \bigcup_{\xi<\alpha} \mathbf{c}_\epsilon^\xi$.

(d) $\{\mathbf{c}_\epsilon^\xi \mid \xi\}$ is a continuous sequence of non-decreasing subsets of A^2. If $\mathbf{c}_\epsilon^\alpha = \mathbf{c}_\epsilon^{\alpha+1}$ holds for some ordinal α, then $\mathbf{c}_\epsilon^\alpha = \mathbf{c}_\epsilon^\beta$ holds for all $\beta > \alpha$. □

By part (d) of the previous fact the sequence $\{\mathbf{c}_\epsilon^\xi(A) \mid 0 \leqslant \xi\}$ (of subsets of A^2) is non-decreasing, hence it is eventually constant. The smallest ordinal α for which $\mathbf{c}_\epsilon^{\alpha+1} = \mathbf{c}_\epsilon^\alpha$ holds is called the $\mathbf{c}_\epsilon$-rank of $\mathbf{A}$. Linear orderings for which $\mathbf{c}_\epsilon(\mathbf{A}) = id_A$ are called $\mathbf{c}_\epsilon$*-inert*. The $\mathcal{L}_{\infty,\infty}$-definability of $\mathbf{c}_\epsilon^\xi$'s guarantees that the $\mathbf{c}_\epsilon$-rank depends only on the isomorphism type of a linear ordering, hence we have a well-defined $\mathbf{c}_\epsilon$-rank of linear order types. The iteration process is used to analyse the structure of $\mathbf{A}$ and is represented by a sequence of condensing relations, which increase until some point, at which the homomorphic image is $\mathbf{c}_\epsilon$-inert.

Of special interest are uniformly $\mathcal{L}_{\omega,\omega}$-definable condensations. For a given liner order $\mathbf{A}$ we can consider the adequate part of $\mathbf{A}^{eq}$, where we have all the definable homomorphisms named by projection maps. In particular, $\mathbf{c}(\mathbf{A}), \mathbf{c}^2(\mathbf{A}), \ldots$ may be identified with $\mathbf{A}^{eq}$-sorts.

Remark 7. The general context in which the previous discussion takes place has universal algebra flavour. The above construction works for any class of algebras which is closed under isomorphism and homomorphism, and a family of 'congruence formulas' (i.e. uniformly definable congruences on the class). If $\epsilon(x, y)$ is a congruence formula, then we can define ϕ_ϵ by replacing atomic formulas in ϕ adequately, so that conclusions of Facts 4 and 5 hold for congruences in place of condensations.

4. Discrete/dense condensation $\mathbf{c}_\delta$

We say that a convex subset of a linear ordering $(A, <)$ is discretely ordered if the restriction of $<$ makes it into a discrete order; it is densely ordered if it has neither minimal nor maximal element and the restriction makes it into a dense order.

Definition 8. A closed interval in a linear ordering is of *dense type* if it is properly contained in a densely ordered convex subset (recall that a dense order has no endpoints); it is of *discrete type* if it is not of dense type and is discretely ordered by $<$.

Interval $[a, a]$ has a type specified: if it is not of dense type, because $[a, a]$ is discretely ordered, it is of discrete type. We define a to be of *dense (discrete)* type if $[a, a]$ is of dense (discrete) type. Not all intervals are of one of the types. Consider the unit interval $([0, 1], <)$ and notice that $[0, 1]$ is neither of dense nor of discrete type.

Lemma 9. *Let $(A, <)$ be a linear ordering.*

(a) A closed interval is of dense type if and only if each of its points is of dense type.

(b) A closed interval of discrete type has all points of discrete type.

(c) If an interval is of dense (discrete) type, then each of its closed subintervals is of dense (discrete) type.

(d) Intervals of distinct types are disjoint.

(e) If two non-disjoint closed intervals have the same discrete/dense type, then their union is an interval of that type.

Proof. (a) We prove only non-trivial direction. Assume that every point of $[a, b]$ is of dense type. For each $c \in [a, b]$ pick a densely ordered convex I_c containing c, and let I be the union of all I_c's. No element of I is an endpoint, because no I_c has an endpoint. It is straightforward to verify that I is convex and densely ordered. Since I_a has points outside $[a, b]$, we conclude that I witnesses that $[a, b]$ is of dense type.

(b) Suppose that $[a, b]$ is of discrete type and $c \in [a, b]$. If $a = b$ then, by definition of types of points, c is of discrete type. Assume $a \neq b$. Then c has an immediate neighbour, hence it cannot be contained in a densely ordered convex set. Therefore, c is of discrete type.

(c) Suppose that $[c, d] \subseteq [a, b]$. In the first case assume that $[a, b]$ is of dense type. By part (a) its elements are all of dense type, hence all elements of $[c, d]$ are of dense type, too. By part (a) again, $[c, d]$ is of dense type. In case when $[a, b]$ is of discrete type it suffices to note that any convex subset of a discrete order is discretely ordered.

(d) Any point of an interval of a dense type is, by part (a), of dense type. By part (b) that point cannot be contained in an interval of discrete type. The conclusion follows.

(e) Assume that $[a, b]$ and $[c, d]$ have the same type. If one of them is contained in the other, the conclusion trivially follows, hence we may assume that it is not the case. Then, without loss of generality, we may assume that $a < c \leqslant b < d$ holds. We will show that $[a, d]$ has the same type as $[a, b]$. In the first case suppose that each of $[a, b]$ and $[c, d]$ is of dense type. Then, by part (a), all points of $[a, d]$ are of dense type. By part (a) again, $[a, d]$ is of dense type. In the second case assume that $[a, b]$ and $[c, d]$ are of discrete type. Clearly, every inner point c' of $[a, d]$ has an immediate predecessor and an immediate successor, and each endpoint of $[a, d]$ has a neighbour. Hence $[a, d]$ is of discrete type. □

The relation '$[x, y]$ is of dense type' is definable: there is a first-order formula $\phi(x, y)$ such that in any linear ordering $(A, <) \models \phi(a, b)$ if and only if $[a, b]$ is of dense type. Similarly, there is a formula saying that '$[x, y]$ is not of dense type' and '$[x, y]$ is discretely ordered' (recall that the theory of discrete linear orderings is finitely axiomatizable). Therefore there is a formula $\delta(x, y)$ in the language $\{<\}$ expressing that:

'the interval $[\min(x, y), \max(x, y)]$ is either of discrete or of dense type'.

In particular 'x is of dense type' is a definable relation on any linear ordering.

Proposition 10. *$\delta(x, y)$ is a condensing formula.*

Proof. Let $(A, <)$ be a linear ordering and let $E \subseteq A^2$ be defined by $\delta(x, y)$. Reflexivity and symmetry are immediate. To prove transitivity, assume that $(a, b), (b, c) \in E$, i.e. that each of intervals $[a, b]$ and $[b, c]$ is of dense or discrete type. They have a common point so, by Lemma 9, they have the same type and their union is of that type, too. Since $[a, c]$ is a subinterval of the union, by Lemma 9 again, it has the same type as $[a, b]$ and $[b, c]$ do. This proves the transitivity, so E is an equivalence relation. It remains to note that its classes are convex. Indeed, $[a]_E$ is the union of all intervals having a as one endpoint and the other endpoint in $[a]_E$. Since a union of convex pairwise intersecting subsets is convex, $[a]_E$ is convex. Thus E is a convex equivalence relation. □

Proposition 11. *Let $\mathbf{A} = (A, <)$ be a linear ordering.*

(a) Any pair of non-trivial, closed subintervals of a fixed $\mathbf{c}_\delta$-class have the same type.

(b) Each class is either a discrete or dense order.

(c) If $a \in A$ is of discrete (dense) type, then $[a]_{\mathbf{c}_\delta}$ is the maximum under inclusion in the set of all convex, discretely (densely) ordered subsets of A that contain a.

Proof. (a) Suppose that two non-trivial closed intervals are contained in the same $\mathbf{c}_\delta$-class. The convex closure of their union is a closed interval which is contained in the class; in particular, it is of either dense or discrete type. Since our intervals are contained in the closure, by Lemma 9, each of them has the same type as the closure. In particular they have the same type.

(b) Consider the class $[a]_{\mathbf{c}_\delta}$. The conclusion is obvious if it has a single element, so assume that $b \in [a]_{\mathbf{c}_\delta}$ and $a < b$. We have two cases. In the first case assume that $[a, b]$ is of discrete type. Suppose that $c \in [a]_{\mathbf{c}_\delta}$ is not maximal in $[a]_{\mathbf{c}_\delta}$, and pick $d \in [a]_{\mathbf{c}_\delta}$ such that $c < d$. Since intervals $[a, b]$ and $[c, d]$ of $[a]_{\mathbf{c}_\delta}$, by part (a), they have the same type so $[c, d]$ is of discrete type. In particular, c has an immediate successor. Similarly, every non-minimal element of $[a]_{\mathbf{c}_\delta}$ has an immediate predecessor, so $[a]_{\mathbf{c}_\delta}$ is discretely ordered by $<$.

In the second case assume that $[a, b]$ is of dense type. It suffices to show that all the points of $[a]_{\mathbf{c}_\delta}$ are of dense type. Indeed, if $c \in [a]_{\mathbf{c}_\delta}$ then $c < b$ or $a < c$. If $c < b$ then the interval $[c, b] \subseteq [a]_{\mathbf{c}_\delta}$ is, by part (a), of dense type because it meets $[a, b]$. Hence c is of dense type. Similarly for $a < c$.

(c) It remains to prove that $[a]_{\mathbf{c}_\delta}$ cannot be a proper subset of a dense (discrete) order; Otherwise, there would be $c \notin [a]_{\mathbf{c}_\delta}$ (say $c < a$) such that $[c, a]$ is of dense (discrete) type because it is a convex subset of a dense (discrete) order. Then, by definition, we would have $\models \delta(c, a)$ and hence $c \in [a]_{\mathbf{c}_\delta}$. A contradiction. □

Example 12. (1) The $\mathbf{c}_\delta$-classes decompose $([0, 1]_\mathbb{Q}, <)$ in the form $\mathbf{1} + \boldsymbol{\eta} + \mathbf{1}$.

(2) Consider $\boldsymbol{\eta} \times \mathbf{2}$. This order type is obtained from a countable dense linear ordering by replacing each of its elements by a two-element order. There are no points of dense type, because every point has a neighbour. Every $\mathbf{c}_\delta$-class consists of two elements, after factoring we deduce $\mathbf{c}_\delta(\boldsymbol{\eta} \times \mathbf{2}) = \boldsymbol{\eta}$.

Let $\mathbf{A} = (A, <)$ be a linear ordering. By Proposition 11, any $\mathbf{c}_\delta$-equivalence class is either a dense or a discrete linear ordering. That produces a simpler description of formula defining types of points:

x is of dense (discrete) type if and only if 'the $\mathbf{c}_\delta$-class of x is a dense (discrete) order'

We may distinguish discretely ordered classes by their complete first-order theories as Remark 1 suggests. That we do by expanding the language by adding adequate unary predicates: let $\mathcal{L}^\# = \{<, W,^* W, Z\} \cup \{C_n \,|\, n \in \mathbb{N}\}$. We intentionally omit predicate for C_∞-types because, unlike the other ones, in a general linear ordering the classes of that type are not definable; in other words, the class of linear orderings whose order type is C_∞ is not finitely axiomatizable (easy application of compactness). However, points of type C_∞ are type-definable (by a countable conjunction).

Definition 13. For any linear ordering $\mathbf{A} = (A, <)$ the $\mathcal{L}^\#$-structure $\mathbf{c}_\delta^\#(\mathbf{A}) = (\mathbf{c}_\delta(A), <_{\mathbf{c}_\delta}, W^{\mathbf{A}},^* W^{\mathbf{A}}, C_n^{\mathbf{A}})_{n\in\mathbb{N}}$ where each unary relation $X^{\mathbf{A}}$ is defined by: $a \in X^{\mathbf{A}}$ if and only if the class $[a]_{\mathbf{c}_\delta}$ is of order type (X) in $\mathbf{A}$.

Proposition 14. *The $\mathcal{L}^\#$-structure $\mathbf{c}_\delta^\#(\mathbf{A})$ is interpretable in any pure linear ordering $\mathbf{A}$.*

Proof. The interpreting equivalence relation is $\mathbf{c}_\delta$. The rest follows from the definition of $\mathbf{c}_\delta^\#(\mathbf{A})$, having on mind that all unary predicates are definable in the the language $\{<\}$. □

Theorem 15. *Any linear ordering with countably many unary predicates whose interpretations are pairwise disjoint is interpretable in a pure linear ordering. Moreover, if the original order is scattered then the pure order may be chosen scattered, too.*

Proof. Let $\mathcal{L} = \{<\} \cup \{P_n \,|\, n \in \mathbb{N}\}$ where each P_n is unary. Suppose that $\mathbf{A} = (A, <, P_n^{\mathbf{A}})_{n\in\mathbb{N}}$ is an $\mathcal{L}$-structure and that $P_n^{\mathbf{A}}$'s are pairwise disjoint subsets of A. By adding an additional predicate if necessary, we may assume that the union of all $P_n^{\mathbf{A}}$'s is A.

Let $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{F}_n$ be linear orderings of types $\boldsymbol{\zeta}, \boldsymbol{\omega}^*$ and $\mathbf{n}$ respectively (for all $n \in \mathbb{N}$). For each $a \in A$ let $\mathbf{B}_a = \mathbf{L} + \mathbf{F}_{n_a} + \mathbf{R}$ where n_a is uniquely determined by $a \in P_{n_a}^{\mathbf{A}}$. Hence each $\mathbf{B}_a$ is of type $\boldsymbol{\zeta} + \mathbf{n}_a + \boldsymbol{\omega}^*$. Let $\mathbf{B} = \sum_{a\in A} \mathbf{B}_a$. We will prove that $\mathbf{A}$ is interpretable in $\mathbf{B} = (B, <')$.

First, we show that $\{a\} \times \mathbf{L}, \{a\} \times \mathbf{F}_{n_a}$ and $\{a\} \times \mathbf{R}$ are $\mathbf{c}_\delta$-classes. Consider $\{a\} \times \mathbf{L}$ which is of dense type. By Proposition 11(c), it suffices to prove that it is not properly contained in a dense, convex subset of B.Let C be a dense convex set containing it. Then C has no points to the right of $\{a\} \times \mathbf{L}$, because the convexity implies that some point from $\{a\} \times \mathbf{F}_{n_a}$ (which is of discrete type) would be in C. Similarly, C has no points to the left: if $a' < a$ and $(a', d) \in C$ then, by convexity, there would be $d' \in R$ such that $(a', d') \in C$, which is not possible because (a', d') is of discrete type. Therefore $\{a\} \times \mathbf{L}$ is a maximal, convex, dense subset of B; by Proposition 11 it is a $\mathbf{c}_\delta$-class. Similar arguments prove that $\{a\} \times \mathbf{F}_{n_a}$ and $\{a\} \times \mathbf{R}$ are $\mathbf{c}_\delta$-classes.

Let $\phi(x)$ be a formula saying that '$[x]_{\mathbf{c}_\delta}$ has both endpoints' (i.e. satisfies $dLO_{(1,1)}$); clearly, $\phi(x)$ defines $\bigcup_{a\in A}\{a\} \times F_{n_a}$ in $\mathbf{B}$. Let $E(x, y)$ be a formula saying that:

the classes $[x]_{\mathbf{c}_\delta}$ and $[y]_{\mathbf{c}_\delta}$ have a common neighbour whose elements satisfy $\phi(x)$.

Then E is a convex equivalence relation on B whose classes are of the form $\{a\} \times B_a$. Consider the function $f : B/E \longrightarrow A$, defined by $f(\{a\} \times B_a) = a$. We show that it is an interpretation of $\mathbf{A}$ in $\mathbf{B}$. For, it suffices to check that $<$ and $\mathbf{P}_n^{\mathbf{A}}$'s are definable. Clearly, $<$ is definable because E is convex. Let $\psi_n(x)$ be the formula saying that 'either x or one of its neighbouring E-classes (in $\mathbf{B}$) has exactly n elements'. Then $P_n^{\mathbf{A}}$ is the projection of the set of all solutions of $\psi_n(x)$ in $\mathbf{B}$. This shows that $\mathbf{A}$ is interpretable in $\mathbf{B}$.

The 'moreover' part follows from the definition of $\mathbf{B}$. □

References

[1] Makkai, M., A survey of basic stability theory with particular emphasis on orthogonality and regular types. Israel J. Math. 49 (1984), 181-238.

[2] Marcja, A., Toffalori, A., A Guide to Classical and Modern Model Theory. Kluwer Academic Publishers 2003.

[3] Marker, D., Model Theory. New York: Springer Verlag 2002.

[4] Rosenstein, J.G., Linear Orderings. New York: Academic Press 1982.

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