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APPROXIMATION BY LINEAR SUMMABILITY MEANS IN ORLİCZ SPACES

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Abstract. In the present work we estimate of deviations of periodic functions from linear operators constructed on basis of its Fourier series in terms of the best approximation of these functions in Orlicz space. Specifically, we study the problem of the effect of metric of space on order of change of deviations.

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1. Introduction and the main results

We suppose that [1] Φ is the class of strictly increasing functions $\Phi : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\phi(\infty) := \lim_{x\to\infty} \phi(x) = \infty$. Let $Y[p,q]$, $-\infty < p \leq q < \infty$ be the class of even functions $\phi \in \Phi$ satisfying the following conditions

1. $\phi(t)/t^p$ is non- decreasing as $|t|$ increases;
2. $\phi(t)/t^q$ is non- increasing as $|t|$ increases.

Let $p < q$. The class of functions ϕ belonging to $Y[p+\epsilon,\, q-\delta]$ for some small numbers ϵ, $\delta > 0$ we will denote by $Y\langle p,q\rangle$. If $1 < p \leq q$, the class of functions M belonging to the class $Y\langle p,q\rangle$ will be denoted by Φ_p^*.

We use c, c_1, $c_2,...$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the question of our interest.

Let $\mathbb{T}$ denote the interval $[0,2\pi]$. We suppose that $M \in \Phi_p^*$, $p > 1$ and we put $\phi_M(u) = M(u)/u$. Note that $1 < p < q < \infty$, then $\phi_M(u) \to \infty$ as $u \to \infty$. Let

$$\Phi_M(x) = \int_0^x \phi_M(u)du.$$

For some positive real constant c let $L_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{R}$ for which

$$\int_{\mathbb{T}} \Phi_M(c\,|f(x)|)dx < \infty.$$

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$L_M(\mathbb{T})$ is called an *Orlicz space* and is a Banach function space with the norm

$$\|f\|_{L_M(\mathbb{T})} := \inf\left\{\lambda > 0 : \int\limits_{\mathbb{T}} \Phi_M\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

Every function in $L_M(\mathbb{T})$ is integrable on $\mathbb{T}$ [22, p. 50], i.e. $L_M(\mathbb{T}) \subset L^1(\mathbb{T})$. Detailed information on properties Orlicz spaces can be found in [5, 16, 22]. Generally, the approximation problems in Orlicz spaces have been investigated, when M is a convex and quasiconvex Young function. According to [6] the condition $M \in \Phi_p^*$, $p > 1$, need not imply M to be convex. Therefore, when $M \in \Phi_p^*$, $p > 1$ it is important to study the approximation of the functions in Orlicz spaces.

Definition 1.1. Let X be a normed space. X is said to be $q-$concave if for an arbitrary system of functions $\{\phi_i(x)\}_{i=1}^n$, $0 \le \phi_i \in X$, the following inequality holds:

$$\left\{\sum_{i=1}^n \|\phi_i\|_X^q\right\}^{\frac{1}{q}} \le c_1 \left\|\left(\sum_{i=1}^n \phi_i^q\right)^{\frac{1}{q}}\right\|_X,$$

X is said to be $p-$convex if for an arbitrary system of functions $\{\phi_i(x)\}_{i=1}^n$, $0 \le \phi_i \in X$, the following inequality holds:

$$\left\{\sum_{i=1}^n \|\phi_i\|_X^p\right\}^{\frac{1}{p}} \ge c_2 \left\|\left(\sum_{i=1}^n \phi_i^p\right)^{\frac{1}{p}}\right\|_X.$$

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x;f),\ A_k(x;f) := a_k(f)\cos kx + b_k(f)\sin kx \tag{1.1}$$

be the Fourier series of the function $f \in L_1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f. The *nth partial sum* of the series (1.1) is defined as:

$$S_n(x;\ f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x;f).$$

We consider the sequence of the functions $\{\lambda_k(r)\}$ defined in the set E of the number line, satisfying the conditions that

$$\lambda_0(r) = 1,\ \lim_{r \longrightarrow r_0} \lambda_\nu(r) = 1$$

for an arbitrary fixed $\nu = 0, 1, 2, \ldots$

For an arbitrary $r \in E$ and for every function $f \in L_M(\mathbb{T})$ the series

$$U(\ f;\ x\,;\lambda) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \lambda_k(r)\ A_k(x;\ f) \tag{1.2}$$

converges in the space $L_M(\mathbb{T})$.

For each linear operator $U_r(f; x; \lambda)$ we set

$$R_r(f;\ \lambda)_M := \|f - U_r(f;\ x;\ \lambda)\|_{L_M(\mathbb{T})}.$$

If we substitute the following

$$\lambda_\nu(r) = \begin{cases} 1 - \frac{\nu}{r+1}, & 0 \le \nu \le r, \\ 0, & \nu > r. \end{cases}, \tag{1.3}$$

$$\lambda_\nu(r) = \begin{cases} 1 - \frac{\nu^k}{(r+1)^k}, & 0 \le \nu \le r, \\ 0, & \nu > r. \end{cases}, \tag{1.4}$$

where $k \ge 1$,

$$\lambda_\nu(r) = r^\nu,\ (\nu = 0, 1, 2, ...)\ (0 \le r \le 1) \tag{1.5}$$

into (1.2) we obtain *Fejér means, Zygmund means of order k and Abel-Poisson means* of the series (1.1) respectively.

We denote by $E_n(f)_M$ the best approximation of $f \in L_M(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_M = \inf\{\| f - T_n \|_{L_M(\mathbb{T})} : T_n \in \Pi_n\}$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

The approximation problems by trigonometric polynomials in Orlicz spaces were investigated by several authors (see, for example, [1, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 21, 23, 29]). In the present paper we investigate the problems of estimating the deviation of the functions from the linear operators constructed on the basis of its Fourier series in terms of the best approximation of these functions in Orlicz spaces. Obtained results show that the estimates of $R_r(f;\lambda)_M$ depends on both the rate of decrease of the sequence $\{E_n(f)_M\}$ and in some cases the metric of the considered space. This is valid for the upper and lower estimates of the quantity $R_r(f;\lambda)_M$. The similar problems of the approximation theory in the different spaces were investigated in [2, 3, 18, 19, 20, 24, 25, 26, 27, 28].

Our main results are the following.

Theorem 1.2. *Let $\{\lambda_\nu(r)\}$ be an arbitrary triangular matrix ($r = 0, 1, 2, 3, ...$; $\lambda_0(r) = 1$; $\lambda_\nu(r) = 0,\ \nu > r$). Let $M \in \Phi_p^*$, $p > 1$ and $f \in L_M(\mathbb{T})$, then the following inequality holds:*

$$\begin{aligned} R_r(f;\ \lambda)_M \quad &\le \quad c_3\{(1 + K_r)E_r(f)_M + \sum_{\nu=0}^{m-1} \delta(2^{\nu+1}; r)\ E_{2^\nu - 1}(f)_M \\ &+\delta(r;\ r)\ E_{2^m}(f)_M\}, \end{aligned} \tag{1.6}$$

where $2^m \le r < 2^{m+1}$, c_3 *is a constant not depending on* r,

$$K_r = \frac{2}{\pi} \int_0^{\pi} \left| \frac{1}{2} + \sum_{\nu=1}^{r} \lambda_\nu(r) \cos \nu\theta \right| d\theta,$$

$$\delta(\mu; r) = \int_0^{\pi} \left| \frac{1-\lambda_\mu(r)}{2} + \sum_{\nu=1}^{\mu-1} \{1-\lambda_{\mu-\nu}(r)\} \cos \nu\theta \right| d\theta, \ \mu \le r. \tag{1.7}$$

Corollary 1.3. *Suppose that the conditions of Theorem 1.2 are satisfied.*

1. Let $\lambda_\nu(r)$, $\nu = 0, 1, 2, ...$ *be a system of numbers defined by relations* (1.3)*. Then the following inequality holds:*

$$R_r(f;\ \lambda)_M \le \frac{c_4}{r+1} \sum_{\nu=0}^{r} E_\nu(f)_M. \tag{1.8}$$

2. Let $\lambda_\nu(r)$, $\nu = 0, 1, 2, ..$ *be a system of numbers defined by relations* (1.4)*. Then the following inequality holds:*

$$R_r(f;\ \lambda)_M \le \frac{c_5}{(r+1)^k} \sum_{\nu=0}^{r} (\nu+1)^{k-1} E_\nu(f)_M, \tag{1.9}$$

where c_5 *is a positive constant depending on* k.

Theorem 1.4. *Let* $M \in \Phi_p^*$, $1 < p \le q$, $\gamma = \max\{2,\ q-\delta\}$ *and* $f \in L_M(\mathbb{T})$, *then for the system of numbers defined by* (1.4) *the following inequality holds:*

$$R_r(f;\ \lambda)_M \ge \frac{c_6}{(r+1)^k} \left\{ \sum_{\nu=1}^{r} \nu^{k\gamma-1} E_\nu^\gamma(f)_M \right\}^{\frac{1}{\gamma}},$$

where δ *is some small positive number and* c_6 *is a constant depending on* p *and* k.

Theorem 1.5. *Let* $M \in \Phi_p^*$, $1 < p \le q$, $\gamma = \max\{2,\ q-\delta\}$*and* $f \in L_M(\mathbb{T})$, *then for the system of numbers defined by* (1.5) *the following inequality holds:*

$$R_r(f;\ \lambda)_M \ge c_7 (1-r) \left\{ \sum_{\nu=0}^{\infty} r^\nu (\nu+1)^{\gamma-1} E_\nu^\gamma(f)_M \right\}^{\frac{1}{\gamma}},$$

where δ *is some small positive number and* c_7 *is a constant depending on* p.

2. Proofs of theorems

We need the following [1] theorems:

Theorem 2.1. *Let a sequence* λ_k *satisfy the conditions*

$$|\lambda_k| \le A, \quad \sum_{k=2^{j-1}}^{2^j-1} |\lambda_k - \lambda_{k+1}| \le A \tag{2.1}$$

where $A > 0$ *does not depend on* k *and* j. *Suppose that satisfied the conditions of Theorem 1.2 For given* $f \in L_M(\mathbb{T})$ *there exists a function* $F \in L_M(\mathbb{T})$ *such that the series*

$$\frac{\lambda_0 a_0}{2} + \sum_{k=0}^{\infty} \lambda_k (a_k \cos kx + b_k \sin kx)$$

is Fourier series for F *and*

$$\| F \|_{L_M(\mathbb{T})} \le c_8 A \| f \|_{L_M(\mathbb{T})}, \tag{2.2}$$

where $c_8 > 0$ *does not depend on* $f \in L_M(\mathbb{T})$.

Theorem 2.2. *Under the conditions of Theorem 1.2 there exist constants* $c_9 > 0$ *and* $c_{10} > 0$ *such that*

$$c_{10} \| f \|_{L_M(\mathbb{T})} \le \left\| \left(\sum_{j=0}^{\infty} \left| \sum_{k=2^{j-1}}^{2^j-1} A_k(x, f) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_M(\mathbb{T})} \le c_9 \| f \|_{L_M(\mathbb{T})}. \tag{2.3}$$

for all $f \in L_M(\mathbb{T})$.

Proof of Theorem 1.2. We consider the trigonometric polynomial

$$T_r(x) = \sum_{\nu=o}^{r} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x).$$

The following inequality holds:

$$\begin{aligned}
&R_r(f;\ \lambda)_M \\
&= \left\| f(x) - \sum_{\nu=0}^{r} \lambda_\nu(r) A_\nu(x;\ f) \right\|_{L_M(\mathbb{T})} \\
&\le \|f(x) - T_r(x)\|_{L_M(\mathbb{T})} + \left\| T_r(x) - \sum_{\nu=0}^{r} \lambda_\nu(r)(\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) \right\|_{L_M(\mathbb{T})} \\
&\quad + \left\| \sum_{\nu=0}^{r} \lambda_\nu(r) A_\nu(x;\ f) - \sum_{\nu=0}^{r} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x) \lambda_\nu(r) \right\|_{L_M(\mathbb{T})} \\
&= \|f(x) - T_r(x)\|_{L_M(\mathbb{T})} + R_r(T_r; \lambda)_M \\
&\quad + \left\| \frac{1}{\pi} \int_0^{2\pi} \{f(x+\theta) - T_r(x+\theta\} \left\{ \frac{1}{2} + \sum_{\nu=1}^{r} \lambda_\nu(r) \cos \nu\theta \right\} d\theta \right\|_{L_M(\mathbb{T})}.
\end{aligned}$$

Therefore, we obtain the following inequality

$$R_r(f,\lambda)_M \le \|f(x)-T_r(x)\|_{L_M(\mathbb{T})}(1+K_r)+R_r(T_r;\lambda)_M, \tag{2.4}$$

where

$$K_r=\frac{2}{\pi}\int_0^\pi\left|\frac{1}{2}+\sum_{\nu=1}^{r}\lambda_\nu(r)\cos\nu\theta\right|d\theta.$$

According to [27] the following identity holds:

$$\sum_{\nu=1}^{n}\{1-\lambda_\nu(r)\}(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)=\frac{2}{\pi}\int T_n(x+\theta)\cos n\theta B_n(r,\theta)d\theta, \tag{2.5}$$

where $\lambda_0(r)=1$ and

$$T_n(x)=\sum_{\nu=o}^{n}(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x).$$

$$B_n(r,\theta)=\frac{1-\lambda_n(r)}{2}+\sum_{\nu=0}^{n-1}(1-\lambda_{n-\nu}(r))\cos\nu\theta.$$

Let $f\in L_M(\mathbb{T})$ and let $T_n\in\Pi$ $(n=0,12,...)$ be the polynomial of best approximation to f i. e.

$$E_n(f)_M=\|f(x)-T_n(x)\|_{L_M(\mathbb{T})}.$$

We set

$$\rho_k(\nu;r;x)=\frac{1}{\pi}\int_0^{2\pi}T_k(x+\theta)\sum_{\mu=1}^{\nu}\{1-\lambda_\mu(r)\}\cos\mu\theta,\ (0\le k\le\nu\le r), \tag{2.6}$$

It is clear that

$$R_r(T_r;\lambda)_{M.}=\|\rho_r(r;r;x)\|_{L_M(\mathbb{T})},$$

$$\rho_0(2;r;x)=0,\rho_k(\nu;r;x)=0,\rho_k(k;r;x)=0,\ (\nu>k).$$

We suppose that the number $m\in N$ satisfies condition $2^m\le r<2^{m+1}$. We have

$$\begin{aligned}R_r(T_r;\lambda)_M \le\ & \|\rho_2(2;r;x)-\rho_0(2;r;x)\|_{L_M(\mathbb{T})}\\ &+\sum_{\mu=1}^{m-1}\left\|\rho_{2^{\mu+1}}(2^{\mu+1};r;x)-\rho_{2^\mu}(2^{\mu+1};r;x)\right\|_{L_M(\mathbb{T})}\\ &+\|\rho_r(r;r;x)-\rho_{2^m}(r;r;x)\|_{L_M(\mathbb{T})}.\end{aligned} \tag{2.7}$$

By (2.5) and (2.6) we get

$$\begin{aligned}
&\left\|\rho_{2^{\mu+1}}(2^{\mu+1};r;x)-\rho_{2^\mu}(2^{\mu+1};r;x)\right\|_{L_M(\mathbb{T})}\\
&= \left\|\frac{1}{\pi}\int_0^{2\pi}\{T_{2^{\mu+1}}(x+\theta)-T_{2^\mu}(x+\theta)\}\sum_{j=1}^{2^{m+1}}\{1-\lambda_j(r)\}\cos j\theta d\theta\right\|_{L_M(\mathbb{T})}\\
(2.8)&= \left\|\frac{2}{\pi}\int_0^{2\pi}\{T_{2^{\mu+1}}(x+\theta)-T_{2^\mu}(x+\theta)\}\cos 2^{\mu+1}\theta B_{2^{\mu+1}}(r;\theta)\right\|_{L_M(\mathbb{T})}\\
&\leq c_{11}\delta(2^{\mu+1};r)E_{2^\mu}(f)_M.
\end{aligned}$$

By (2.7) and (2.8) we find

$$\begin{aligned}
R_r(T_r;\lambda)_M &\leq c_{12}\delta(2;r)E_0(f)_M+\sum_{\mu=1}^{m-1}\delta(2^{\mu+1};r)E_{2^\mu}(f)_M\\
(2.9)\qquad &\quad +\delta(r;r)E_{2m}(f)_M.
\end{aligned}$$

According to [27] $K_r \leq c_{12}$. The inequality (2.4) and (2.9) yield (1.6). □

Proof of Corollary 1.3. If we put

$$\lambda_\nu(r)=1-\frac{\nu^k}{(\nu+1)^k},\ (0\leq\nu\leq r) \text{ and } \lambda_\nu(r)=0,\ \nu>r$$

in the inequality (2.5) we have

$$\begin{aligned}
&\sum_{\nu=1}^{n}\nu^k(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\\
(2.10)\qquad &= \frac{2n^k}{\pi}\int_0^{2\pi}T_n(x+\theta)\cos n\theta\left\{\frac{1}{2}+\sum_{\nu=1}^{n-1}(1-\frac{\nu}{n})^k\cos\nu\theta\right\}d\theta.
\end{aligned}$$

From (2.10) it is follows that

$$\left\|\sum_{\nu=1}^{n}\nu^k(\alpha_\nu\cos\nu x+\beta_\nu\sin\nu x)\right\|_{L_M(\mathbb{T})}\leq c_{13}n^k\left\|T_n(x)\right\|_{L_M(\mathbb{T})}.$$

If we put

$$\lambda_{2^{\mu+1}}(r)=1-\frac{2^{(\mu+1)}}{(r+1)^k}$$

in (1.7) we have

$$\delta(2^{\mu+1};r) = \int_0^\pi \left| \frac{1-\lambda_{2^{\mu+1}}(r)}{2} + \sum_{\nu=1}^{2^{\mu+1}} \{1-\lambda_{2^{\mu+1}-\nu}(r)\}\cos\nu\theta \right| d\theta$$

$$(2.11) \quad = \frac{2^{(\mu+1)k}}{(r+1)^k} \int_0^\pi \left| \frac{1}{2} + \sum_{\nu=1}^{2^{\mu+1}-1} (1-\frac{\nu}{2^{\mu+1}})^k \cos\nu\theta \right| d\theta \le c_{14} \frac{2^{(\mu+1)k}}{(r+1)^k}.$$

Then from (2.11) and (1.6) we obtain the inequalities (1.8) and (1.9) of Corollary 1.3. □

Proof of Theorem 1.4. We suppose that the number $m \in N$ satisfies condition $2^m \le n < 2^{m+1}$. From $E_n(f)_M \downarrow 0$ we get

$$\begin{aligned} \sigma_{n,k}^\gamma &= \frac{C}{(n+1)^{k\gamma}} \left\{ \sum_{\nu=1}^{\infty} \nu^{k\gamma-1} E_\nu^\gamma(f)_M \right\}^{\frac{1}{\gamma}} \\ &\le \frac{c_{15}}{(n+1)^{k\gamma}} \left\{ \sum_{\nu=0}^{m+1} \sum_{\mu=2^\nu}^{2^{\nu+1}-1} \mu^{k\gamma-1} E_n^\gamma(f)_M \right\}^{\frac{1}{\gamma}} \\ &\le \frac{c_{16}}{(n+1)^{k\gamma}} \left\{ \sum_{\nu=0}^{m+1} 2^{\nu\gamma k} E_{2^\nu}^\gamma(f)_M \right\}^{\frac{1}{\gamma}}. \end{aligned}$$

Using the estimate [1]

$$(2.12) \qquad \|f(x) - S_n(x,f)\|_{L_M(\mathbb{T})} \le c_{17} E_n(f)_M$$

and (2.3) we have

$$\begin{aligned} \sigma_{n,k}^\gamma &\le \frac{c_{18}}{(n+1)^{k\gamma}} \left\{ \sum_{\nu=0}^{m+1} 2^{\nu\gamma k} \left\| \sum_{\mu=2^\nu}^{\infty} A_\mu(x;f) \right\|_{L_M(\mathbb{T})}^\gamma \right\}^{\frac{1}{\gamma}} \\ &\le \frac{c_{19}}{(n+1)^{k\gamma}} \left\{ \sum_{\nu=0}^{m+1} 2^{\nu\gamma k} \left\| \left(\sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right)^{\frac{1}{2}} \right\|_{L_M(\mathbb{T})}^\gamma \right\}^{\frac{1}{\gamma}}. \end{aligned}$$

By the Minkowski's inequality we get

$$\sigma_{n,k}^\gamma \le c_{20} \left\{ \sum_{\nu=0}^{m+1} \left\| \left(\frac{2^{2\nu k}}{(n+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right)^{\frac{1}{2}} \right\|_{L_M(\mathbb{T})}^\gamma \right\}^{\frac{1}{\gamma}}.$$

We suppose that $\gamma = 2$. In this case we obtain $2 \geq (q-\delta)$. Then we get

$$\sigma_{n,k}^2 \leq c_{21} \left\{ \sum_{\nu=0}^{m+1} \left\| \left(\frac{2^{2\nu k}}{(n+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right)^{\frac{1}{2}} \right\|_{L_M(\mathbb{T})}^2 \right\}^{\frac{1}{2}} .$$

Its clear that the norm l_p decreases with $p \uparrow$. Then

$$\sigma_{n,k}^2 \leq c_{22} \left\{ \sum_{\nu=0}^{m+1} \left\| \left(\frac{2^{2\nu k}}{(n+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right)^{\frac{1}{2}} \right\|_{L_M(\mathbb{T})}^{q-\delta} \right\}^{\frac{1}{q-\delta}} .$$

The space $L_M(T)$ is of concavity $(q-\delta)$. Then we obtain

$$\begin{aligned} \sigma_{n,k}^2 &\leq c_{23} \left\| \left(\sum_{\nu=0}^{m+1} \left(\frac{2^{2\nu k}}{(n+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right)^{(q-\delta)/2} \right)^{1/(q-\delta)} \right\|_{L_M(\mathbb{T})} \\ &\leq c_{24} \left\| \sum_{\nu=0}^{m+1} \frac{2^{\nu k}}{(n+1)^k} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1} \right\|_{L_M(\mathbb{T})} . \end{aligned}$$

Using Abel's transformation and Minkowski's inequality, we find that

$$\begin{aligned} \sigma_{n,k}^2 &\leq c_{25} \left\| \left(\sum_{\nu=0}^{m} \frac{2^{\nu k}}{(n+1)^k} \Delta_{\nu+1} + \frac{2^{(m+1)k}}{(n+1)^k} \sum_{\mu=m+1}^{\infty} \Delta_{\mu+1} \right) \right\|_{L_M(\mathbb{T})} \\ (2.13) \quad &\leq c_{26} \left\| \sum_{\nu=0}^{m} \frac{2^{\nu k}}{(n+1)^k} \Delta_{\nu+1} \right\|_{L_M(\mathbb{T})} + c_{27} \left\| \frac{2^{(m+1)k}}{(n+1)^k} \sum_{\mu=m+1}^{\infty} \Delta_{\mu+1} \right\|_{L_M(\mathbb{T})} . \end{aligned}$$

Taking the relations (2.3) and (2.12) into account we get

$$(2.14) \quad \left\| \sum_{\mu=m+1}^{\infty} \Delta_{\mu+1} \right\|_{L_M(\mathbb{T})} \leq c_{28} \left\| \sum_{\mu=2^{m+1}}^{\infty} A_\mu(x;f) \right\|_{L_M(\mathbb{T})} \leq c_{29} E_n(f)_M .$$

Then from (2.13) and (2.14) we conclude that

$$\sigma_{n,k}^2 \leq c_{30} \left\| \sum_{\nu=0}^{m} \frac{2^{\nu k}}{(n+1)^k} \Delta_{\nu+1} \right\|_{L_M(\mathbb{T})} + c_{31} E_n(f)_M .$$

Note that system of multipliers

$$\begin{aligned} \lambda_\mu &= \frac{2^{\nu k}}{\mu^k (n+1)^k} \ (2^\nu \leq \mu \leq 2^{\nu+1}-1, \ \nu = 1,2,..., \ 2^{m+1}-1), \\ \lambda_\mu &= 0 \ (\mu \geq 2^{m+1}) \end{aligned}$$

satisfies the conditions (2.1). Therefore, by (2.2) we obtain

$$\sigma_{n,k}^2 \le c_{32} \left\| \left| \sum_{\mu=0}^{n} \frac{\mu^k}{(n+1)^k} A_\mu(x;f) \right| \right\|_{L_M(\mathbb{T})} + c_{33} E_n(f) \le c_{34} R_n(f;\lambda)_M .$$

Let $\gamma = q-\delta$. Then $2 \le (q-\delta)$. Using $(q-\delta)$ concavity of $L_M(\mathbb{T})$ we get

$$\begin{aligned} \sigma_{n,k}^{q-\delta} &\le c_{35} \left\{ \sum_{\nu=0}^{m+1} \left\| \left(\frac{2^{2\nu k}}{(n+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right)^{\frac{1}{2}} \right\|_{L_M(\mathbb{T})}^{q-\delta} \right\}^{\frac{1}{q-\delta}} \\ &\le c_{36} \left\| \left(\sum_{\nu=0}^{m+1} \left(\frac{2^{2\nu k}}{(n+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right)^{(q-\delta)/2} \right)^{1/q-\delta)} \right\|_{L_M(\mathbb{T})} \\ &\le c_{37} \left\| \left(\sum_{\nu=0}^{m+1} \frac{2^{2\nu k}}{(n+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right)^{\frac{1}{2}} \right\|_{L_M(\mathbb{T})} . \end{aligned}$$

Further, using the same Abel's transformation and reasoning as in the case $2 \ge (q-\delta)$ we have

$$\sigma_{n,k}^{q-\delta} \le c_{38} R_n(f;\lambda)_M .$$

Proof of Theorem 1.4 is completed. □

Proof of Theorem 1.5 is similar to proof of Theorem 1.4.

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