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VARIATION OF PARAMETERS FOR NABLA FRACTIONAL DIFFERENCE EQUATIONS

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Abstract. The initial data associated with mathematical models or equations that describe physical phenomena, may have errors. It is important to know the effect of these errors on the desired behaviour of the solutions of initial value problems. In this paper, we discuss the continuous dependence of solutions on the initial conditions for nabla fractional difference equations. We also obtain the linear variation of parameters formula for nabla fractional difference equations involving Riemann-Liouville type fractional differences.

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1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Many scientists have paid a lot of attention to this calculus because of its interesting applications in various fields of science and engineering, such as viscoelasticity, diffusion, neurology, control theory and statistics [19]. The analogous theory for discrete fractional calculus was initiated by Miller and Ross [18] and Gray and Zhang [14], where basic approaches, definitions, and properties of the theory of fractional sums and differences were discussed. After then, several authors [1–3, 5–9, 11–13, 15–17, 20] started to deal with discrete fractional calculus on the lines of time scales calculus.

The present article is organized as follows: Section 2 contains basic definitions and results concerning nabla discrete fractional calculus. In section 3, we discuss the continuous dependence of solutions of nabla fractional difference equations on the initial conditions. We derive the linear variation of parameters formula for nabla fractional difference equations in Section 4.

2. Nabla Discrete Fractional Calculus

Throughout the article, we shall consider the discrete time scale

$$\mathbb{T} = \mathbb{N}_a = \{a, a+1, a+2, ...\}, \quad \text{where} \quad a \in \mathbb{R} \quad \text{is} \quad \text{fixed}.$$

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For any function $f : \mathbb{N}_a \to \mathbb{R}$, the backward difference or nabla operator is defined as $\nabla f(t) = f(t) - f(t-1)$ for $t \in \mathbb{N}_{a+1}$ and the higher order differences are defined recursively by $\nabla^n f(t) = \nabla\big(\nabla^{n-1} f(t)\big)$ for $t \in \mathbb{N}_{a+n}, n \in \mathbb{N}$. In addition, we take ∇^0 as the identity operator. Based on these preliminary definitions, we say F is an anti-nabla difference of f on $\mathbb{N}_a$ if and only if $\nabla F(t) = f(t)$ for $t \in \mathbb{N}_{a+1}$. We then define the definite nabla integral of $f : \mathbb{N}_a \to \mathbb{R}$ by

$$\int_c^d f(s)\nabla s = \begin{cases} \sum_{s=c+1}^d f(s), & \text{if } c < d, \\ 0, & \text{if } c = d, \\ -\sum_{s=d+1}^c f(s), & \text{if } c > d. \end{cases}, \quad \text{where } \ c, d \in \mathbb{N}_a. \tag{2.1}$$

Definition 2.1. For any real numbers α and t, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} \setminus \{......, -2, -1, 0\}, \quad 0^{\overline{\alpha}} = 0. \tag{2.2}$$

Definition 2.2. (Nabla Fractional Sum [2, 15]) Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$ be given. Then the α^{th}-order nabla fractional sum of f is given by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s) \quad \text{for} \quad t \in \mathbb{N}_a \tag{2.3}$$

where $\rho(s) = s - 1$. Also, we define the trivial sum by $\nabla_a^{-0} f(t) = f(t)$ for $t \in \mathbb{N}_a$.

Definition 2.3. (R-L Nabla Fractional Difference [2, 15]) Let $f : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ be given, and $N \in \mathbb{N}$ be chosen such that $N - 1 < \alpha \leq N$. Then the α^{th}-order Riemann-Liouville type nabla fractional difference of f is given by

$$\nabla_a^{\alpha} f(t) = \nabla^N \nabla_a^{-(N-\alpha)} f(t) \quad \text{for} \quad t \in \mathbb{N}_{a+N}. \tag{2.4}$$

For $\alpha = 0$, we set $\nabla_a^0 f(t) = f(t)$ for $t \in \mathbb{N}_a$.

The unified definition for fractional sums and differences is as follows.

Remark 2.4. Let $f : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ be given and $N \in \mathbb{N}$ be chosen such that $N - 1 < \alpha \leq N$. Then

1. the α^{th}-order nabla fractional sum of f is given by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s) \quad \text{for} \quad t \in \mathbb{N}_a. \tag{2.5}$$

2. the α^{th}-order fractional difference of f is given by

$$\nabla_a^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{-\alpha-1}} f(s), & \alpha \notin \mathbb{N} \\ \nabla^N f(t), & \alpha = N \in \mathbb{N}, \end{cases} \tag{2.6}$$

for $t \in \mathbb{N}_{a+N}$.

We adopt the following notation given by Atici and Eloe [8].

Definition 2.5. For any functions $y(t), \phi(t) : \mathbb{N}_a \to \mathbb{R}$, define

$$E_y\phi = \nabla_a^{-\alpha} y(t)\phi(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s)\phi(s) \tag{2.7}$$

and

$$E_y^k\phi = E_y^{k-1}[E_y\phi], \quad k = 1, 2, ... \tag{2.8}$$

3. Continuous Dependence of Solutions

Let $f(t,r) : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$, $u(t) : \mathbb{N}_a \to \mathbb{R}$ and $0 < \alpha < 1$. Consider a nonlinear fractional difference equation together with an initial condition of the form

$$\nabla_{a-1}^{\alpha} u(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1}, \tag{3.1}$$

$$\nabla_{a-1}^{-(1-\alpha)} u(t)\Big|_{t=a} = u(a) = u_0. \tag{3.2}$$

Abdeljawad and Atici [2] established the following result.

Lemma 3.1. *u(t) is a solution of the initial value problem* (3.1) - (3.2) *if and only if u(t) has the following representation*

$$u(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)). \tag{3.3}$$

The recursive iteration to this sum equation implies the existence of unique solution of (3.1) - (3.2). Atici and Eloe [8] proved the following theorem which is an analogue of the Gronwalls inequality in discrete fractional calculus.

Theorem 3.2. *Let $u(t)$ and $y(t)$ be nonnegative real valued functions such that $0 \le y(t) < 1$ for all $t \in \mathbb{N}_a$ and*

$$u(t) \le \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s)u(s). \tag{3.4}$$

Then

$$u(t) \le \frac{u_0}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_y^k (t-a+1)^{\overline{\alpha-1}} \tag{3.5}$$

where

$$E_y(t-a+1)^{\overline{\alpha-1}} = \nabla_a^{-\alpha}(t-a+1)^{\overline{\alpha-1}} y(t). \tag{3.6}$$

Theorem 3.3. *Let the following condition be satisfied.*

$$|f(t,u(t)) - f(t,v(t))| \leq \lambda(t)|u(t) - v(t)| \tag{3.7}$$

where $v(t)$, $\lambda(t) : \mathbb{N}_a \to \mathbb{R}$ *such that* $0 \leq \lambda(t) < 1$. *Then, for the solutions* $u(t)$ *and* $v(t)$ *of the initial value problems* (3.1) - (3.2) *and*

$$\nabla^{\alpha}_{a-1} v(t) = f(t, v(t)), \quad t \in \mathbb{N}_{a+1}, \tag{3.8}$$

$$\nabla^{-(1-\alpha)}_{a-1} v(t)\Big|_{t=a} = v(a) = v_0 \tag{3.9}$$

respectively, the following inequality holds

$$|u(t) - v(t)| \leq \frac{|u_0 - v_0|}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E^k_{\lambda}(t-a+1)^{\overline{\alpha-1}}. \tag{3.10}$$

Proof. Using (3.3), the initial value problems (3.1) - (3.2) and (3.8) - (3.9) are equivalent to

$$u(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)),$$

$$v(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} v_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, v(s)).$$

Then

$$u(t) - v(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}[u_0 - v_0] + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}}[f(s,u(s) - f(s,v(s))]. \tag{3.11}$$

Thus, from (3.7), it follows that

$$|u(t)-v(t)| \leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}|u_0-v_0| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}}[\lambda(s)|u(s)-v(s)|]. \tag{3.12}$$

Now an application of Theorem 3.2 yields (3.10). □

Hereafter, to emphasize the dependence of the initial point (a, u_0) we shall denote the solution of the initial value problem (3.1) - (3.2) as $u(t, a, u_0)$.

Theorem 3.4. *Assume*

$$|f(t,u(t)) - f(t,v(t))| \leq g(t, |u(t) - v(t)|) \tag{3.13}$$

for all $(t, u(t)), (t, v(t)) \in \mathbb{N}_a \times \mathbb{R}$ *where* $g(t,r)$ *is defined on* $\mathbb{N}_a \times \mathbb{R}$ *and non-decreasing in* r *for any fixed* $t \in \mathbb{N}_a$. *Further, let* $u(t,a,u_1)$ *and* $u(t,a,u_2)$ *be solutions of* (3.1). *Then, for all* $t \in \mathbb{N}_a$,

$$|u(t,a,u_1) - u(t,a,u_2)| \leq r(t,a,r_0) \tag{3.14}$$

where $r(t) = r(t, a, r_0)$ is the solution of the initial value problem

$$\nabla_{a-1}^{\alpha} r(t) = g(t, r(t)), \quad t \in \mathbb{N}_{a+1}, \tag{3.15}$$

$$\nabla_{a-1}^{-(1-\alpha)} r(t)\Big|_{t=a} = r(a) = r_0 (= |u_1 - u_2|). \tag{3.16}$$

Proof. Since $u(t, a, u_1)$ and $u(t, a, u_2)$ are solutions of (3.1), we have

$$u(t, a, u_1) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_1 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s, a, u_1)),$$

$$u(t, a, u_2) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_2 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s, a, u_2)).$$

Then

$$\begin{aligned} |u(t, a, u_1) - u(t, a, u_2)| \le{}& \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |u_1 - u_2| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} |f(s, u(s, a, u_1)) - f(s, u(s, a, u_2))|. \end{aligned}$$

Let $z(t) = |u(t, a, u_1) - u(t, a, u_2)|$. Then,

$$z(t) \le \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} z_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} g(s, z(s)). \tag{3.17}$$

Further, $z_0 \le r_0$ and

$$r(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} r_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} g(s, r(s)). \tag{3.18}$$

Suppose that $z(t) \le r(t)$ is not true. Then, because of $z_0 \le r_0$, there exists a $k \in \mathbb{N}_a$ such that $z(m) \le r(m)$ for all $m \le k$ and

$$z(k+1) > r(k+1). \tag{3.19}$$

From the monotone property of g, for $m \le k$,

$$g(m, z(m)) \le g(m, r(m)). \tag{3.20}$$

Using (3.17) - (3.20), we get

$$\begin{aligned} z(k+1) &\le \frac{(k-a+2)^{\overline{\alpha-1}}}{\Gamma(\alpha)} z_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{k+1} (k+1-\rho(s))^{\overline{\alpha-1}} g(s, z(s)) \\ &= \frac{(k-a+2)^{\overline{\alpha-1}}}{\Gamma(\alpha)} z_0 \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{k} (k+1-\rho(s))^{\overline{\alpha-1}} g(s, z(s)) + g(k+1, z(k+1)) \end{aligned}$$

$$\leq \frac{(k-a+2)^{\overline{\alpha-1}}}{\Gamma(\alpha)} r_0$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{k} (k+1-\rho(s))^{\overline{\alpha-1}} g(s, r(s)) + g(k+1, z(k+1))$$
$$= r(k+1) - g(k+1, r(k+1)) + g(k+1, z(k+1))$$

implies

$$g(k+1, z(k+1)) < g(k+1, r(k+1)) \tag{3.21}$$

which is a contradiction to the monotone property of g. Hence the proof. □

Remark 3.5. If $r(t, a, 0) = 0$ for all $t \in \mathbb{N}_{a+1}$ and $r(t, a, r_0) \to 0$ as $r_0 \to 0$, then from (3.14) it is clear that the solution $u(t, a, u_0)$ continuously depends on u_0.

4. Variation of constants

Let $u(t)$, $v(t)$, $x(t)$, $y(t) : \mathbb{N}_a \to \mathbb{R}$ such that $|x(t)| < 1$ and $0 < \alpha < 1$. Consider a linear homogeneous fractional difference equation of the form

$$\nabla_{a-1}^{\alpha} u(t) = x(t) u(t), \quad t \in \mathbb{N}_{a+1}. \tag{4.1}$$

If we take the initial condition as

$$\nabla_{a-1}^{-(1-\alpha)} u(t)\Big|_{t=a} = u(a) = u_0, \tag{4.2}$$

then using Lemma 3.1, we have

$$u(t, a, u_0) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s) u(s), \quad t \in \mathbb{N}_a. \tag{4.3}$$

Now we use the following result given by Atici and Eloe [8].

Theorem 4.1. *Assume that $|x(t)| < 1$ for $t \in \mathbb{N}_a \cap [a, b]$. Then the discrete fractional sum equation*

$$u(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s) u(s) \tag{4.4}$$

for $t \in \mathbb{N}_a \cap [a, b]$, where $b \in \mathbb{R}$, has a solution

$$u(t) = \frac{u_0}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_x^k (t-a+1)^{\overline{\alpha-1}}. \tag{4.5}$$

Here

$$E_x (t-a+1)^{\overline{\alpha-1}} = \nabla_a^{-\alpha} (t-a+1)^{\overline{\alpha-1}} x(t). \tag{4.6}$$

Using Theorem 4.1,

$$u(t,a,u_0) = \frac{u_0}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_x^k (t-a+1)^{\overline{\alpha-1}}, \quad \text{for } t \in \mathbb{N}_a \cap [a,b], \text{ where } b \in \mathbb{R}, \tag{4.7}$$

is the solution of the initial value problem (4.1) - (4.2). Now we consider a linear nonhomogeneous fractional difference equation of the form

$$\nabla^{\alpha}_{a-1} v(t) = x(t)v(t) + y(t), \quad t \in \mathbb{N}_{a+1}. \tag{4.8}$$

Theorem 4.2. *(Superposition Principle) Let $v(t)$ be a given solution of* (4.8) *and $u(t)$ be a solution of* (4.1)*. Then the function $w(t) = u(t) + v(t)$ is a solution of* (4.8)*.*

Proof. Since $u(t)$ satisfies (4.1) and $v(t)$ satisfies (4.8), we have

$$\nabla^{\alpha}_{a-1} u(t) = x(t)u(t), \tag{4.9}$$

$$\nabla^{\alpha}_{a-1} v(t) = x(t)v(t) + y(t), \quad t \in \mathbb{N}_{a+1}. \tag{4.10}$$

We show that $w(t)$ satisfies the equation (4.8). From the definition of $w(t)$ it follows that

$$\begin{aligned} \nabla^{\alpha}_{a-1} w(t) = \nabla^{\alpha}_{a-1}[u(t) + v(t)] &= \nabla^{\alpha}_{a-1} u(t) + \nabla^{\alpha}_{a-1} v(t) \\ &= x(t)u(t) + x(t)v(t) + y(t) \\ &= x(t)w(t) + y(t). \end{aligned}$$

Therefore,

$$\nabla^{\alpha}_{a-1} w(t) = x(t)w(t) + y(t), \quad t \in \mathbb{N}_{a+1}.$$

Hence $w(t)$ is a solution of the equation (4.8). □

Variation of constants is a very important technique in obtaining the asymptotic behavior of solutions of linear and nonlinear fractional difference equations under perturbations. In this section we develop the variation of parameters formula to represent the solution $v(t,a,u_0)$ of the perturbed problem (4.8) in terms of the solution $u(t,a,u_0)$ of the unperturbed problem (4.1).

Theorem 4.3. *Let $u(t,a,u_0)$ and $v(t,a,u_0)$ denote the solutions of the equations* (4.1) *and* (4.8) *respectively. Then,*

$$v(t,a,u_0) = u(t,a,u_0) + \sum_{s=a+1}^{t} u(t,s,y(s)). \tag{4.11}$$

Proof. Let

$$p(t) = \sum_{s=a+1}^{t} u(t,s,y(s)). \tag{4.12}$$

It is sufficient to show that $p(t)$ satisfies equation (4.8). Then we apply superposition principle to conclude that $v(t,a,u_0)$ satisfies (4.8).

Clearly $p(a)=0$. We use the method of verification to show that $p(t)$ is a solution of (4.8). We show that

$$\nabla_{a-1}^{\alpha} p(t) = x(t)p(t) + y(t), \quad t \in \mathbb{N}_{a+1} \tag{4.13}$$

and then

$$p(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} [x(s)p(s)+y(s)], \quad t \in \mathbb{N}_a. \tag{4.14}$$

Consider

$$\begin{aligned}
p(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} [x(s)p(s)+y(s)] \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s)p(s) + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s) \Big[\sum_{r=a+1}^{s} u(s,r,y(r)) \Big] \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s) \\
&= \sum_{r=a+1}^{t} \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=r}^{t} (t-\rho(s))^{\overline{\alpha-1}} x(s) u(s,r,y(r)) \Big] \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s) \\
&= \sum_{r=a+1}^{t} \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=r+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \nabla_{r-1}^{\alpha} u(s,r,y(r)) \Big] \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s) \\
&= \sum_{r=a+1}^{t} \Big[\nabla_{r}^{-\alpha} \nabla_{r-1}^{\alpha} u(t,r,y(r)) \Big] + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s) \\
&= \sum_{r=a+1}^{t} \Big[u(t,r,y(r)) - \frac{(t-r+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(r) \Big] + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} y(s) \\
&= \sum_{r=a+1}^{t} u(t,r,y(r)) = p(t).
\end{aligned}$$

The proof is complete. □

Remark 4.4. (Variation of Constants) For $t \in \mathbb{N}_a \cap [a,b]$ and $b \in \mathbb{R}$, the solution of (4.1) - (4.2) is

$$u(t,a,u_0) = \frac{u_0}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_x^k (t-a+1)^{\overline{\alpha-1}}.$$

Then

$$u(t,s,y(s)) = \frac{y(s)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_x^k (t-s+1)^{\overline{\alpha-1}}. \tag{4.15}$$

Substituting these expressions in (4.11), we get the solution of (4.8)-(4.2) as (4.16)

$$v(t,a,u_0) = \frac{u_0}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_x^k (t-a+1)^{\overline{\alpha-1}} + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left[y(s) \sum_{k=0}^{\infty} E_x^k (t-s+1)^{\overline{\alpha-1}} \right]$$

for $t \in \mathbb{N}_a \cap [a,b]$, where $b \in \mathbb{R}$.

5. Conclusion

If we take $x(t) = \lambda$, using [8], the solution of the initial value problem

$$\nabla_{a-1}^{\alpha} u(t) = \lambda u(t), \quad t \in \mathbb{N}_{a+1}, \tag{5.1}$$

$$\nabla_{a-1}^{-(1-\alpha)} u(t)\Big|_{t=a} = u(a) = u_0, \tag{5.2}$$

is given by (5.3)

$$u(t,a,u_0) = \frac{u_0}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_\lambda^k (t-a+1)^{\overline{\alpha-1}} = (t-a+1)^{\overline{\alpha-1}} u_0 F_{\alpha,\alpha}(\lambda (t-a+\alpha)^{\overline{\alpha}}).$$

Here F is the discrete Mittag - Leffler function defined by

$$F_{\alpha,\beta}(\lambda t^{\overline{\nu}}) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\overline{k\nu}}}{\Gamma(k\alpha+\beta)} \tag{5.4}$$

where α and β are positive real numbers, ν is any real number and $|\lambda| < 1$. Using (5.3), the solution of

$$\nabla_{a-1}^{\alpha} v(t) = x(t)v(t) + y(t), \quad t \in \mathbb{N}_{a+1} \tag{5.5}$$

is given by

$$\begin{aligned} v(t,a,u_0) &= u(t,a,u_0) + \sum_{s=a+1}^{t} u(t,s,y(s)) \\ &= (t-a+1)^{\overline{\alpha-1}} u_0 F_{\alpha,\alpha}(\lambda (t-a+\alpha)^{\overline{\alpha}}) \\ &+ \sum_{s=a+1}^{t} (t-s+1)^{\overline{\alpha-1}} y(s) F_{\alpha,\alpha}(\lambda (t-s+\alpha)^{\overline{\alpha}}). \end{aligned} \tag{5.6}$$

For a particular value of a, Atici and Eloe [7] have obtained the same solution for (5.5) using N-transform. Further, Abdeljawad et.al. [3] found the solution of (5.5) for $a = 0$ by recursion. But the solution obtained in (5.6) is the solution of the linear nonhomogeneous fractional difference equation (5.5) for any a using the variations of constants method. We have also obtained the variation of constants formula for any function $x(t)$. To my knowledge, this method is not used explicitly elsewhere.

The variation of parameters formula for linear and nonlinear differential and difference equations is an important tool in the study of qualitative properties of perturbed problems. The present work can be extended to a more generalized discrete time scales discussed in [4,13]. Further, one can establish the nonlinear variation of parameters formula for nabla fractional difference equations on discrete time scales [4, 10, 13].

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