Novi Sad J. Math.
Vol. 44, No. 2, 2014, 121-135

APPLICATIONS OF THE THICK DISTRIBUTIONAL CALCULUS

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Abstract. We give several applications of the thick distributional calculus. We consider homogeneous thick distributions, point source fields, and higher order derivatives of order 0.

AMS Mathematics Subject Classification (2010): 46F10

Key words and phrases: thick points, delta functions, distributions, generalized functions

1. Introduction

The aim of this note is to give several applications of the recently introduced calculus of thick distributions in several variables [16], generalizing the thick distributions of one variable [3]. The thick distributional calculus allows us to study problems where a finite number of special points are present; it is the distributional version of the analysis of Blanchet and Faye [1], who employed the concepts of Hadamard finite parts as developed by Sellier [13] to study dynamics of point particles in high post-Newtonian approximations of general relativity. We give a short summary of the theory of thick distributions in Section 2.

Our first application, given in Section 3, is the computation of the distributional derivatives of homogeneous distributions in $\mathbb{R}^n$ by first computing the thick distributional derivatives and then projecting onto the space of standard distributions. Our analysis makes several delicate points quite clear.

Next, in Section 4, we consider an application to point source fields. In [2], Bowen computed the derivative of the distribution

$$g_{j_1,\ldots,j_k}(\mathbf{x}) = \frac{n_{j_1}\cdots n_{j_k}}{r^2}\,, \tag{1.1}$$

of $\mathcal{D}'\left(\mathbb{R}^3\right)$, where $r = |\mathbf{x}|$ and $\mathbf{n} = (n_i)$ is the unit normal vector to a sphere centered at the origin, that is, $n_i = x_i/r$. Following the notation introduced by the late Professor Farassat [8] of denoting distributional derivatives with an overbar, Bowen's result can be written as,

$$\frac{\overline{\partial}}{\partial x_i} g_{j_1,\ldots,j_k} = \left\{ \sum_{q=1}^{k} \delta_{ij_q} \frac{n_{j_1}\cdots n_{j_k}}{n_{j_q}} - (k+2)\, n_i n_{j_1}\cdots n_{j_k} \right\} \frac{1}{r^3} + A\delta(\mathbf{x})\,, \tag{1.2}$$

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where $n_i n_{j_1} \cdots n_{j_k} = n_1^a n_2^b n_3^c$, and $A = 0$ if a, b, or c is odd, while

$$A = \frac{2\Gamma\left(\left(a+1\right)/2\right)\Gamma\left(\left(b+1\right)/2\right)\Gamma\left(\left(c+1\right)/2\right)}{\Gamma\left(\left(a+b+c+3\right)/2\right)}, \tag{1.3}$$

if the three exponents are even. Interestingly, he observes that if one tries to compute this formula by induction, employing the product rule for derivatives, the result obtained is *wrong.* In this article we show that one can actually apply the product rule in the space of thick distributions, obtaining (1.2) by induction; furthermore, our analysis shows *why* the wrong result is obtained when applying the product rule in [2].

Finally in Section 5 we show how the thick distributional calculus allows one to avoid mistakes in the computation of higher order derivatives of thick distributions of order 0.

2. Thick distributions

We now recall the basic ideas of the thick distributional calculus [16]. If $\mathbf{a}$ is a fixed point of $\mathbb{R}^n$, then the space of test functions with a thick point at $\mathbf{x} = \mathbf{a}$ is defined as follows.

Definition 2.1. Let $\mathcal{D}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$ denote the vector space of all smooth functions ϕ defined in $\mathbb{R}^n \setminus \{\mathbf{a}\}$, with support of the form $K \setminus \{\mathbf{a}\}$, where K is compact in $\mathbb{R}^n$, that admit a strong asymptotic expansion of the form

$$\phi\left(\mathbf{a}+\mathbf{x}\right) = \phi\left(\mathbf{a}+r\mathbf{w}\right) \sim \sum_{j=m}^{\infty} a_j\left(\mathbf{w}\right) r^j, \quad \text{as } \mathbf{x} \to \mathbf{0}\,, \tag{2.1}$$

where m is an integer (positive or negative), and where the a_j are smooth functions of $\mathbf{w}$, that is, $a_j \in \mathcal{D}\left(\mathbb{S}\right)$. The subspace $\mathcal{D}_{*,\mathbf{a}}^{[m]}\left(\mathbb{R}^n\right)$ consists of those test functions ϕ whose expansion (2.1) begins at m. For a fixed compact K whose interior contains $\mathbf{a}$, $\mathcal{D}_{*,\mathbf{a}}^{[m;K]}\left(\mathbb{R}^n\right)$ is the subspace formed by those test functions of $\mathcal{D}_{*,\mathbf{a}}^{[m]}\left(\mathbb{R}^n\right)$ that vanish in $\mathbb{R}^n \setminus K$.

Observe that we require the asymptotic development of $\phi\left(\mathbf{x}\right)$ as $\mathbf{x} \to \mathbf{a}$ to be "strong". This means [7, Chapter 1] that for any differentiation operator $\left(\partial/\partial\mathbf{x}\right)^{\mathbf{p}} = \left(\partial^{p_1}...\partial^{p_n}\right)/\partial x_1^{p_1}...\partial x_n^{p_n}$, the asymptotic development of $\left(\partial/\partial\mathbf{x}\right)^{\mathbf{p}}\phi\left(\mathbf{x}\right)$ as $\mathbf{x} \to \mathbf{a}$ exists and is equal to the term-by-term differentiation of $\sum_{j=m}^{\infty} a_j\left(\mathbf{w}\right) r^j$. Observe that saying that the expansion exists as $\mathbf{x} \to \mathbf{0}$ is the same as saying that it exists as $r \to 0$, uniformly with respect to $\mathbf{w}$.

We call $\mathcal{D}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$ the space of test functions on $\mathbb{R}^n$ with a thick point located at $\mathbf{x} = \mathbf{a}$. We denote $\mathcal{D}_{*,\mathbf{0}}\left(\mathbb{R}^n\right)$ as $\mathcal{D}_{*}\left(\mathbb{R}^n\right)$.

The topology of the space of thick test functions is constructed as follows.

Definition 2.2. Let m be a fixed integer and K a compact subset of $\mathbb{R}^n$ whose interior contains $\mathbf{a}$. The topology of $\mathcal{D}_{*,\mathbf{a}}^{[m;K]}\left(\mathbb{R}^n\right)$ is given by the seminorms

$\left\{\| \quad \|_{q,s}\right\}_{q>m,s\geq 0}$ defined as

$$(2.2) \qquad \|\phi\|_{q,s} = \sup_{\mathbf{x}-\mathbf{a}\in K}\ \sup_{|\mathbf{p}|\leq s} \frac{\left|\dfrac{\partial^{\mathbf{p}}\phi}{\partial\mathbf{x}}(\mathbf{a}+\mathbf{x}) - \displaystyle\sum_{j=m-|\mathbf{p}|}^{q-1} a_{j,\mathbf{p}}(\mathbf{w})\, r^{j}\right|}{r^{q}},$$

where $\mathbf{x} = r\mathbf{w}$ and

$$(2.3) \qquad \frac{\partial^{\mathbf{p}}\phi}{\partial\mathbf{x}}(\mathbf{a}+\mathbf{x}) \sim \sum_{j=m-|\mathbf{p}|}^{\infty} a_{j,\mathbf{p}}(\mathbf{w})\, r^{j}.$$

The topology of $\mathcal{D}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ is the inductive limit topology of the $\mathcal{D}_{*,\mathbf{a}}^{[m;K]}(\mathbb{R}^n)$ as $K \nearrow \infty$. The topology of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ is the inductive limit topology of the $\mathcal{D}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ as $m \searrow -\infty$.

A sequence $\{\phi_l\}_{l=0}^{\infty}$ in $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ converges to ψ if and only there exists $l_0 \geq 0$, an integer m, and a compact set K with $\mathbf{a}$ in its interior, such that $\phi_l \in \mathcal{D}_{*,\mathbf{a}}^{[m;K]}(\mathbb{R}^n)$ for $l \geq l_0$ and $\|\psi - \phi_l\|_{q,s} \to 0$ as $l \to \infty$ if $q > m, s \geq 0$. Notice that if $\{\phi_l\}_{l=0}^{\infty}$ converges to ψ in $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ then ϕ_l and the corresponding derivatives converge uniformly to ψ and its derivatives in any set of the form $\mathbb{R}^n \setminus B$, where B is a ball with center at $\mathbf{a}$; in fact, $r^{|\mathbf{p}|-m}(\partial/\partial\mathbf{x})^{\mathbf{p}}\phi_l$ converges uniformly to $r^{|\mathbf{p}|-m}(\partial/\partial\mathbf{x})^{\mathbf{p}}\psi$ over all $\mathbb{R}^n$. Furthermore, if $\left\{a_j^l\right\}$ are the coefficients of the expansion of ϕ_l and $\{b_j\}$ are those for ψ, then $a_j^l \to b_j$ in the space $\mathcal{D}(\mathbb{S})$ for each $j \geq m$.

We can now consider distributions in a space with one thick point, the "thick distributions."

Definition 2.3. The space of distributions on $\mathbb{R}^n$ with a thick point at $\mathbf{x} = \mathbf{a}$ is the dual space of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$. We denote it $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$, or just as $\mathcal{D}'_{*}(\mathbb{R}^n)$ when $\mathbf{a} = \mathbf{0}$.

Observe that $\mathcal{D}(\mathbb{R}^n)$, the space of standard test functions, is a closed subspace of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$; we denote by

$$(2.4) \qquad i : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n),$$

the inclusion map and by

$$(2.5) \qquad \Pi : \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n),$$

the projection operator, dual of the inclusion (2.4).

The derivatives of thick distributions are defined in much the same way as the usual distributional derivatives, that is, by duality.

Definition 2.4. If $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ then its thick distributional derivative $\partial^* f/\partial x_j$ is defined as

$$\left\langle \frac{\partial^* f}{\partial x_j}, \phi \right\rangle = -\left\langle f, \frac{\partial \phi}{\partial x_j} \right\rangle , \qquad \phi \in \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n) \ . \tag{2.6}$$

We denote by $\mathcal{E}_*(\mathbb{R}^n)$ the space of smooth functions in $\mathbb{R}^n \setminus \{\mathbf{a}\}$ that have a strong asymptotic expansion of the form (2.1); alternatively, $\psi \in \mathcal{E}_*(\mathbb{R}^n)$ if $\psi = \psi_1 + \psi_2$, where $\psi_1 \in \mathcal{E}(\mathbb{R}^n)$, the space of all smooth functions in $\mathbb{R}^n$, and where $\psi_2 \in \mathcal{D}_*(\mathbb{R}^n)$. The space $\mathcal{E}_*(\mathbb{R}^n)$ is the space of *multipliers* of $\mathcal{D}_*(\mathbb{R}^n)$ and of $\mathcal{D}'_*(\mathbb{R}^n)$. Furthermore [16], the product rule for derivatives holds,

$$\frac{\partial^*(\psi f)}{\partial x_j} = \frac{\partial \psi}{\partial x_j} f + \psi \frac{\partial^* f}{\partial x_j} , \tag{2.7}$$

if f is a thick distribution and ψ is a multiplier. Notice that $\partial\psi/\partial x_j$ is the *ordinary* derivative in (2.7).

Let $g(\mathbf{w})$ is a distribution in $\mathbb{S}$. The thick delta function of degree q, denoted as $g\delta_*^{[q]}$, or as $g(\mathbf{w})\,\delta_*^{[q]}$, acts on a thick test function $\phi(\mathbf{x})$ as

$$\left\langle g\delta_*^{[q]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C} \left\langle g(\mathbf{w}), a_q(\mathbf{w}) \right\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} , \tag{2.8}$$

where $\phi(r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j$, as $r \to 0^+$, and where

$$C = \frac{2\pi^{n/2}}{\Gamma(n/2)} , \tag{2.9}$$

is the surface area of the unit sphere $\mathbb{S}$ of $\mathbb{R}^n$. If g is locally integrable function in $\mathbb{S}$, then

$$\left\langle g\delta_*^{[q]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C} \int_{\mathbb{S}} g(\mathbf{w})\, a_q(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \ . \tag{2.10}$$

Thick deltas of order 0 are called just thick deltas, and we shall use the notation $g\delta_*$ instead of $g\delta_*^{[0]}$.

Let $g \in \mathcal{D}'(\mathbb{S})$. Then

$$\frac{\partial^*}{\partial x_j}\left(g\delta_*^{[q]}\right) = \left(\frac{\delta g}{\delta x_j} - (q+n)\, n_j g\right) \delta_*^{[q+1]}. \tag{2.11}$$

Here $\delta g/\delta x_j$ is the $\delta-$derivative of g [4, 6]; in general the $\delta-$derivatives can be applied to functions and distributions defined only on a smooth hypersurface Σ of $\mathbb{R}^n$. Suppose now that the surface is $\mathbb{S}$, the unit sphere in $\mathbb{R}^n$ and let f be a smooth function defined in $\mathbb{S}$, that is, $f(\mathbf{w})$ is defined if $\mathbf{w} \in \mathbb{R}^n$ satisfies $|\mathbf{w}| = 1$. Observe that the expressions $\partial f/\partial x_j$ are not defined and, likewise, if $\mathbf{w} = (w_j)_{1\le j\le n}$ the expressions $\partial f/\partial w_j$ do not make sense either; the derivatives that are always defined and that one should consider are the

$\delta f/\delta x_j$, $1 \le j \le n$. Let F_0 be the extension of f to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ that is homogeneous of degree 0, namely, $F_0(\mathbf{x}) = f(\mathbf{x}/r)$ where $r = |\mathbf{x}|$. Then [16]

$$\frac{\delta f}{\delta x_j} = \left.\frac{\partial F_0}{\partial x_j}\right|_{\mathbb{S}}. \tag{2.12}$$

Also, if we use polar coordinates, $\mathbf{x} = r\mathbf{w}$, so that $F_0(\mathbf{x}) = f(\mathbf{w})$, then $\partial F_0/\partial x_j$ is homogeneous of degree -1, and actually $\partial F_0/\partial x_j = r^{-1}\delta f/\delta x_j$ if $\mathbf{x} \neq \mathbf{0}$.

The matrix $\mu = (\mu_{ij})_{1\le i,j\le n}$, where $\mu_{ij} = \delta n_i/\delta x_j$, plays an important role in the study of distributions on a surface Σ. If $\Sigma = \mathbb{S}$ then $\mu_{ij} = \delta n_i/\delta x_j = \delta_{ij} - n_i n_j$. Observe that $\mu_{ij} = \mu_{ji}$, an identity that holds in any surface.

The differential operators $\delta f/\delta x_j$ are initially defined if f is a smooth function defined on Σ, but we can also define them when f is a distribution. We can do this if we use the fact that smooth functions are dense in the space of distributions on Σ.

3. The thick distribution $\mathcal{P}f(1)$

Let us consider one of the simplest functions, namely, the function 1, defined in $\mathbb{R}^n$. Naturally this function is locally integrable, and thus it defines a regular distribution, also denoted as 1, and the ordinary derivatives and the distributional derivatives both coincide and give the value 0. On the other hand, 1 does not automatically give an element of $\mathcal{D}'_*(\mathbb{R}^n)$ since if $\phi \in \mathcal{D}_*(\mathbb{R}^n)$ the integral $\int_{\mathbb{R}^n} \phi(\mathbf{x})\,\mathrm{d}\mathbf{x}$ could be divergent, and thus we consider the *spherical* finite part thick distribution $\mathcal{P}f(1)$ given as

$$\langle \mathcal{P}f(1), \phi\rangle = \text{F.p.} \int_{\mathbb{R}^n} \phi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \text{F.p.} \lim_{\varepsilon\to 0^+} \int_{|\mathbf{x}|\ge\varepsilon} \phi(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{3.1}$$

The derivatives of $\mathcal{P}f(1)$ do not vanish, since actually we have the following formula [16].

Lemma 3.1. *In $\mathcal{D}'_*(\mathbb{R}^n)$,*

$$\frac{\partial^*}{\partial x_i}(\mathcal{P}f(1)) = C n_i \delta_*^{[-n+1]}, \tag{3.2}$$

where C is given by (2.9).

Proof. One can find a proof of a more general statement in [16], but in this simpler case the proof can be written as follows,

$$\begin{aligned}\left\langle \frac{\partial^*}{\partial x_i}(\mathcal{P}f(1)), \phi\right\rangle &= -\left\langle \mathcal{P}f(1), \frac{\partial\phi}{\partial x_i}\right\rangle \\ &= -\text{F.p.} \lim_{\varepsilon\to 0^+} \int_{|\mathbf{x}|\ge\varepsilon} \frac{\partial\phi}{\partial x_i}\,\mathrm{d}\mathbf{x} \\ &= \text{F.p.} \lim_{\varepsilon\to 0^+} \int_{\varepsilon\mathbb{S}^{n-1}} n_i\phi\,\mathrm{d}\sigma,\end{aligned}$$

so that if $\phi \in \mathcal{D}_* (\mathbb{R}^n)$ has the expansion $\phi(\mathbf{x}) \sim \sum_{j=m}^{\infty} a_j (\mathbf{w}) r^j$, as $\mathbf{x} \to \mathbf{0}$, then

$$\int_{\varepsilon \mathbb{S}^{n-1}} n_i \phi \, \mathrm{d}\sigma \sim \sum_{j=m}^{\infty} \left(\int_{\mathbb{S}} n_i a_j (\mathbf{w}) \, \mathrm{d}\sigma (\mathbf{w}) \right) \varepsilon^{n-1+j},$$

as $\varepsilon \to 0^+$. The finite part of the limit is equal to the coefficient of ε^0, thus

$$\begin{aligned} \mathrm{F.p.} \lim_{\varepsilon \to 0} \int_{\varepsilon \mathbb{S}^{n-1}} n_i \phi \, \mathrm{d}\sigma &= \int_{\mathbb{S}} n_i a_{1-n} (\mathbf{w}) \, \mathrm{d}\sigma (\mathbf{w}) \\ &= \left\langle C n_i \delta_*^{[1-n]}, \phi \right\rangle , \end{aligned}$$

as required. □

If $\psi \in \mathcal{E}_* (\mathbb{R}^n)$ is a multiplier of $\mathcal{D}_* (\mathbb{R}^n)$, then we define, in a similar way, the thick distribution $\mathcal{P}f(\psi) \in \mathcal{D}'_* (\mathbb{R}^n)$, and we clearly have the useful formula

$$\mathcal{P}f(\psi) = \psi \mathcal{P}f(1) , \tag{3.3}$$

which immediately gives the thick distributional derivative of $\mathcal{P}f(\psi)$ as

$$\frac{\partial^*}{\partial x_i} (\mathcal{P}f(\psi)) = \frac{\partial \psi}{\partial x_i} \mathcal{P}f(1) + \psi \frac{\partial^*}{\partial x_i} (\mathcal{P}f(1)) ,$$

so that we obtain the ensuing formula.

Proposition 3.2. *If $\psi \in \mathcal{E}_* (\mathbb{R}^n)$ then*

$$\frac{\partial^*}{\partial x_i} (\mathcal{P}f(\psi)) = \mathcal{P}f\left(\frac{\partial \psi}{\partial x_i} \right) + C n_i \psi \delta_*^{[1-n]}. \tag{3.4}$$

Notice that, in general, the term $C n_i \psi \delta_*^{[1-n]}$ is *not* a thick delta of order $1-n$. Indeed, let us now consider the case when $\psi \in \mathcal{E}_* (\mathbb{R}^n)$ is homogeneous of order $k \in \mathbb{Z}$. Then $\psi(\mathbf{x}) = r^k \psi_0 (\mathbf{x})$, where ψ_0 is homogeneous of order 0. Since $r^k \delta_*^{[q]} = \delta_*^{[q-k]}$[16, Eqn. (5.16)] we obtain the following particular case of (3.4), where now the term $C n_i \psi_0 \delta_*^{[1-n-k]}$ *is* a thick delta of order $1-n-k$.

Proposition 3.3. *If $\psi \in \mathcal{E}_* (\mathbb{R}^n)$ is homogeneous of order $k \in \mathbb{Z}$, then*

$$\frac{\partial^*}{\partial x_i} (\mathcal{P}f(\psi)) = \mathcal{P}f\left(\frac{\partial \psi}{\partial x_i} \right) + C n_i \psi_0 \delta_*^{[1-n-k]} , \tag{3.5}$$

where $\psi_0 (\mathbf{x}) = |\mathbf{x}|^{-k} \psi (\mathbf{x})$.

If we now apply the projection Π onto the usual distribution space $\mathcal{D}' (\mathbb{R}^n)$, we obtain the formula for the distributional derivatives of homogeneous distributions. Observe first that if $k > -n$ then ψ is integrable at the origin, and thus ψ is a regular distribution and $\Pi (\mathcal{P}f(\psi)) = \psi$. If $k \leq -n$ then $\Pi (\mathcal{P}f(\psi)) = \mathcal{P}f(\psi)$, since in that case the integral $\int_{\mathbb{R}^n} \psi (\mathbf{x}) \phi (\mathbf{x}) \, \mathrm{d}\mathbf{x}$ would be divergent, in general, if $\phi \in \mathcal{D}(\mathbb{R}^n)$. A particularly interesting case is when $k = -n$, since if ψ is homogeneous of degree $-n$ and

$$\int_{\mathbb{S}} \psi(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) = 0\,, \tag{3.6}$$

then the *principal value* of the integral

$$\text{p.v.} \int_{\mathbb{R}^n} \psi(\mathbf{x})\, \phi(\mathbf{x})\, \mathrm{d}\mathbf{x} = \lim_{\varepsilon \to 0^+} \int_{|\mathbf{x}| \geq \varepsilon} \psi(\mathbf{x})\, \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}\,, \tag{3.7}$$

actually exists for each $\phi \in \mathcal{D}(\mathbb{R}^n)$, so that $\mathcal{P}f(\psi) = \text{p.v.}(\psi)$, the principal value distribution. Note however that if Σ is a closed surface in $\mathbb{R}^n$ that encloses the origin, described by an equation of the form $g(\mathbf{x}) = 1$, where $g(\mathbf{x})$ is continuous in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and homogeneous of degree 1, then $\langle \mathcal{R}_\Sigma(\psi(\mathbf{x})), \phi(\mathbf{x}) \rangle = \lim_{\varepsilon \to 0} \int_{g(\mathbf{x}) \geqslant \varepsilon} \psi(\mathbf{x})\phi(\mathbf{x})\, \mathrm{d}\mathbf{x}$, defines another regularization of ψ, but in general $\mathcal{R}_\Sigma(\psi(\mathbf{x})) \neq \text{p.v.}(\psi(\mathbf{x}))$ [15], a fact observed by Farassat [8], who indicated its importance in numerical computations, and studied by several authors [11, 15].

Condition (3.6) holds whenever $\psi = \partial\xi/\partial x_j$ for some ξ homogeneous of order $-n+1$.

Proposition 3.4. *Let ψ be homogeneous of order $k \in \mathbb{Z}$ in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Then, in $\mathcal{D}'(\mathbb{R}^n)$ the distributional derivative $\overline{\partial}\psi/\partial x_i$ is given as follows:*

$$\frac{\overline{\partial}\psi}{\partial x_i} = \frac{\partial \psi}{\partial x_i}\,, \qquad k > 1 - n\,, \tag{3.8}$$

equality of regular distributions;

$$\frac{\overline{\partial}\psi}{\partial x_i} = \text{p.v.}\left(\frac{\partial \psi}{\partial x_i}\right) + A\delta(\mathbf{x})\,, \qquad k = 1 - n\,, \tag{3.9}$$

where $A = \int_{\mathbb{S}} n_i \psi_0(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) = \langle \psi_0, n_i \rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})}$, while

$$\frac{\overline{\partial}\psi}{\partial x_i} = \mathcal{P}f\left(\frac{\partial \psi}{\partial x_i}\right) + D(\mathbf{x})\,, \qquad k < 1 - n\,, \tag{3.10}$$

where $D(\mathbf{x})$ is a homogeneous distribution of order $k-1$ concentrated at the origin and given by

$$D(\mathbf{x}) = (-1)^{-k-n+1} \sum_{j_1 + \cdots + j_n = -k-n+1} \frac{\left\langle n_i \psi_0, \mathbf{w}^{(j_1, \ldots, j_n)} \right\rangle}{j_1! \cdots j_n!} \mathbf{D}^{(j_1, \ldots, j_n)} \delta(\mathbf{x})\,. \tag{3.11}$$

Proof. It follows from (3.4) if we observe [16, Prop. 4.7] that if $g \in \mathcal{D}'(\mathbb{S})$ then

$$\Pi\left(g\delta_*^{[q]}\right) = \frac{(-1)^q}{C} \sum_{j_1 + \cdots + j_n = q} \frac{\left\langle g(\mathbf{w}), \mathbf{w}^{(j_1, \ldots, j_n)} \right\rangle}{j_1! \cdots j_n!} \mathbf{D}^{(j_1, \ldots, j_n)} \delta(\mathbf{x})\,, \tag{3.12}$$

and, in particular,

$$\Pi\left(g\delta_*\right)=\frac{1}{C}\left\langle g\left(\mathbf{w}\right),1\right\rangle\delta\left(\mathbf{x}\right)\ , \tag{3.13}$$

if $q=0$. □

Our next task is to compute the second order thick derivatives of homogeneous distributions. Indeed, if ψ is homogeneous of degree k then we can iterate the formula (3.5) to obtain

$$\begin{aligned}\frac{\partial^{*2}}{\partial x_i\partial x_j}\left(\mathcal{P}f\left(\psi\right)\right)&=\frac{\partial^*}{\partial x_i}\left(\mathcal{P}f\left(\frac{\partial\psi}{\partial x_j}\right)+Cn_j\psi_0\delta_*^{[1-n-k]}\right)\\&=\mathcal{P}f\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)+Cn_i\xi_0\delta_*^{[2-n-k]}+\frac{\partial^*}{\partial x_i}\left(Cn_j\psi_0\delta_*^{[1-n-k]}\right)\ ,\end{aligned} \tag{3.14}$$

where $\xi=\partial\psi/\partial x_j$ is homogeneous of degree $k-1$ and $\xi_0\left(\mathbf{x}\right)=\left|\mathbf{x}\right|^{1-k}\xi\left(\mathbf{x}\right)$ is the associated function which is homogeneous of degree 0. Use of (2.11) allows us to write

$$\begin{aligned}\frac{\partial^*}{\partial x_i}\left(Cn_j\psi_0\delta_*^{[1-n-k]}\right)&=C\left(\frac{\delta}{\delta x_i}\left(n_j\psi_0\right)+\left(k-1\right)n_in_j\psi_0\right)\delta_*^{[2-n-k]}\\&=C\left(\left(\delta_{ij}-n_in_j\right)\psi_0+n_j\frac{\delta\psi_0}{\delta x_i}+\left(k-1\right)n_in_j\psi_0\right)\delta_*^{[2-n-k]}\\&=C\left(\left(\delta_{ij}+\left(k-2\right)n_in_j\right)\psi_0+n_j\frac{\delta\psi_0}{\delta x_i}\right)\delta_*^{[2-n-k]}\ ,\end{aligned} \tag{3.15}$$

while the equation $\psi=r^k\psi_0$ yields $\partial\psi/\partial x_j=r^{k-1}\{kn_j\psi_0+\delta\psi_0/\delta x_j\}$, so that

$$\xi_0=kn_j\psi_0+\frac{\delta\psi_0}{\delta x_j}\ . \tag{3.16}$$

Collecting terms we thus obtain the following formula.

Proposition 3.5. *If $\psi\in\mathcal{E}_*\left(\mathbb{R}^n\right)$ is homogeneous of order $k\in\mathbb{Z}$, then*

$$\begin{aligned}\frac{\partial^{*2}}{\partial x_i\partial x_j}\left(\mathcal{P}f\left(\psi\right)\right)&=\mathcal{P}f\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)\\&\quad+C\left(\left(\delta_{ij}+2\left(k-1\right)n_in_j\right)\psi_0+n_j\frac{\delta\psi_0}{\delta x_i}+n_i\frac{\delta\psi_0}{\delta x_j}\right)\delta_*^{[2-n-k]}\ .\end{aligned} \tag{3.17}$$

where $\psi_0\left(\mathbf{x}\right)=\left|\mathbf{x}\right|^{-k}\psi\left(\mathbf{x}\right)$.

Projection onto $\mathcal{D}'\left(\mathbb{R}^n\right)$ of (3.17) gives the formula for the distributional derivatives $\overline{\partial}^2/\partial x_i\partial x_j(\mathcal{P}f\left(\psi\right))$ if $\psi\in\mathcal{E}_*\left(\mathbb{R}^n\right)$ is homogeneous of order $k\in\mathbb{Z}$. In case $k=2-n$ we obtain the following formula.

Proposition 3.6. *If $\psi \in \mathcal{E}_*(\mathbb{R}^n)$ is homogeneous of order $2-n$, then*

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}(\psi) = \text{p.v.}\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right) + B\delta(\mathbf{x}) , \tag{3.18}$$

where

$$B = \langle \psi_0, 2n_i n_j - \delta_{ij} \rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})} . \tag{3.19}$$

Proof. If we apply the operator Π to (3.17) and employ (3.13) we obtain (3.18) with

$$B = \left\langle \left(\delta_{ij} + 2(k-1) n_i n_j\right) \psi_0 + n_j \frac{\delta \psi_0}{\delta x_j} + n_i \frac{\delta \psi_0}{\delta x_j}, 1 \right\rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})} .$$

But [16, (2.6)] yields

$$\left\langle n_j \frac{\delta \psi_0}{\delta x_j}, 1 \right\rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})} = \langle \psi_0, n\, n_i n_j - \delta_{ij} \rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})} , \tag{3.20}$$

and (3.19) follows since $k = 2-n$. □

We would like to observe that while ψ_0 has been supposed smooth, a continuity argument immediately gives that ψ_0 could be any distribution of $\mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$ that is homogeneous of degree 0.

4. Bowen's formula

If we apply formula (3.9) to the function $\psi = n_{j_1} \cdots n_{j_k}/r^2$, which is homogeneous of degree -2 in $\mathbb{R}^3$ we obtain at once that

$$\begin{aligned} &\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_1} \cdots n_{j_k}}{r^2}\right) = \\ &\quad \text{p.v.}\left(\left\{\sum_{q=1}^{k} \delta_{ij_q} \frac{n_{j_1} \cdots n_{j_k}}{n_{j_q}} - (k+2) n_i n_{j_1} \cdots n_{j_k}\right\} \frac{1}{r^3}\right) + A\delta(\mathbf{x}) , \end{aligned} \tag{4.1}$$

where

$$A = \int_{\mathbb{S}} n_i n_{j_1} \cdots n_{j_k} \, d\sigma(\mathbf{w}) . \tag{4.2}$$

This integral was computed in [5, (3.13)], the result being

$$A = \frac{2\Gamma((a+1)/2)\,\Gamma((b+1)/2)\,\Gamma((c+1)/2)}{\Gamma((a+b+c+3)/2)} , \tag{4.3}$$

if $n_i n_{j_1} \cdots n_{j_k} = n_1^a n_2^b n_3^c$, and a, b, or c are even, while $A = 0$ if any exponent is odd. Bowen [2, Eqn. (A5)] also computes the integral, and obtains a different but equivalent expression; in particular, his formula for $k = 3$ reads as

$$A = \frac{4\pi}{15}\left(\delta_{ij_1}\delta_{j_2 j_3} + \delta_{ij_2}\delta_{j_1 j_3} + \delta_{ij_3}\delta_{j_1 j_2}\right) , \tag{4.4}$$

so that (4.3) or (4.4) would yield that if (a,b,c) is a permutation of $(2,2,0)$ then $A=4\pi/15$ while if (a,b,c) is a permutation of $(4,0,0)$ then $A=4\pi/5$.

Our main aim is to point out why the product rule for derivatives, as employed in [2] does not produce the correct result. Indeed, if we use [2, Eqn. (16)] written as

$$\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_1}}{r^2}\right)=\text{p.v.}\left(\frac{\delta_{ij_1}-3n_in_{j_i}}{r^3}\right)+\frac{4\pi}{3}\delta_{ij_1}\delta\left(\mathbf{x}\right)\ , \tag{4.5}$$

and then try to proceed as in [2, Eqn. (18)],

$$\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_1}n_{j_2}n_{j_3}}{r^2}\right)\text{ ``¿ = ?'' }n_{j_1}n_{j_2}\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_3}}{r^2}\right)+\frac{n_{j_3}}{r^2}\frac{\overline{\partial}}{\partial x_i}\left(n_{j_1}n_{j_2}\right)\ . \tag{4.6}$$

Thus (4.5) and the formula

$$\frac{\overline{\partial}}{\partial x_i}\left(n_{j_1}n_{j_2}\right)=\frac{\delta_{ij_1}n_{j_2}+\delta_{ij_2}n_{j_1}-2n_in_{j_1}n_{j_2}}{r}\ , \tag{4.7}$$

give

$$n_{j_1}n_{j_2}\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_3}}{r^2}\right)+\frac{n_{j_3}}{r^2}\frac{\overline{\partial}}{\partial x_i}\left(n_{j_1}n_{j_2}\right)=\text{``Normal''}+\text{``Src''}\ , \tag{4.8}$$

where

$$\text{``Normal''}=\text{p.v.}\left(\frac{\delta_{ij_1}n_{j_2}n_{j_3}+\delta_{ij_2}n_{j_1}n_{j_3}+\delta_{ij_3}n_{j_1}n_{j_2}-5n_in_{j_1}n_{j_2}n_{j_3}}{r^3}\right)\ , \tag{4.9}$$

coincides with the first term of (4.1) while

$$\text{``Src''}=\frac{4\pi}{3}\delta_{ij_3}n_{j_1}n_{j_2}\delta\left(\mathbf{x}\right)\ . \tag{4.10}$$

The right hand side of (4.10) is not a well defined distribution, of course, but Bowen suggested that we treat it as what we now call the projection of a thick distribution, that is, as

$$\text{``Src''}=\Pi\left(\frac{4\pi}{3}\delta_{ij_3}n_{j_1}n_{j_2}\delta_*\right)=\frac{4\pi}{9}\delta_{ij_3}\delta_{j_1j_2}\delta\left(\mathbf{x}\right)\ , \tag{4.11}$$

since $\Pi\left(n_{j_1}n_{j_2}\delta_*\right)=(1/3)\,\delta_{j_1j_2}\delta\left(\mathbf{x}\right)$ [16, Example 5.10]. In order to compare with (4.1) and (4.4) we observe that by symmetry the same result would be obtained if j_3 and j_1, or j_3 and j_2, are exchanged, so that if in the term "Src" we do these exchanges, add the results and divide by 3, we would get

$$\text{``SrcSym''}=\frac{4\pi}{27}\left(\delta_{ij_1}\delta_{j_2j_3}+\delta_{ij_2}\delta_{j_1j_3}+\delta_{ij_3}\delta_{j_1j_2}\right)\delta\left(\mathbf{x}\right)\ , \tag{4.12}$$

and thus the symmetric version of the (4.8) is "Normal"+"SrcSym", which of course is different from (4.1) since the coefficient in (4.4) is $4\pi/15$, while that in (4.12) is $4\pi/27$. Therefore, the relation "¿=?" in (4.6) cannot be replaced by $=$.

Hence the product rule for derivatives fails in this case. *The question is why?* Indeed, when computing the right side of (4.6), that is, the left side of (4.8), we found just one irregular product, namely $n_{j_1}n_{j_2}\delta(\mathbf{x})$, but using the average value $(1/3)\,\delta_{j_1j_2}\delta(\mathbf{x})$ seems quite reasonable.

In order to see what went wrong let us compute $\overline{\partial}/\partial x_i\left(n_{j_1}n_{j_2}n_{j_3}/r^2\right)$ by computing the thick derivative $\partial^*/\partial x_i\mathcal{P}f\left(n_{j_1}n_{j_2}n_{j_3}/r^2\right)$, applying the product rule for thick derivatives, and then taking the projection π of this. We have,

$$\begin{aligned}\frac{\partial^*}{\partial x_i}\mathcal{P}f\left(\frac{n_{j_1}n_{j_2}n_{j_3}}{r^2}\right) &= \frac{\partial^*}{\partial x_i}\left[n_{j_1}n_{j_2}\mathcal{P}f\left(\frac{n_{j_3}}{r^2}\right)\right] \\ &= n_{j_1}n_{j_2}\frac{\partial^*}{\partial x_i}\mathcal{P}f\left(\frac{n_{j_3}}{r^2}\right)+\frac{\partial\left(n_{j_1}n_{j_2}\right)}{\partial x_i}\mathcal{P}f\left(\frac{n_{j_3}}{r^2}\right)\,,\end{aligned}$$

and taking (3.5) into account, we obtain

$$n_{j_1}n_{j_2}\left\{\mathcal{P}f\left(\frac{\delta_{ij_3}-3n_in_{j_3}}{r^3}\right)+4\pi n_{j_3}n_i\delta_*\right\} + \frac{\delta_{ij_1}n_{j_2}+\delta_{ij_2}n_{j_1}-2n_in_{j_1}n_{j_2}}{r}\mathcal{P}f\left(\frac{n_{j_3}}{r^2}\right)\,,$$

that is, $\partial^*/\partial x_i\mathcal{P}f\left(n_{j_1}n_{j_2}n_{j_3}/r^2\right)$ equals

$$\mathcal{P}f\left(\frac{\delta_{ij_1}n_{j_2}n_{j_3}+\delta_{ij_2}n_{j_1}n_{j_3}+\delta_{ij_3}n_{j_1}n_{j_2}-5n_in_{j_1}n_{j_2}n_{j_3}}{r^3}\right) + 4\pi n_{j_1}n_{j_2}n_{j_3}n_i\delta_*\,. \tag{4.13}$$

Applying the projection operator Π we obtain that the $\mathcal{P}f$ becomes a p.v., so that the term "Normal" given by (4.9) is obtained, while (3.13) yields that the projection of thick delta is exactly $A\delta(\mathbf{x})$ where $A=\int_{\mathbb{S}}n_in_{j_1}n_{j_2}n_{j_3}\,\mathrm{d}\sigma(\mathbf{w})$, that is, the *correct* term

$$\frac{4\pi}{15}\left(\delta_{ij_1}\delta_{j_2j_3}+\delta_{ij_2}\delta_{j_1j_3}+\delta_{ij_3}\delta_{j_1j_2}\right)\delta(\mathbf{x})\,.$$

The reason we now obtain the correct result is while it is true that $\Pi\left(n_{j_1}n_{j_2}\delta_*\right)=(1/3)\,\delta_{j_1j_2}\delta(\mathbf{x})$ and that $\Pi\left(n_{j_3}n_i\delta_*\right)=(1/3)\,\delta_{ij_3}\delta(\mathbf{x})$, it is *not* true that the projection $\Pi\left(4\pi n_{j_1}n_{j_2}n_{j_3}n_i\delta_*\right)$ can be obtained as $4\pi(1/3)\,\delta_{ij_3}\Pi\left(n_{j_1}n_{j_2}\delta_*\right)$ nor as $4\pi(1/3)\,\delta_{j_1j_2}\Pi\left(n_{j_3}n_i\delta_*\right)$, and actually not even the symmetrization of such results, given by (4.12), works. Put in simple terms, it is not true that the average of a product is the product of the averages!

One can, alternatively, compute $\partial^*/\partial x_i\mathcal{P}f\left(n_{j_1}n_{j_2}n_{j_3}/r^2\right)$ as

$$\frac{\partial}{\partial x_i}\left(\frac{n_{j_3}}{r^2}\right)\mathcal{P}f\left(n_{j_1}n_{j_2}\right)+\left(\frac{n_{j_3}}{r^2}\right)\frac{\partial^*}{\partial x_i}\mathcal{P}f\left(n_{j_1}n_{j_2}\right)\,, \tag{4.14}$$

since

$$\frac{\partial^*}{\partial x_i}\mathcal{P}f\left(n_{j_1}n_{j_2}\right)=\mathcal{P}f\left(\frac{\delta_{ij_1}n_{j_2}+\delta_{ij_2}n_{j_1}-2n_in_{j_1}n_{j_2}}{r}\right)+4\pi n_{j_1}n_{j_2}n_i\delta_*^{[-2]}\,. \tag{4.15}$$

Here the thick delta term in (4.14) is $4\pi\left(n_{j_3}/r^2\right)n_{j_1}n_{j_2}n_i\delta_*^{[-2]}$, which becomes, as it should, $4\pi n_{j_1}n_{j_2}n_{j_3}n_i\delta_*$.

Complications in the use of the product rule for derivatives in one variable were considered in [3] when analysing the formula [14]

$$\frac{d}{dx}\left(H^n\left(x\right)\right)=nH^{n-1}\left(x\right)\delta\left(x\right)\,, \tag{4.16}$$

where H is the Heaviside function; see also [12].

5. Higher order derivatives

We now consider the computation of higher order derivatives in the space $\left(\mathcal{D}_*^{[0]}\left(\mathbb{R}^n\right)\right)'$. If $f\in\mathcal{D}_*'\left(\mathbb{R}^n\right)$ then, of course, the thick derivative $\partial^*f/\partial x_i$ is defined by duality, that is,

$$\left\langle\frac{\partial^*f}{\partial x_i},\phi\right\rangle=-\left\langle f,\frac{\partial\phi}{\partial x_i}\right\rangle\,, \tag{5.1}$$

for $\phi\in\mathcal{D}_*\left(\mathbb{R}^n\right)$. Suppose now that $\mathcal{A}$ is a subspace of $\mathcal{D}_*\left(\mathbb{R}^n\right)$ that has a topology such that the imbedding $i:\mathcal{A}\hookrightarrow\mathcal{D}_*\left(\mathbb{R}^n\right)$ is continuous; then the transpose $i^T:\mathcal{D}_*'\left(\mathbb{R}^n\right)\to\mathcal{A}'$ is just the restriction operator $\Pi_{\mathcal{A}}$. If $\mathcal{A}$ is closed under the differentiation operators. Note that, the space $\mathcal{A}'$ would be a space of (thick) distributions in the sense of Zemanian [18]. Then we can also define the derivative of any $f\in\mathcal{A}'$, say $\partial_{\mathcal{A}}f/\partial x_i$, by employing (5.1) for $\phi\in\mathcal{A}$. Then

$$\Pi_{\mathcal{A}}\left(\frac{\partial^*f}{\partial x_i}\right)=\frac{\partial_{\mathcal{A}}}{\partial x_i}\left(\Pi_{\mathcal{A}}\left(f\right)\right)\,, \tag{5.2}$$

for any thick distribution $f\in\mathcal{D}_*'\left(\mathbb{R}^n\right)$. In the particular case when $\mathcal{A}=\mathcal{D}\left(\mathbb{R}^n\right)$, then $\partial_{\mathcal{A}}f/\partial x_i=\overline{\partial}f/\partial x_i$, the usual distributional derivative, and thus (5.2) becomes [16, Eqn. (5.22)],

$$\Pi\left(\frac{\partial^*f}{\partial x_i}\right)=\frac{\overline{\partial}\Pi\left(f\right)}{\partial x_i}\,. \tag{5.3}$$

What this means is that one can use thick distributional derivatives to compute $\partial_{\mathcal{A}}f/\partial x_i$, as we have already done to compute distributional derivatives.

When $\mathcal{A}$ is not closed under the differentiation operators then $\partial_{\mathcal{A}}f/\partial x_i$ cannot be defined by (5.1) if $f\in\mathcal{A}'$ since in general $\partial\phi/\partial x_i$ does not belong to $\mathcal{A}$ and thus the right side of (5.1) is not defined. However, if $f\in\mathcal{A}'$ has a *canonical* extension $\widetilde{f}\in\mathcal{D}_*'\left(\mathbb{R}^n\right)$ then we could define $\partial_{\mathcal{A}}f/\partial x_i$ as $\Pi_{\mathcal{A}}\left(\partial^*\widetilde{f}/\partial x_i\right)$. This applies, in particular when $\mathcal{A}=\mathcal{D}_*^{[0]}\left(\mathbb{R}^n\right)$: if $f\in\left(\mathcal{D}_*^{[0]}\left(\mathbb{R}^n\right)\right)'$ then $\partial_0^*f/\partial x_i=\partial_{\mathcal{A}}f/\partial x_i$ cannot be defined, in general, but if f has a canonical extension $\widetilde{f}\in\mathcal{D}_*'\left(\mathbb{R}^n\right)$, then $\partial_0^*f/\partial x_i$ is understood as $\Pi_{\mathcal{D}_*^{[0]}\left(\mathbb{R}^n\right)}\left(\partial^*\widetilde{f}/\partial x_i\right)$.

Our aim is to point out that, in general, if $P=RS$ is the product of two differential operators with constant coefficients, then while, with obvious notations, $P^* = R^*S^*$, $P_{\mathcal{A}} = R_{\mathcal{A}}S_{\mathcal{A}}$, if $\mathcal{A}$ is closed under differential operators, and $\overline{P} = \overline{R}\,\overline{S}$, it is *not* true that $P_0^* = R_0^*S_0^*$. Therefore the space $\left(\mathcal{D}_*^{[0]}(\mathbb{R}^n)\right)'$ is not a convenient framework to generalize distributions to thick distributions; the whole $\mathcal{D}'_*(\mathbb{R}^n)$ is needed if we want a theory that includes the possibility of differentiation.

Example 5.1. Let us consider the second order derivatives of the distribution $\mathcal{P}f(1)$. Formula (3.17) yields

$$\frac{\partial^{*2}}{\partial x_i \partial x_j}(\mathcal{P}f(1)) = C(\delta_{ij} - 2n_in_j)\,\delta_*^{[-n+2]}. \tag{5.4}$$

In particular, in $\mathbb{R}^2$, $\partial^{*2}/\partial x_i\partial x_j(\mathcal{P}f(1)) = 2\pi(\delta_{ij} - 2n_in_j)\,\delta_*$. If we consider the function 1 as an element of $\left(\mathcal{D}_*^{[0]}(\mathbb{R}^2)\right)'$ then it has the canonical extension $\mathcal{P}f(1) \in \mathcal{D}'_*\left(\mathbb{R}^2\right)$ and so

$$\frac{\partial_0^*(1)}{\partial x_j} = \Pi_{\mathcal{D}_*^{[0]}(\mathbb{R}^2)}\left(2\pi n_j\delta_*^{[-1]}\right) = 0\,,$$

and consequently,

$$\frac{\partial_0^*}{\partial x_i}\left(\frac{\partial_0^*(1)}{\partial x_j}\right) = \frac{\partial_0^*}{\partial x_i}(0) = 0 \neq 2\pi(\delta_{ij} - 2n_in_j)\,\delta_* = \frac{\partial^{*2}(1)}{\partial x_i\partial x_j}\,. \tag{5.5}$$

Observe that $\Pi(2\pi(\delta_{ij} - 2n_in_j)\,\delta_*) = 0$, but observe also that this means very little.

Example 5.2. It was obtained in [16, Thm. 7.6] that in $\mathcal{D}'_*\left(\mathbb{R}^3\right)$

$$\frac{\partial^{*2}\mathcal{P}f\left(r^{-1}\right)}{\partial x_i\partial x_j} = \left(3x_ix_j - \delta_{ij}r^2\right)\mathcal{P}f\left(r^{-5}\right) + 4\pi(\delta_{ij} - 4n_in_j)\,\delta_*\,. \tag{5.6}$$

Since $\Pi(n_in_j\delta_*) = (1/3)\,\delta_{ij}\delta(\mathbf{x})$ in $\mathbb{R}^3$, this yields the well known formula of Frahm [9]

$$\frac{\overline{\partial}^2}{\partial x_i\partial x_j}\left(\frac{1}{r}\right) = \text{p.v.}\left(\frac{3x_ix_j - r^2\delta_{ij}}{r^5}\right) - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta(\mathbf{x})\,. \tag{5.7}$$

We also immediately obtain that

$$\frac{\partial_0^{*2}\mathcal{P}f\left(r^{-1}\right)}{\partial x_i\partial x_j} = \mathcal{P}f\left(\frac{3x_ix_j - r^2\delta_{ij}}{r^5}\right) + 4\pi(\delta_{ij} - 4n_in_j)\,\delta_*\,, \tag{5.8}$$

a formula that can also be proved by other methods [17]. On the other hand, in [10] one can find the computation of

$$\frac{\partial_0^*}{\partial x_i}\left(\frac{\partial_0^*}{\partial x_j}\left(\frac{1}{r}\right)\right) = \mathcal{P}f\left(\frac{3x_ix_j - r^2\delta_{ij}}{r^5}\right) - 4\pi n_in_j\delta_*\,. \tag{5.9}$$

The fact that $\frac{\partial_0^*}{\partial x_i}\left(\frac{\partial_0^*}{\partial x_j}\right) \neq \frac{\partial_0^{*2}}{\partial x_i \partial x_j}$ is obvious in the Example 5.1, but it is harder to see it in cases like this one. In fact, the fact that the two results are different is overlooked in [10]. Observe that the projection of both $4\pi\left(\delta_{ij} - 4n_i n_j\right)\delta_*$ and of $-4\pi n_i n_j \delta_*$ onto $\mathcal{D}'\left(\mathbb{R}^3\right)$ is given by $-\left(4\pi/3\right)\delta_{ij}\delta\left(\mathbf{x}\right)$, but this does not mean that they are equal; observe also that one needs the finite part in (5.8) and in (5.9) since the principal value, as used in (5.7), exists in $\mathcal{D}'\left(\mathbb{R}^3\right)$ but not in $\left(\mathcal{D}_*^{[0]}\left(\mathbb{R}^3\right)\right)'$.

Acknowledgement

The authors gratefully acknowledge support from NSF, through grant number 0968448.

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Received by the editors September 16, 2013