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Ratko Tošić

AN OPTIMAL SEARCH PROCEDURE

The problem is the following:

(P) There are exactly two defective (unknown) elements in the set $X=\{x_1, x_2, \ldots, x_n\}$, all possibilities with equal probabilities. We want to identify the unknown (defective) elements testing some subsets A of X whether A contains any of them. The individual test informs us that either all elements of the tested set are good or at least one of them is defective (but we do not know which ones or how many). A set containing at least one defective element is said to be defective. Our aim is to minimize the maximal number of tests.

For the optimal strategy the maximal test length is denoted by $l_2(n)$. Then we have the following result:

Theorem. *Let*

$$t_k=F_{\left[\frac{k}{2}\right]}+2^{\left[\frac{k}{2}\right]}+(1+(-1)^{k+1})\cdot 2^{\frac{k-5}{2}} \tag{1}$$

for $k=2, 3, \ldots$, *where* F_j *is the* j*-th member of the Fibonacci sequence*

$$F_1=1,\ F_2=1;\ F_j=F_{j-1}+F_{j-2}\ (j=3, 4, \ldots). \tag{2}$$

Then $l_2(t_k)=k$.

Proof. The sequence (1) can be written in the form:

$$\begin{aligned} t_{2m}&=F_m+2^m \\ t_{2m+1}&=F_m+2^m+2^{m-1} \end{aligned} \qquad (m=1, 2, \ldots) \tag{3}$$

Let $S_n^2(l)$ denote any strategy for identification both defective elements of the set X with maximal test length l. The inequality

$$\binom{t_k}{2}>2^{k-1} \qquad (k=2, 3, \ldots) \tag{4}$$

which can be easily verified, implies the optimality of the strategy $S_{t_k}^2(k)$, if any such exists.

Now, we shall prove that the strategy $S_{t_k}^2(k)$ always exists. The proof will be by mathematical induction, using the next relations which can be easily verified:

$$F_m\leqslant 2^{m-2} \qquad (m=2, 3, \ldots) \tag{5}$$

$$2^m\leqslant t_{2m} \qquad (m=1, 2, \ldots) \tag{6}$$

(7) $$t_{2m-4} \leqslant 2^{m-1} \qquad (m=4, 5, \ldots)$$

(8) $$t_{2m-1} \leqslant 2^m \leqslant t_{2m} \qquad (m=2, 3, \ldots)$$

(9) $$t_{2m+1} = t_{2m} + 2^{m-1} \qquad (m=1, 2, \ldots)$$

(10) $$t_{2m} = t_{2m-1} + t_{2m-4} \qquad (m=3, 4, \ldots)$$

For the first three members of the sequence (1):$t_2=3$, $t_3=4$, $t_4=5$, the statement is true, the optimal strategies are in fact strategies "element by element".

Suppose that the statement is true for all $k \leqslant 2m$ and that the corresponding optimal strategies $S^2_{t_k}(k)$ are constructed ($2 \leqslant k \leqslant 2m$; $m \geqslant 2$).

Then the optimal strategies $S^2_{t\,2m+1}(2m+1)$ and $S^2_{t\,2m+2}(2m+2)$ can be constructed according to the following schemes:

(i) $S^2_{t\,2m+1}(2m+1)$

(1) Test $A=\{x_1, x_2, \ldots, x_{2^{m-1}}\}$ and go to (2).

(2) If A is not defective then both defective elements are among the remaining $t_{2m+1}-2^{m-1}=t_{2m}$ elements and in that case we continue the procedure by applying the strategy $S^2_{t\,2m}(2m)$ which exists by the induction hypothesis.

If A is defective, test $A_1=\{x_{2^{m-1}+1}, \ldots, x_{2^{m-1}+2^m}\}$ and go to (3).

(3) If A_1 is defective, we conclude that each of the sets A and A_1 (with cardinalities 2^{m-1} and 2^m respectively) contains exactly one defective element, so the procedure can be prolonged by successive applying of two independent strategies $S^1_{2^{m-1}}(m-1)$ and $S^1_{2^m}(m)$. We denote by $S^1_n(r)$ a strategy which enables the identification of the unknown element, if there is exactly one, in the set of n elements, the maximal test length of the strategy being r. It is well known that if $S^1_n(r)$ is an optimal strategy then the maximal test length r satisfies the inequality:

$$2^{r-1} < n \leqslant 2^r.$$

If A_1 is not defective, test $A_{10}=\{x_{2^{m-1}+2^m+1}, \ldots, x_{t_{2m+1}}\}$ whose cardinality s $t_{2m+1}-2^{m}-2^{m-1}=F_m$ and go to (4).

(4) If A_{10} is defective, we conclude that each of the sets A and A_{10} contains exactly one defective element and we continue the procedure by successive applying of two independent strategies $S^1_{2^{m-1}}(m-1)$ and $S^1_{F_m}(\leqslant m-2)$. The estimate of the maximal test length of the latter strategy is due to inequality (5).

If A_{10} is not defective, we conclude that both defective elements are in the set A of cardinality 2^{m-1}; now, using (6), we see that for the identification of defective elements we can apply the strategy $S^2_{2^{m-1}}(\leqslant 2m-2)$

So, the strategy $S^2_{t\,2m+1}(2m+1)$ is constructed.

(ii) $S^2_{t\,2m+2}(2m+2)=S^2_{t\,2(m+1)}(2(m+1))$

(1′) Test $B=\{x_1, x_2, \ldots, x_{t_{2m-2}}\}$ and go to (2′).

(2′) If B is not defective then both defective elements are among the remaining $t_{2m+2}-t_{2m-2}=t_{2m+1}$ elements (we use (10)), and we continue the procedure by applying the strategy $S^2_{t_{2m+1}}(2m+1)$ just constructed in (i).

If B is defective, test $B_1=\{x_{t_{2m-2}+1}, \ldots, x_{t_{2m-2}+2m}\}$ and go to (3′).

(3′) If B_1 is defective, it means that each of the sets B and B_1 (cardinalities t_{2m-2} and 2^m respectively) contains exactly one defective element and we continue the procedure by successive applying of two independent strategies $S^1_{t_{2m-2}}(\leqslant m)$ and $S^1_{2^m}(m)$. The estimate of the maximal test length of the former strategy follows from (7).

If B_1 is not defective, test

$B_{10}=\{x_{t_{2m-2}+2^m+1}, \ldots, x_{t_{2m-2}+2^m+2^{m-1}}\}$ of the cardinality 2^{m-1} and go to (4′).

(4′) If B_{10} is defective, it means that each of the sets B and B_{10} contains exactly one defective element and we continue the procedure by successive applying of two independent strategies $S^1_{t_{2m-2}}(\leqslant m)$ and $S^1_{2^{m-1}}(m-1)$.

If B_{10} is not defective, test the set

$B_{100}=\{x_{t_{2m-2}+2^m+2^{m-1}+1}, \ldots, x_{t_{2m+2}}\}$ i.e. the set of F_m remaining elements not tested yet and go to (5′).

(5′) If B_{100} is defective, we conclude that each of the sets B and B_{100} (cardinalities t_{2m-2} and F_m respectively) contains exactly one defective element and we continue the procedure by successive applying of two independent strategies $S^1_{t_{2m-2}}(\leqslant m)$ and $S^1_{F_m}(\leqslant m-2)$. In estimating the maximal test lengths of these strategies we use the relations (5) and (7).

If B_{100} is not defective, it means that both defective elements are in B whose cardinality is t_{2m-2}. Now, we can use the strategy $S^2_{t_{2m-2}}(2m-2)$ which exists by the induction hypothesis.

So, the strategy $S^2_{t_{2m+2}}(2m+2)$ is constructed and the theorem is proved.

As an illustration we give in the following table values of t_k for $k=2, \ldots,$ 28:

Table

k	t_k	k	t_k	k	t_k
2	3	11	53	20	1079
3	4	12	72	21	1591
4	5	13	104	22	2137
5	7	14	141	23	3161
6	10	15	205	24	4240
7	14	16	277	25	6288
8	19	17	405	26	8425
9	27	18	546	27	12521
10	37	19	802	28	16761

It is easy to see that for $t_{k-1}<n<t_k$ the schemes (i) and (ii) give us either an optimal or an almost optimal strategy, say $S_n^2(k)$, i.e. if $S_n^2(l)$ is an optimal strategy then: $k-l\leqslant 1$.

REFERENCES

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Ratko Tošić

JEDNA OPTIMALNA ISTRAŽNA PROCEDURA

Rezime

U radu je posmatran sledeći problem:

(P) U skupu $X=\{x_1, x_2, \ldots, x_n\}$ su tačno dva neispravna (nepoznata) elementa. Naš cilj je da identifikujemo oba neispravna elementa testirajući ispravnost nekih podskupova A skupa X, pri čemu za skup A kažemo da je neispravan ako sadrži barem jedan neispravan element. Svaki pojedinačni test informiše nas o ispravnosti testiranog skupa, ali u slučaju neispravnosti ne i o broju neispravnih elemenata u njemu. Kriterijum optimalnosti istražnog postupka (strategije) je minimum maksimalnog broja testova za identifikaciju dvostruke neispravnosti.

Ako maksimalan broj testova optimalne strategije obeležimo sa $l_2(n)$, tada važi:

Teorema. Neka je

(1) $$t_k=F_{\left[\frac{k}{2}\right]}+2^{\left[\frac{k}{2}\right]}+(1+(-1)^{k+1})\cdot 2^{\frac{k-5}{2}}$$

za $k=2, 3, \ldots$, gde je F_j j-ti član Fibonačijevog niza

(2) $$F_1=1.\ F_2=1;\ F_j=F_{j-1}+F_{j-2}\ (j=3, 4, \ldots).$$

Tada je $l_2(t_k)=k$.

Dokaz je konstruktivan i omogućava efektivno nalaženje optimalne strategije.

Za one prirodne brojeve n, koji nisu članovi niza (1), dobija se optimalna ili skoro optimalna strategija, tj. takva koja zahteva samo jedan test više od optimalne strategije.