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Dragan M. Acketa

ON THE ENUMERATION OF MATROIDS OF RANK 2

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Abstract. A formula for the enumeration of the matroids of rank 2 is given (Theorem 2.). This formula is based on the bijection between the class of the matroids of rank 2 and a special class of graphs.

A *matroid* M on a finite set S is a pair $(S, \mathcal{B})$, where $\mathcal{B}$ is a non-empty collection of subsets of S, which satisfies the following condition (*the axiom of bases*) [1]:

$$(B_1, B_2 \in \mathcal{B} \wedge x \in B_1 \setminus B_2) => (\exists y) (y \in B_2 \setminus B_1 \wedge (B_1 \setminus x) \cup y \in \mathcal{B})$$

The set S is the *carrier* and the subsets from $\mathcal{B}$ are the *bases* of the matroid M. We can restate the given axiom in the following way:

Each element from any base can be replaced by some element from any other base so that a base is again obtained.

An *n-set* (base) is a set (base), which has n elements.

It can be verified that all bases of a matroid have the same cardinality, which is called the *rank of matroid*. The ranks of the matroids, whose carrier is an n-set, are between 0 and n.

Two matroids are *isomorphic* if there is a bijection between their carriers which „preserves" the bases (i.e., which maps the bases of one matroid onto the bases of the other).

Let $m_k(n)$ denote the number of non-isomorphic matroids of rank k on an n-set.

We see that $m_0(n)=1$ (if the rank is zero, then the empty set is the only base) and $m_1(n)=n$ (all collections of 1-sets satisfy the axiom of bases).

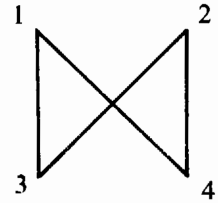
This paper gives a formula, which enables a relatively quick evaluation of $m_2(n)$.

By a „graph" we shall always mean a non-oriented graph without loops and parallel edges.

When adjoining a graph to the collection of bases of a matroid, it is usual (as, for example, in [3]) to map the bases of the matroid onto the vertices of a graph, which are adjacent if and only if the corresponding bases differ in just one element.

In the case of the matroids of rank 2, however, we choose to map the elements of the carrier and the 2-bases of a matroid onto the vertices and the edges of a graph respectively.

For example, if $S=\{1, 2, 3, 4\}$ and $\mathcal{B}=\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$, then we adjoin the following graph to the matroid $(S, \mathcal{B})$:



We shall define several concepts which we shall use later.

A graph is *matroidic* if the set of its vertices and the collection of adjacent pairs of vertices are, respectively, the carrier and the collection of 2-bases of a matroid.

Note. The term „matroidic" is used in order to avoid confusion with the matroidal graphs, defined in [4].

A *3-point* is a set of three vertices of a graph. We differentiate 3-points on the basis of the number of adjacent pairs of vertices. So we have 3-points with zero, one, two and three vertices.

An *isolated vertex* of a graph is a vertex, which is not an endpoint of an edge.

A *complete graph* is a graph in which an edge joins each pair of vertices. The complete graph on n vertices is denoted by K_n.

An *empty graph* is a graph, which has no vertices.

Two graphs are *disjoint* if they have no common vertex.

A *path* in a graph is a finite sequence of edges of the form $\{[v_0, v_1], [v_1, v_2], \ldots, [v_{m-1}v_m]\}$, where from $i \neq j$ follows $v_i \not\equiv v_j$.

A *connected component* of a graph is a part of the graph in which each pair of vertices belongs to a common path.

A 2-set of vertices of a graph *determines an edge* if the vertices of that set are adjacent.

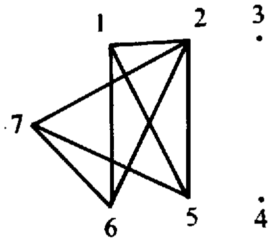
The *deletion of the edge* $[xy]$ from a graph G is the operation which transforms the graph G into a graph G_1, which differs from G solely by the fact that the vertices x and y in G_1 are not adjacent.

We similarly define the *deletion of an isolated vertex.*

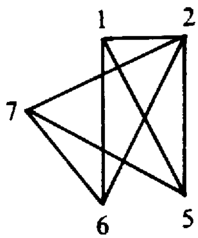
An $\overline{M}$-*3-point* in a graph is a 3-point with exactly one edge, the opposite vertex of which is non-isolated.

We adjoin two subgraphs to a graph G. The first one is G', which remains after the deletion of all isolated vertices from G, and the second is G'', which has no isolated vertices and the edges of which exactly complement the graph G' up to a complete graph. We give an example for the graphs G, G' and G'':

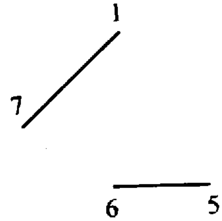
.G



G'



G''



We shall prove the following theorem, which is the main result of this paper:

Theorem 1. A graph G is matroidic if and only if the graph G' is complete or G' can be obtained from a complete graph by deleting all edges of some disjoint complete proper subgraphs.

Note. We use the word "proper" in order to avoid the case when the set of edges of the graph G' and the family of bases of the corresponding matroid are empty.

The proof of Theorem 1. is based on three lemmas.

Lemma 1. A graph G is matroidic if and only if the graph G' contains at least one edge and G' does not contain an $\overline{M}$-3-point.

Note. The designation "$\overline{M}$-3-point" is motivated by the fact that the appearance of such a 3-point in a graph "cancels Matroidicity".

Proof. If a graph G is matroidic, then the graph G' contains at least one edge, because the matroid, which corresponds to the graph G, contains at least one 2-base.

Suppose that 3-point xyz of G' is such that $[xy]$ is the only edge in xyz and the opposite vertex z is non-isolated (i.e., z is adjacent to another vertex t). The sets $\{x, y\}$ and $\{z, t\}$ are bases of the above mentioned matroid. However, if the element t is replaced in $\{z, t\}$, either by x, or by y, then an adjacent pair of vertices is not obtained. This means that the axiom of bases is not satisfied. A contradiction.

Conversely, if a graph G is not matroidic, then either G is edgeless or the collection of non-ordered pairs of adjacent vertices of G does not satisfy the axiom of bases.

In the second case there exist two sets $\{x, y\}$ and $\{z, t\}$ in that collection such that, for example, neither $\{x, z\}$, nor $\{y, z\}$ are in the same collection. Then xyz is an $\overline{M}$-3-point of the graph G. As all vertices of an $\overline{M}$-3-point are non-isolated, the $\overline{M}$-3-point xyz occurs just in the graph G'.

Lemma 2. If a graph G' is complete or can be obtained from a complete graph by deleting all the edges of some disjoint complete proper subgraphs, then G' contains at least one edge and G' does not contain a 3-point with exactly one edge.

Note. As there are no isolated vertices in G', the 3-point is just an $\overline{M}$-3--point.

Proof. The proof of the first statement is straightforward, so we shall prove just the second one.

If G' is complete, then each 3-point of G' has three edges.

If, after deleting all the edges of a subgraph K_n, two edges are left in a 3--point, then these two edges also have to be left in the graph G' (otherwise the complete graphs, the edges of which are deleted, would not be disjoint).

If, when deleting all edges of a subgraph K_n, two edges are deleted from a 3-point xyz (for example, $[xy]$ and $[yz]$), then the third edge $[xz]$ is also deleted (because the complete graph K_n contains the edge, which joins the vertices x and z).

We, conclude that 3-points of the graph G' may have three, two or zero edges, but by no means just one, which completes the proof.

It seems convenient to use the graph G'' when proving the converse of Lemma 2. Therefore we state the following facts:

G' is complete	$\leftrightarrow$	*G'' is empty*
G' can be obtained from a complete graph by deleting all the edges of some disjoint complete proper subgraphs.	$\leftrightarrow$	G'' is an union of disjoint complete graphs, which is different from the complete graph over all the non--isolated vertices of G.
G' is a graph which contains at least one edge and which does not contain a 3-point with exactly one edge.	$\leftrightarrow$	G'' is a graph, which is different from the complete graph over all non-isolated vertices of G, and which does not contain a 3-point with exactly two edges.

The converse of Lemma 2. can be restated as follows:

If G'' is a graph, which is different from the complete graph over all non--isolated vertices of G, and which does not contain a 3-point with exactly two edges, then either G'' is empty or G'' is a union of disjoint complete graphs, which is different from the complete graph over all non-isolated vertices of G.

If G'' is empty, then there is nothing to prove.

If G'' is non-empty, then there is at least one connected component of G''.

As connected components of the graph G'' are disjoint graphs, which are different from the complete graph over all non-isolated vertices of G, so it suffices to prove the following lemma:

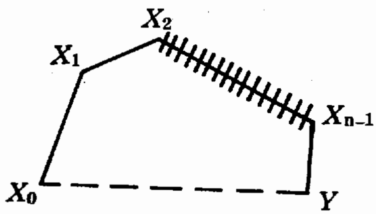
Lemma 3. If there is not a 3-point with exactly two edges in a graph G'', then each connected component of G'' is a complete graph.

Proof. Suppose that there exists a connected component in G'', such that two vertices of that component, say x_0 nd y, are not adjacent. By the definition of connected component, then there exists a path $\{[x_0x_1], [x_1x_2], \ldots, [x_{n-1}y]\}$, which joins x_0 and y.

Let us examine the sets $\{x_i, y\}$ $(0 \leqslant i \leqslant n-1)$

The set $\{x_{n-1}, y\}$ determines an edge, but $\{x_0, y\}$ does not. We immediately obtain that there exists an i $(1 \leqslant i \leqslant n-1)$, such that $\{x_i, y\}$ determines an edge, but $\{x_{i-1}, y\}$ does not. The 3-point $x_{i-1}x_i y$ has exactly two edges. A contradiction.

This completes the proof of Theorem 1.



By the use of the Theorem 1, we shall give a formula which considerably simplifies the evaluation of $m_2(n)$.

Let $h(k, t)$ denote the number of *partitions* of a natural number k (=the number of separations of the number k into a sum of natural addends, without regard to order), such that all addends exceed the non-negative integer t.

The number of unrestricted partitions of k is $h(k, 0)$.

We denote $h(k, 1)$ by $g(k)$. Then we have:

Theorem 2.

$$m_2(n) = \sum_{p=2}^{n} \sum_{k=2}^{p} g(k)$$

Proof. The graph obtained from K_5 by deleting all the edges of disjoint proper subgraphs K_3 and K_2, for example, will be denoted by "K_5-(K_3+K_2)".

As non-isomorphic matroids of rank 2 correspond exactly to the non-isomorphic matroidic graphs, the number of non-isomorphic matroidic graphs (characterized by Theorem 1.) with n vertices is just $m_2(n)$.

Let p denote the number of non-isolated vertices in a matroidic graph with n vertices. As a matroid of rank 2 has at least one 2-base, we have that $p \geqslant 2$ (the graphs of the form $K_p - K_p$ are "forbidden" in Theorem 1. for the same reason).

It is easy to check that the number of non-isolated vertices cannot be reduced by deleting the edges of disjoint proper complete subgraphs from a complete graph. There are two consequences of this fact:

1. A matroidic graph with p non-isolated vertices is either K_p or is obtained by deleting edges from K_p.

2. We cannot obtain isomorphic matroidic graphs by deleting edges from different complete graphs (if the graphs of the form $K_p - K_p$ were matroidic, then this would not be true).

We establish two more important facts:

3. We do not delete edges from a graph K_1, because K_1 has no edges.

4. The sum of numbers of vertices of the complete proper subgraphs, the edges of which are deleted from K_p, cannot exceed p (because these subgraphs are disjoint).

From these facts we conclude that all matroidic graphs with n vertices appear in one of the following two forms:

a) $$K_p \quad (2 \leqslant p \leqslant n)$$

b) $$K_p - \sum_{j=1}^{s} K_{i_j} (2 \leqslant p \leqslant n,\ 2 \leqslant i_j < p,\ \sum_{j=1}^{s} i_j = k \leqslant p)$$

We adjoin the following partitions to these classes of graphs for fixed p:

a) $$p = p$$

b) $$k = \sum_{j=1}^{s} i_j (2 \leqslant k \leqslant p,\ 2 \leqslant i_j < p)$$

The order of addends in the sum is of no importance, because the order of deleting edges of complete subgraphs is of no importance, too.

If we also fix k, then case b) gives $g(k) - 1$ partitions of the natural number k. The only excluded partition is $p = p$, for the condition $i_j < p$. This partition is, however, provided by case a).

Hence the number of matroidic graphs with p non-isolated vertices and with a fixed number of isolated vertices amounts to:

$$\sum_{k=2}^{p} g(k)$$

For p between 2 and n, we get the assertion of the theorem.

Theorem 2. requires a method for the evaluation of $g(k)$. It seems necessary to evaluate also $h(k, t)$, as the following theorem suggests:

Theorem 3.

$$g(k) = 1 + \sum_{m=2}^{k-1} h(k-m, m-1)$$

Proof. The number 1 corresponds to the partition $k = k$. We divide the remaining partitions of the number k (with all addends exceeding 1) into classes according to the minimal addend.

The number of the partitions of the form

$$k = m + x_1 + \ldots + x_s$$

with the minimal number m, equals the number of partitions of the number $k-m$ subject to the condition that all addends exceed $m-1$, that is, $h(k-m, m-1)$.

As m is between 2 and $k-1$, the theorem is proved.

The function $h(k, t)$ satisfies the following conditions:

$$h(2, 1)=h(3, 1)=1$$

$$h(k, t)=0 \qquad \text{for } t \geqslant k$$

We shall prove the following recursive relation for the same function:

Theorem 4.

$$h(k, t)=h(k, t-1)-h(k-t, t-1)$$

Proof. Let $A_{k,t}$ denote the set of the partitions, which are counted by $h(k, t)$. Then it holds:

$$A_{k,t} \subseteq A_{k,t-1}$$

Let us observe the set $B=A_{k,t-1}-A_{k.t}$.

Each partition (of the number k) from B contains an addend t, i.e., is of the form

$$k=t+x_1+\ldots+x_s$$

There is a one-to-one correspodence between the partitions of this form and the partitions of the form

$$k-t=x_1+\ldots+x_s$$

from $A_{k-t,t-1}$.

Hence we have the equality of the cardinalities:

$$|B|=|A_{k-t,\ t-1}|$$

that is: $|A_{k,\ t-1}|-|A_{k,t}|=|A_{k-t,\ t-1}|$

or: $h(k, t-1)-h(k, t)=h(k-t, t-1)$

which proves the theorem.

We evaluate $m_2(n)$ by the use of Theorem 2. and just proved auxilliary theorems. For example, if n is between 2 and 25, then the corresponding values of $m_2(n)$ are in order: 1, 3, 7, 13, 23, 37, 58, 87, 128, 183, 259, 359, 493, 668, 898, 1194, 1578, 2067, 2693, 3484, 4485, 5739, 7313, 9270.

Final Remarks

1. The idea for the characterization of the matroidic graphs by the use of complete graphs appeared after the examination of the tabel with 208 non-isomorphic graphs, having not more than 6 vertices, in [2]. $\overline{M}$-3-points were primarily sought in these graphs. The obtained matroidic graphs were analysed afterwards.

2. The supplements of the bases of a matroid (according to the carrier) are the bases of another, dual, matroid. This gives the equality

$$m_{n-k}(n)=m_k(n)$$

By the use of this equality and the values of $m_0(n)$, $m_1(n)$ and $m_2(n)$, we find the number of all non-isomorphic matroids on an n-set — $m(n)$, up to $n=5$ inclusive. For n between 0 and 5, the values of $m(n)$ are in order

1, 2, 4, 8, 17, 38

3. The attempts, made in order to generalize the obtained results for the case of matroids of higher ranks, have been unsuccessful so far. It seems that the enumeration of matroids of higher ranks should be "attacked" starting from other axioms for matroids.

4. We could enumerate non-isomorphic matroids directly, using solely the axiom of bases. Such an investigation is, however, very difficult in the case when the rank exceeds 1.

In order to find $m_k(n)$, we are to check whether the axiom of bases is satisfied at $2^{\binom{n}{k}}$ collections of k-subsets of an n-set.

The confirmation of the axiom of bases for each single collection is rather complicated. When this is accomplished, then an even more difficult problem arises: how to choose just one representative from each class of mutually isomorphic collections that satisfy the axiom of bases.

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Dragan M. Acketa

O PREBRAJANJU MATROIDA RANGA 2

Rezime

U radu je dokazana sledeća teorema:

Broj neizomorfnih matroida ranga 2, na skupu od n elemenata — $m_2(n)$, nalazi se po formuli:

$$m_2(n) = \sum_{p=2}^{n} \sum_{k=2}^{p} g(k)$$

gde je sa $g(k)$ označen broj mogućnosti za rastavljanje prirodnog broja k u zbir prirodnih brojeva ne manjih od 2, pri čemu smatramo istim one mogućnosti koje se razlikuju isključivo u poretku sabiraka.