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A NOTE ON THE MONOTONICITY AND REGULARITY OF GENERALIZED PSEUDO-BOOLEAN FUNCTIONS

D.I. Wilde and I. M. Sanchez-Antoan [3] have studied monotonically increasing pseudo-Boolean functions, and have given a very efficient method for their minimization.

P. L. Hammer [2] has proved that if a pseudo-Boolean function f has a unique minimizing point C then f increases monotonically if and only if it is regular.

We shall prove in our paper that a generalized pseudo-Boolean function f has a unique minimizing point if and only if it is regular with a restriction.

Let $L=\{a_0, a_1, \ldots, a_{p-1}\}$ and $L^n=\{x/x=(x_1, \ldots, x_n), x_i \in L\}$ be two sets. A real function $f: L^n \to R$ (R is a set of real numbers) is called a generalized pseudo-Boolean function.

A generalized pseudo-Boolean function f has a unique minimizing point C if for every

$$X \in L^n \setminus \{C\},\ f(C)<f(X).$$

It is proved [4] that every generalized pseudo-Boolean function can be written in the form

$$f(x)=\sum_{B \in L^n} f(B)\, x^B \tag{1}$$

where

$$B=(b_1, \ldots, b_n)$$

$$X^B=x_1^{b_1} \cdot x_2^{b_2} \cdot \ldots x_n^{b_n}$$

$$x_i^{b_i}=\begin{cases}1, & x_i=b_i\\ 0, & x_i \neq b_i\end{cases} \tag{2}$$

$$\sum_{b_i \in L} x_i^{b_i}=1,\ i=1, \ldots, n. \tag{2'}$$

Let us fix a point $C=(c_1, \ldots, c_n) \in L^n$ then immediately from (2′) follows

$$x_i^{c_i}=1-\sum_{b_i \in L \setminus \{c_i\}} x_i^{b_i},\ i=1, \ldots, n. \tag{3}$$

If we replace (3) in (1) the generalized pseudo-Boolean function (1) is transformed in the form

$$f(x)=d+\sum_{j=i}^{i_n}\sum_{b\in L\setminus\{c_{[j]}\}} d_{jb}x_j^b,\quad (f(c)=d) \tag{4}$$

where: $i_i, i_2, \ldots, i_n$ is a permutation of the set $1, 2, \ldots, n$; $[j]=\text{index}\, j, j\in\{i_1, \ldots i_n\}$; and

$$d_{jb}(x_{i_{[j]+1}}, \ldots, x_{i_n}),\ j=i_1, \ldots, i_n,\ b\in L\setminus\{c_{[j]}\} \tag{5}$$

$$d_{i_1c_1}(x_1, \ldots x_n)=\ldots=d_{i_nc_n}(x_1, \ldots, x_n)=0 \tag{5'}$$

are derivatives.

A generalized pseudo-Boolean function is regular if all the derivatives $d_1a_0, \ldots$ have constant signs (i.e. each of them is either nonnegative or nonpositive) for all points in L^n.

A generalized pseudo-Boolean function increase monotonically afte; the coordinates. If there exists a unique point $C=(c_1, \ldots c_n)\in L^n$, such that for every point

$$(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_n)\in L^{n-1}$$

$$f(x_1, \ldots, x_{i-1}, \alpha, x_{i+1}, \ldots, x_n)>f(x_1, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_n) \tag{6}$$

$$i=1, \ldots, n;\ \alpha\in L\setminus\{c_i\}.$$

Lemma 1. A monotonically increasing generalized pseudo-Boolean function f upon the coordinates has a unique minimizing point.

Proof 1. Let f be a monotonically increasing function after the coordinates. Condition (6) is satisfied. There exists a point $C=(c_1, \ldots, c_n)\in L^n$. If we replace it in (6)

$$f(\alpha_1 \ldots \alpha_n)>f(C),\ i=1, \ldots, n;\ \alpha_i\in L\setminus\{c_i\}$$

I.e. for every $x\in L^n\setminus\{C\}$, $f(x)>f(C)$.

Lemma 2. A monotonically increasing generalized pseudo-Boolean function f upon the coordinates is regular.

Proof. Let f be a monotonically increasing funcion after the coordinates.

Let $$x'=(\alpha_1, x_2, \ldots x_n)$$

$$C'=(c_1, x_2, \ldots x_n)\ \text{replace in (4).}$$

From (4) we see that

$$f(x')-f(C')=\sum_{b\in L\setminus\{c_1\}} dj_b(x_2, \ldots, x_n)\,\alpha^b$$

as $\alpha^b=1$ for $\alpha=b$ and $\alpha^b=0$ in other cases and $f(x')-f(c')>0$ from (6), it follows that for every $(x_2, \ldots, x_n)\in L^{b-1}$.

Let $x=(c_1, \ldots, c_n)\in L^n$ from (4) we get condition (5').

Lemma 3. If the derivates from (5) are positive and from (5′) are 0, then a regular generalized pseudo-Boolean function f has a unique minimizing point C.

Proof. Let f be a regular function written if the form (4). If the derivates from (5) are positive, and from (5′) are 0, and relation (2) is satisfied, then for every $x \in L^n \{C\}$

$$f(X) - f(C) > 0 \text{ and } f_{min}(C) = d$$

C is a unique minimizing point. Let us suppose that it is not (i.e. that there exists a minimizing point C', $C' \neq C$). From (4) it follows

$$f'_{min}(C') = f_{min}(C) + d_{i_1c'_1}(C') + \ldots + d_{i_nc'_n}(C')$$

(the derivates are positive) it is a contradiction.

Lemma 4. If a generalized pseudo-Boolean function f has a unique minimizing point, then it is regular and the derivates from (5) are positive and the derivates from (5′) are 0.

Proof. Let f be a function with a unique minimizing point $C = (c_1, \ldots, c_n)$, i.e. for any $x \in L^n \setminus \{C\}$ $f(x) - f(C) > 0$. If we put function f in the form (4), then we see that $f_{min}(C) = d$ and condition (5′) is satisfied.

Let us suppose that some of the derivates from (5) are 0 in a point $A = (a_1, \ldots, a_n)$

$$d_{jb}(a_s, a_{s+1}, \ldots, a_n) = 0,$$

$$j = 1, 2, \ldots, n, \; b \in \{a, \ldots, a_n\}.$$

If we put $C' = (c_1, c_2, \ldots, c_{s-1}, a_s, \ldots, a_n) \neq C$ in (4) we get

$$f'_{min}(C') = d.$$

Contradiction, because $f_{min}(C) = d$.

Now let us suppose that there exist negative derivates in X'. If we put X' in (4), then

$$f(X') = f(C) = \sum d_{ij}(X'), \; d_{ij}(X') < 0.$$

It follows that $f(X') < f_{min}(C)$ (Contradiction).

From lemma 3 and lemma 4 we have

Theorem. A generalized pseudo-Boolean function f has a unique minimizing point if and only if it is regular with positive derivates from (5) and the derivates from (5′) are 0.

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O REGULARNOSTI I MONOTONIJI GENERALISANIH PSEUDO-BULOVIH FUNKCIJA

Rezime

Dokazano je da generalisana pseudo-Bulova funkcija $f:L^n \to R$ (L-konačan skup, R-skup realnih brojeva) ima jedinstvenu tačku minimazacije ako i samo ako je regularna sa restrikcijom.