Zbornik radova Prirodno-matematičkog fakulteta — Univerzitet u Novom Sadu, knjiga 8 (1978)
Review of Research Faculty of Science — University of Novi Sad, Volume 8 (1978)

O. Hadžić, Đ. Paunić

AN EXISTENCE THEOREM FOR THE SYSTEM $x=H(x, y)$, $y=K(x, y)$ IN PROBABILISTIC LOCALLY CONVEX SPACES

The notion of probabilistic locally convex spaces was introduced by V. Istratescu in [10], and in [5] and [6] some fixed point theorems in such spaces are proved. In this paper we shall prove an existence theorem for the system $x=H(x, y)$, $y=K(x, y)$ in probabilistic locally convex spaces, using the theorem from [7].

First, we shall give some definitions and notations which we shall use later.

Let S be a vector space over the real or complex field K and for every $i \in J$, $\mathcal{F}^i: S \to \Delta^+$ where Δ^+ is the family of distribution functions such that $F(0)=0$ for all $F \in \Delta^+$. The function $\mathcal{F}^i(p)$ will be denoted by F_p^i. The triplet $(S, \mathcal{F}^i, t)$ is a *probabilistic locally convex space* iff the following conditions are satisfied for every $i \in J$:

1. $F_0^i=H$, where $H(u)=\begin{cases} 0 & u \leqslant 0 \\ 1 & u>0 \end{cases}$

2. $F_{\lambda x}^i(u)=F_x^i\left(\dfrac{u}{|\lambda|}\right)$ for every $\lambda \in K (\lambda \neq 0)$, every $x \in S$ and every $u>0$.

3. $F_{x+y}^i(u+v) \geqslant t(F_x^i(u), F_y^i(v))$, for every $x, y \in S$ and every $u, v>0$ where the mapping $t: [0, 1]^2 \to [0, 1]$ is a T-norm [12].

If T-norm t is continuous then $(S, \mathcal{F}^i, t)$ is a topological vector space when the fundamental system of the neighbourhood of zero is defined by:

$$\mathcal{U}=\{U^i(\varepsilon, \delta)\}_{(i, \varepsilon, \delta) \in J \times R^+ \times (0,1)}$$

where $U^i(\varepsilon, \delta)=\{x \mid x \in S, F_x^i(\varepsilon)>1-\delta\}$.

This topology is (ε, δ)-topology of probabilistic locally convex space $(S, \mathcal{F}^i, t)$.

Let E be a locally convex topological vector space and $\{p_i\}_{i \in J}$ be a family of seminorms defining the topology in E and $T: X \to E (X \subseteq E)$ such that for every $i \in J$:

(1) $$p_j(Tx-Ty) \leqslant q(j) p_{f(j)}(x-y) \text{ for every } x, y \in X$$

where $q: J \to R^+$ and $f: J \to J$. In [1], [2], [3] and [4] some fixed point theorems for mapping T are proved and in [3] some applications in the theory of differential equations in the field of Mikusinski's operators are given. In [3] the differential equation:

$$x'(\lambda)=s^{\beta}\omega(\lambda)\,x(\lambda) \quad x(0)=I$$

in the field of Mikusiński's operators is solved, where s is the differential operator and I is the identity operator. The locally convex space E was in [3] a subspace of the field of Mikusiński's operators (introduced by B. Stanković), the index set $J=[N \cup \{0\}]^2$ and the mapping T was:

$$Tx=I+\int_0^{\lambda} s^{\beta}\omega(u)\,x(u)\,du.$$

In [3] it is shown that the mapping T satisfies the following inequality:

$$\| Tx-Ty \|_{(k,m)} \leq q(k,m) \, \| x-y \|_{(k+1,m)}$$

and so the mapping $f: J \to J$ is, in this case, defined in the following way: $f(k, m)=(k+1, m)$ for every $(k, m) \in J$.

In the following Lemma we shall give some sufficient conditions under which there exists one and only one fixed point of the mapping $T: M \to S$ $(M \subseteq S)$ where $(S, \mathcal{F}^i, t)$ is a probabilistic locally convex space and the mapping T satisfies the inequality:

$$F^i_{Tx-Tx}(q(i)\,\varepsilon) \geq F^{f(i)}_{x-y}(\varepsilon) \tag{2}$$

which is probabilistic analogy of the inequality (1).

Lemma. Let $(S, \mathcal{F}^i, t)$ be a Hausdorff sequentially complete probabilistic locally convex space where t is continuous, M be a closed subset of S for which $\sup_{r>0} \inf_{x,y \in M} F^i_{x-y}(r)=1$, for every $i \in J$, $T: M \to M$ such that the following conditions are satisfied:

a) *For every $j \in J$ there exist $q(j)>0$ and $f(j) \in J$ such that:*

$$F^j_{Tx-Ty}(q(j)\,\varepsilon) \geq F^{f(j)}_{x-y}(\varepsilon), \text{ for every } \varepsilon>0$$

and every $x, y \in M$, where for every $j \in J$:

$$\lim_{n\to\infty} \prod_{m=0}^{n} q(f^m(j))=0$$

(b) *For every $j \in J$ there exists $g(j) \in J$ such that:*

$$F^{f^n(j)}_x(\varepsilon) \geq F^{g(j)}_x(\varepsilon), \text{ for every } x \in S, \text{ every } \varepsilon>0$$

and every $n=0, 1, 2, \ldots$ where $f^n(j)=f(f^{n-1}(j))$ for every $n=1, 2, \ldots$ and $j \in J$, $f^0(j)=j$ for every $j \in J$.

Then there exists one and only one fixed point x^* *of the mapping* T *and* $x^* = \lim_{n\to\infty} x_n$, $x_n = Tx_{n-1}$ *for every* $n \in N$ *and* x_0 *is an arbitrary element of* M.

Proof: For every $j \in J$, every $\varepsilon > 0$ and every $n, p \in N$ we have from (a) that:

$$F^j_{x_{n+p}-x_n}(\varepsilon) \geqslant F^{f(j)}_{x_{n+p-1}-x_{n-1}}\left(\frac{\varepsilon}{q(j)}\right) \geqslant \ldots \geqslant F^{f^n(j)}_{x_p-x_0}\left(\frac{\varepsilon}{q_{n-1}(j)}\right)$$

where $q_n(j) = \prod_{p=0}^{n} q(f^p(j))$ for every $n = 0, 1, \ldots$ Using the condition (b) it follows that:

$$F^j_{x_{n+p}-x_n}(\varepsilon) \geqslant F^{g(j)}_{x_p-x_0}\left(\frac{\varepsilon}{q_{n-1}(j)}\right).$$

Since for every $j \in J$: $\sup_{r>0} \inf_{x,y\in M} F^j_{x-y}(r) = 1$ and $\lim_{n\to\infty} q_{n-1}(j) = 0$ we conclude that for every $j \in J$, every $\varepsilon > 0$ and every $\delta \in (0, 1)$ there exists $N(j, \varepsilon, \delta) \in N$ such that:

$$F^j_{x_{n+p}-x_n}(\varepsilon) > 1 - \delta$$

for every $n \geqslant N(j, \varepsilon, \delta)$ and every $p = 1, 2, \ldots$ Let $x^* = \lim_{n\to\infty} x_n$ where $x_n = Tx_{n-1}$ for every $n \in N$ and x_0 is an arbitrary element from M. Then we have:

$$F^j_{x_{n+1}-Tx^*}(\varepsilon) = F^j_{Tx_n-Tx^*}(\varepsilon) \geqslant F^{f(j)}_{x_n-x^*}\left(\frac{\varepsilon}{q(j)}\right)$$

and so $x^* = \lim_{n\to\infty} x_n = Tx^*$. Suppose now that $y = Ty$ and $x^* \neq y$. Then we have:

$$F^j_{y-x^*}(\varepsilon) \geqslant F^{g(j)}_{y-x^*}\left(\frac{\varepsilon}{q_{n-1}(j)}\right)$$

and when $n \to \infty$ it follows that $\dfrac{\varepsilon}{q_{n-1}(j)} \to \infty$ which implies that $F^j_{y-x^*}(\varepsilon) = 1$ for every $\varepsilon > 0$, j $\in J$. Since S is Hausdorff topological vector space it follows that $x^* = y$.

In the next text we shall use the following notation:

$\Phi_n(t, u) = \underbrace{t\,(t\,(\ldots t\,(t}_{n\text{-times}}\,(u, u), u), \ldots, u)$ for every $n \in N$ and $u \in [0, 1]$ where t is a T-norm. We shall suppose in the next Theorem that $(S_1, \mathcal{F}^i_1, t_1)$ is a Hausdorff sequentially complete, probabilistic locally convex space with continuous T-norm t_1, $(S_2, \mathcal{F}^l_2, t_2)$ is a probabilistic locally convex space with continuous T--norm t_2, $U \subseteq S_1$, $V \subseteq S_2$, $H: U \times V \to U$ and $K: U \times V \to V$ are continuous mappings.

Theorem. *Suppose that* U *is closed and* $\sup_{r>0} \sup_{\delta<r} \inf_{x,y\in U} F^i_{x-y}(\delta) = 1$ *for every* $i \in J$, V *is closed, convex and compact and the following two conditions are satisfied:*

(A) *For every* $j \in J$ *there exist* $q(j)>0$ *and* $f(j) \in J$ *such that:*

$$F^{j}_{H(x_1,y)-H(x_2,y)}(q(j)\,\varepsilon) \geq F^{f(j)}_{x_1-x_2}(\varepsilon)$$

for every $x_1, x_2 \in U$, *every* $y \in V$ *and* $\varepsilon>0$, *where for every* $j \in J$:

$$\lim_{n\to\infty} \prod_{s=0}^{n} q(f^s(j))=0.$$

(B) *For every* $j \in J$ *there exists* $g(j) \in J$ *such that:*

$F^{f^n(j)}_x(\varepsilon) \geq F^{g(j)}_x(\varepsilon)$ *for every* $x \in S$, *every* $\varepsilon>0$ *and every* $n=0, 1, 2, \ldots$

If the family $\{\Phi_n(t_2, u)\}_{n\in N}$ *is equicontinuous at the point* $u=1$* *then there exists at least one fixed point of the mapping* $G: U \times V \to U \times V$, *where* $Gz=(Hz, Kz)$ *for every* $z \in U \times V$.

Proof: First of all we shall show that there exists one and only one continuous mapping $\tilde{R}: V \to U$ such that $\tilde{R}y=H(\tilde{R}y, y)$ for every $y \in V$. It is easy to see that if $(S, \mathcal{F}^t, t)$ is a Hausdorff sequentially complete, probabilistic locally convex space with continuous T-norm t and Λ is a compact topological space then the triplet $(C(\Lambda, S), \tilde{\mathcal{F}}^t, t)$ is a Hauscorff sequentially complete probabilistic locally convex space where $C(\Lambda, S)$ is the set of all continuous mapping from Λ into S and the mapping $\tilde{\mathcal{F}}^t: C(\Lambda, S) \to \Delta^+$ is defined in the following way:

$$\tilde{F}^t_{\tilde{x}}(\varepsilon)=\sup_{\delta<\varepsilon} \inf_{\lambda\in\Lambda} F^t_{x(\lambda)}(\delta) \text{ for every } \tilde{x}=\{x(\lambda)\} \in C(\Lambda, S).$$

Let us define the mapping $\tilde{T}: C(\Lambda, U) \to C(\Lambda, S)$, for $\Lambda=V$, in the following way:

$$(\tilde{T}\tilde{x})(\lambda)=H(x(\lambda), \lambda) \quad \text{for every} \quad \lambda \in \Lambda.$$

Then $\tilde{T}(C(V, U)) \subset C(V, U)$ and it is easy to see that the mapping $\tilde{T}$ satisfies all the conditions of the Lemma. This implies that there exists one and only fixed point of the mapping $\tilde{T}$ and let us denote this fixed point by $\tilde{R}$. Then for every $y \in V$ we have:

$$\tilde{R}(y)=H(\tilde{R}(y), y)$$

Let us define the mapping R^*: $V \to V$ in the following way: $R^*y=K(\tilde{R}y, y)$ for every $y \in V$. It is obviuous that R^* is a compact mapping since V is the compact subset of S_2 and R^* is continuous. In [7] is proved that $(S_2, \mathcal{F}^l_2, t_2)$ is, in the (ε, δ)--topology, a locally convex topological vector space since the family $\{\Phi_n(t_2, u)\}_{n\in N}$

---

* The family $\{\Phi_n(t, u)\}_{n\in N}$ is equicontinuous at the point $u=1$ if for every $\lambda \in (0, 1)$ there is a $\eta(\lambda) \in (0, 1)$ such that $x>\eta(\lambda) \Rightarrow \Phi_n(t, x)>1-\lambda$, $\forall_{n\in N}$. The trivial example is $t=\min$ and we shall give a nontrivial example.

is equicontinuous at the point $u=1$. Using Tihonoff's fixed point theorem we conclude that there exists at least one fixed point of the mapping R^*. If y_0 is a fixed point of the mapping R^* then $(\tilde{R}y_0, y_0)$ is a fixed of the mapping G.

Corollary. *Let $(S_1, \mathcal{F}_1, t_1)$ be a complete random normed space with continuous T-norm t_1, $(S_2, \mathcal{F}_2, t_2)$ be a random normed space with continuous T-norm t_2, $U \subseteq S_1$, $V \subseteq S_2$ and $H: U \times V \to U$, $K: U \times V \to V$ such that the following conditions are satisfied:*

(*) $F_{H(x_1, y) - H(x_2, y)}(q\varepsilon) \geqslant F_{x_1 - x_2}(\varepsilon)$ for *every* $(x_1, x_2, y) \in U \times U \times V$ *and every* $\varepsilon > 0$ *where* $q \in (0, 1)$.

(**) *H and K are continuous mappings, $U = \overline{U}$, $V = \overline{co}\ V$ and V is compact. If the family $\{\Phi_n(t_2, u)\}_{n \in N}$ is equicontinuous at the point $u=1$ then there exists at least one fixed point of the mapping G,*

$$Gz = (Hz, Kz) \text{ for every } z \in U \times V.$$

Now, we shall give a nontrivial example of a continuous T-norm t such that the family $\{\Phi_n(t, u)\}_{n \in N}$ is equicontinuous at the point $u=1$ [7].

Let $\bar{t}$ be any continuous T-norm and let us define T-norm t in the following way:

$$t(x, y) = \begin{cases} 1 - 2^{-m} + 2^{-m-1}\, \bar{t}\,(2^{m+1}(x - 1 + 2^{-m}), 2^{m+1}(y - 1 + 2^{-m})), & \text{if } (x, y) \in J_m^2 \\ \min\{x, y\} & \text{if } (x, y) \notin \bigcup_{m=0}^{\infty} J_m^2 \end{cases}$$

where $J_m = [1 - 2^{-m}, 1 - 2^{-m-1}]$ for every $m = 0, 1, 2, \ldots$ It is easy to see that the mapping t is a continuous T-norm [11] such that the family $\{\Phi_n(t, u)\}_{n \in N}$ is equicontinuous at the point $u=1$ since $t: J_m \times J_m \to J_m$ and $\lim_{m \to \infty} 1 - 2^{-m} = 1$.

Remark 1: A subset M of topological vector spase S is *admissible* if for every compact subset $K \subseteq M$ and every neighbourhood $\mathcal{O}$ of zero in S there exists a continuous mapping $h: K \to M$ such that $x - hx \in \mathcal{O}$ for every $x \in K$ and dim (span $h(K)) < \infty$. If we suppose that V is the admissible subset of S_2 in the Theorem it is enough to suppose only that T-norm t_2 is continuous. Namely in this case we can apply the following result of S. Hahn and K.F. Potter: If S a topological vector space, K is a closed, convex and admissible subset of S then every compact mapping $F: K \to K$ has at least one fixed point.

It is known that every convex sub set of a locally convex space is admissible and so the set V in the Theorem is admissible.

Problem: Construct an example of random normed space $(S, \mathcal{F}, t)$ with continuous T-norm t and convex, admissible subset $V \subset S$ such that S is not, in the (ε, δ)-topology, a locally convex topological vector space.

Remark 2: If T-norm t is continuous, in [10] the following proposition is given:

$$I \times I = (\bigcup_{k \in K} J_k \times J_k) \cup C(\bigcup_{k \in K} J_k \times J_k), \quad I = [0,1]$$

where the set K is at most denumerable, for every $k \in K$ is J_k an open interval, $J_k \cap J_r = \emptyset$ for every $k \neq r$ and the restriction $t \mid J_k \times J_k = t_k$ is an Archimedean semigroup for every $k \in K$.

In [8] it is proved that if $(S, \mathcal{F}^i, t)$ is a probabilistic locally convex space with continuous T-norm t such that every t_k $(k \in K)$ is a strict semigroup and the family $\{\Phi_n(t, u)\}_{n \in N}$ is equicontinuous at the point $u=1$ then there exists a sequence $\{a_n\}_{n \in N} \subseteq (0,1)$ such that $\lim_{n \to \infty} a_n = 1$, and the family of seminorms $\{p_{i,n}\}_{i \in J, n \in N}$ defined by:

$$p_{i,n}(x) = \sup \{t \mid F_x^i(t) \leqslant a_n\} \quad (n \in N; i \in J; x \in S)$$

induces the (ε, δ)-topology in S. So if we suppose in the Theorem that the family $\{\Phi_n(t, u)\}_{n \in N}$ is equicontinuous at the point $u=1$ and every t_k $(k \in K)$ is strict, then the assumption that the set U is probabilistically bounded can be dropped. Namely, it is easy to see that in this case the following two inequality are satisfied:

$$p_{i,n}(H(x_1, y) - H(x_2, y)) \leqslant q(i)\, p_{f(i),n}(x_1, x_2) \quad (i \in J; n \in N)$$

for every $x_1, x_2 \in U$ and every $y \in V$ and:

$$p_{f^m(i),n}(x) \leqslant p_{g(i),n}(x) \quad \text{for every } x \in S \text{ and every } m \in N, \; n \in N.$$

So we can apply the Theorem from [4] and from it we conclude that the mapping $\tilde{R}$ in the Theorem is continuous.

REFERENCES

[1] А. Деляну, Г. Маринеску, *Теорема о неподвижной точке и неявных функциях в локально выпуклых пространствах*, Revue de Math. pures et appliq. 8, 1963, 91–99.

[2] F. Gandac, *Teoreme de punct fix in spatii local convexe*, Studii si cerc. mat., T 24, Nr 7, (1972), 1097–1106.

[3] O. Hadžić and B. Stanković, *Some theorems on the fixed point in locally convex spaces*, Publ. Inst. Math., T 10 (24), (1970), 9–19.

[4] O. Hadžić, *Existence theorems for the system* $x = H(x, y), y = K(x, y)$ *in locally convex spaces*, Publ. Inst. Math., T 16 (30), (1973), 65–73.

[5] O. Hadžić, *A fixed point theorem for mappings with a sequentially compact iteration in probabilistic locally convex spaces*, Publ. Inst. Math., 22 (36), (1977), 71–76.

[6] O. Hadžić, *A fixed point theorem in probabilistic locally convex spaces*, Bull. Math. R S R (to appear).

[7] O. Hadžić, *On the* (ε, λ)*-topology of probabilistic locally convex spaces*, Glasnik matematički (to appear).

[8] O. Hadžić and M. Budinčević, *A fixed point theorem in probabilistic metric spaces*, Colloquia Mathematica Societatis Janos Bolyai (to appear).

[9] S. Hahn and K. F. Pötter, *Über Fixpunkte kompakter Abbildungen in topologischen Vektorraümen*, Studia Math., 60, (1974), 1–16.

[10] V. Istratescu, *Introducere in teoria spatiilor metrice probabiliste cu aplicatii*, Ed. Tehnica, Bucuresti, 1974.

[11] B. Schweizer and A. Sklar, *Associative functions and abstract semigroups*, Publicationes Mathematicae, Debrecen, 10 (1963), 69–81.

[12] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math., 10, (1960), 313—334.
[13] V. M. Sehgal and A. T. Bharucha-Reid, *Fixed points of contraction mappings on probabilistic metric space*, Math. Systems Theory, 6, (1972), 97—102.
[14] A. N. Šerstnev, *The notion of random normed space*, DAN USSR, 149 (2), (1963), 280—283 (in Russian).

Olga Hadžić i Đura Paunić

TEOREMA O EGZISTENCIJI REŠENJA SISTEMA $x=H(x,y)$, $y=K(x,y)$ U VEROVATNOSNIM LOKALNO KONVEKSNIM PROSTORIMA

Rezime

Dokazana je teorema o egzistenciji rešenja sistema $x=H(x,y)$ $y=K(x,y)$ u verovatnosnim lokalno konveksnim prostorima, gde je $H:U\times V\to U, K:U\times V\to V$, $U\subseteq S_1$, $V\subseteq S_2$, a $(S_1, \mathcal{F}_1^1, t_1)$ i $(S_2, \mathcal{F}_2^1, t_2)$ su verovatnosni lokalno konveksni prostori sa neprekidnim T-normama t_1 i t_2. U radu je postavljen i sledeći problem:

Konstruisati primer verovatnosnog lokalno konveksnog prostora $(S, \mathcal{F}, t)$ sa neprekidnom T-normom t tako da prostor S nije u (ε, λ)-topologiji lokalno konveksan, a postoji konveksan podskup $V\subseteq S$ koji je dopustiv.