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THE APPLICATION OF AN APPROXIMATION TO A SPECIAL OPERATOR

Abstract: Following the ideas of the author's previous papers [3] [4] here is given the approximation of a special operator which appears by finding the dispersion coefficients [5]. The error of this approximation is also given.

In [1] L. Berg applies his method of finding an approximate inversion o the Laplace transformation to the function

$$F(p)=\exp(-\sqrt{p+a+b-b^2(cp+b)^{-1}})$$

a, $b>0$ and $c>1$ which appears by finding of the dispersion coefficients [5]

B. Stanković, D. Herceg apply a theory of approximation [6] to the corresponding operator of the following form

$$F(s)=\exp(-\sqrt{s+d-b^2(cs+b)^{-1}}) \tag{1}$$

d $b>0$, $c>1$, s-differential operator in the field M of Mikusiński operators.

They give the upper bound for the measure and the error of the approximation of the operator (1) and a program to calculate this bound by computer.

In this paper we shall give the representation of the operator (1) by means of convergent series and its approximation. However in numerical calculation we shall always restrict ourselves to a finite number of the terms of series. The error committed may be estimated by replacing the considered series by its majorant series (4) as we shall do.

We start from the operator

$$\exp(\sqrt{s}-\sqrt{s+d-b^2(cs+b)^{-1}}). \tag{2}$$

Multipling the operator (2) by the operator $\exp(-\sqrt{s})$ we obtain the representation of the operator (1). In the solving of the above problem we use two well known results [2]*

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* C is the ring of continuous complex valued functions defined over $[0, \infty)$ with the operation sum and finite convolution.

L is the ring of locally integrable functions over $[0, \infty)$ with the same operations. The quotient field of these rings is the field M of Mikusiński operators.

(A) *C is an ideal in L*

(B) *Every locally integrable function f is a logarithm and operator exp f expands into a power series just as ordinary exponential function.*

Now if we denote by $a=b/c$ and by w_1 the following operator

$$w_1=\sqrt{s}-\sqrt{s+d-b^2(cs+b)^{-1}}=\sqrt{s}\left(1-\sqrt{1+(d-b)s^{-1}+b(s+a)^{-1}}\right)=$$

$$=-\frac{b}{s}+\frac{ab}{\sqrt{s}(s+a)}-\sum_{k=2}^{\infty}\binom{1/2}{k}\left(\frac{d-b}{s}+\frac{b}{s+a}\right)^k s^{1/2}$$

then w_1 is a function of class L also logarithm.

Using the hypergeometric expansion

$$(\sqrt{1+z}-1)^{k+1}=(k+1)\sum_{i=0}^{\infty}(-1)^i\frac{(k+2i)!\,z^{k+i+1}}{2^{k+2i+1}\,i!\,(k+1+i)!},\qquad k=0,1,2,\ldots$$

we can easily find that

$$\exp\left(\sqrt{s}-\sqrt{s+d-b^2(cs+b)^{-1}}\right)=1-\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}s^{\frac{k+1}{2}}\left(\sqrt{1+\frac{d-b}{s}+\frac{b}{s+a}}-1\right)^{k+1}$$

$$=1-\sum_{k=0}^{\infty}\frac{(-1)^k s^{(k+1)/2}}{k!}\sum_{n=0}^{\infty}(-1)^n\frac{(k+2n)!\left(\frac{d-b}{s}+\frac{b}{s+a}\right)^{k+n+1}}{2^{k+2n+1}\,n!\,(k+1+n)!}$$

(2.1)

$$=1-\sum_{k=0}^{\infty}\frac{(-1)^k s^{(k+1)/2}}{k!}\sum_{\nu=k}^{\infty}(-1)^{\nu-k}\frac{(2\nu-k)!\left(\frac{d-b}{s}+\frac{b}{s+a}\right)^{\nu+1}}{2^{2\nu-k+1}(\nu-k)!\,(\nu+1)!}$$

$$=1-\sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{1}{2^{2\nu+1}(\nu+1)!}\left(\frac{d-b}{s}+\frac{b}{s+a}\right)^{\nu+1}\sum_{k=0}^{\nu}\frac{(2\nu-k)!\,2^k s^{(k+1)/2}}{(\nu-k)!\,k!}$$

Multiplying (2.1) by $\exp(-\sqrt{s})$ we obtain the formula

$$\exp\left(-\sqrt{s+d-b^2(cs+b)^{-1}}\right)=$$

(3)

$$=\exp(-\sqrt{s})\left[1-\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(\nu+1)!\,2^{2\nu+1}}\left(\frac{d-b}{s}+\frac{b}{s+a}\right)^{\nu+1}\sum_{k=0}^{\nu}\frac{(2\nu-k)!\,2^k s^{(k+1)/2}}{(\nu-k)!\,k!}\right].$$

It is well known that generally by it is impossible to say that one of two operators is smaller thar the other. In the particular case of f and g being the function of class L with real values we shall understand by symbol

$$f \leqslant g$$

that

$$f(t) \leqslant g(t)$$

for all values of $t \geqslant 0$ for which both functions are continuous. Thus for instance

$$\exp(-\sqrt{s}) \leqslant 3\sqrt{\frac{6}{\pi e^3}}\, s^{-1}$$

and

$$\frac{b}{s+a} \leqslant \frac{b}{s} \quad (a>0,\ b>0).$$

By the module $|f|$ of a function of class L we shall simply understand

$$|f| = |\{f(t)\}| = \{|f(t)|\}$$

The module f is thus again a function of class L. If f and g are the function of class L then

$$|f+g| \leqslant |f| + |g|, \quad |f\,g| \leqslant |f|\,|g|.$$

Hence for majorization of the operator $F(s) \varepsilon L$ the following expression can be used

$$3\sqrt{\frac{6}{\pi e^3}}\, s^{-1} + \tag{4}$$
$$+3\sqrt{\frac{6}{\pi e^3}}\, s^{-1/2} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu+1}}{2^{2\nu+1}(\nu+1)!} \left(\frac{d-b}{s} + \frac{b}{s+a}\right)^{\nu+1} \sum_{k=0}^{\nu} \frac{2^k (2\nu-k)!\, s^{k/2}}{(\nu-k)!\, k!}$$

One approximation $\tilde{x}$ of the operator F (s) is

$$\tilde{x} = 3\sqrt{\frac{6}{\pi e^3}} \left[\frac{1}{s} + \right. \tag{5}$$
$$\left. + \frac{1}{\sqrt{s}} \sum_{\nu=0}^{N} \frac{(-1)^{\nu+1}}{2^{2\nu+1}(\nu+1)!} \left(\frac{d-b}{s} + \frac{b}{s+a}\right)^{\nu+1} \sum_{k=0}^{\nu} \frac{2^k (2\nu-k)!\, s^{k/2}}{(\nu-k)!\, k!}\right].$$

Because the coefficients of series (4) are with alternate signs it follows

$$\left|\sum_{\nu=N+1}^{\infty}\frac{(-1)^{\nu+1}}{2^{2\nu+1}(\nu+1)!}\left(\frac{d-b}{s}+\frac{b}{s+a}\right)^{\nu+1}\sum_{k=0}^{\nu}\frac{2^k(2\nu-k)!\,s^{k/2}}{(\nu-k)!\,k!}\right|\leqslant$$

$$\leqslant\frac{1}{2^{2N+3}(N+2)!}\left|\left(\frac{d-b}{s}+\frac{b}{s+a}\right)^{N+2}\sum_{k=0}^{N+1}\frac{2^k(2N+2-k)!\,s^{k/2}}{(N+1-k)\,k!}\right|.$$

According to this we have:

$$|F(s)-\tilde{x}|\leqslant 3\sqrt{\frac{6}{\pi e^3}}\left|\frac{1}{\sqrt{s}}\left(\frac{d}{s}\right)^{N+2}\frac{1}{2^{2N+3}(N+2)!}\sum_{k=0}^{N+1}\frac{(2N+2-k)!\,2^k s^{k/2}}{(N+1-k)!\,k!}\right|=$$

$$=3/2\sqrt{\frac{6}{\pi e^3}}\frac{d^{N+2}}{2^{2N+2}(N+2)}\left|\sum_{k=0}^{N+1}\frac{(2N+2-k)!\,2^k}{(N+1-k)!\,k!\,s^{N+5/2-k/2}}\right|=$$

$$=3/2\sqrt{\frac{6}{\pi e^3}}\frac{d^{N+2}}{(N+2)!}\left\{\left|\frac{2t^{N+3/2}}{(2N+3)\sqrt{\pi}}+\frac{(2N+1)!!\,t^{N+1}}{2^{N+1}(N+1)!}+\frac{Nt^{N+1/2}}{(2N+1)\sqrt{\pi}}+\right.\right.$$

$$+\frac{(2N-1)!!\,t^N}{2^N\,3!\,N(N-2)!}+\frac{(N-1)(N-2)\,t^{N-1/2}}{2(2N-1)\,3!\sqrt{\pi}}+\frac{(2N-3)!!\,t^{N-1}}{2^{N-1}(N-1)(N-4)!\,5!}+$$

$$+\frac{(N-2)(N-3)(N-4)\,t^{N-3/2}}{3\cdot 5!\,(2N-3)\sqrt{\pi}}+\frac{(2N-5)!!\,t^{N-2}}{2^{N-2}(N-2)(N-6)!\,7!}+$$

$$+\frac{(N-3)(N-4)(N-5)(N-6)\,t^{N-5/2}}{4\cdot 7!\,(2N-5)\sqrt{\pi}}+\frac{(2N-7)!!\,t^{N-3}}{2^{N-3}(N-3)(N-8)!\,9!}+$$

$$\left.\left.+\ldots+\frac{(N+2)(N+1)\,t^{N/2+3/2}}{2^{N+2}\,\Gamma(N/2+5/2)}+\frac{t^{N/2+1}}{2^{N+1}\,\Gamma(N/2+2)}\right|\right\}.$$

But we know that

If $N=2n_o$ then $\Gamma(N/2+5/2)=\Gamma(n_o+5/2)=\dfrac{(2n_o+3)!\sqrt{\pi}}{2^{2n_o+3}(n_o+1)!}$

and

$$\Gamma(N/2+2)=\Gamma(n_o+2)=(n_o+1)!.$$

If $N=2n_0+1$ then

$$\Gamma(N/2+5/2)=\Gamma(n_o+3)=(n_o+2)!$$

and

$$\Gamma(N/2+2) \quad = \Gamma(n_o+5/2) = \frac{(2n_o+3)!\sqrt{3}}{2^{2n_o+3}(n_o+1)!}.$$

Namely for every t ε [O, T] is

$$|F(s)-\tilde{x}| = |R_N| \leqslant 3/2\sqrt{\frac{6}{3e^3}}\,\frac{(dT^{1/2})^{N+2}}{(N+2)!}\left[\frac{2T^{N/2+1/2}}{(2N+3)\sqrt{\pi}} + \frac{(2N+1)!!T^{N/2}}{2^{N+1}(N+1)!} + \right.$$

$$+\frac{NT^{N/2-1/2}}{(2N+1)\sqrt{\pi}} + \frac{(2N-1)!!T^{N/2-1}}{2^N 3!N(N-2)!} + \frac{(N-1)(N-2)\,T^{N/2-3/2}}{2\cdot 3!\,(2N-1)\sqrt{\pi}} +$$

$$+\frac{(2N-3)!!T^{N/2-2}}{2^{N-1}(N-1)(N-4)!\,5!} + \frac{(N-2)(N-3)(N-4)\,T^{N/2-5/2}}{3\cdot 5!\,(2N-3)\sqrt{\pi}} + \ldots +$$

$$\left. + \frac{(N+2)(N+1)\,T^{1/2}}{2^{N+2}\Gamma(N/2+5/2)} + \frac{1}{2^{N+1}\Gamma(N/2+2)}\right].$$

For the special case $d=1$, $N=12$ we obtain:

T	N	$\lvert F(s)-\tilde{x}\rvert$	T	N	$\lvert F(s)-\tilde{x}\rvert$
	1	·79235151911752D-03		1	·27751310576616D-02
	2	·10057353213407D-03		2	·47066155122122D-03
	3	·66684477875170D-05		3	·53420760363005D-04
	4	·52933177593133D-06		4	·61139053193938D-05
	5	·36569793949070D-07		5	·70406273053938D-06
·125	6	·32946017096719D-08	·250	6	·94166968090408D-07
	7	·13947769175820D-09		7	·63825525336634D-08
	8	·12195688492976D-10		8	·71146012515615D-09
	9	·62740070015803D-12		9	·51517982689917D-10
	10	·54058679251945D-13		10	·56196536215193D-11
	11	·26414284030356C-14		11	·41325735115983D-12
	12	·20464780184825D-15		12	·40613983373099D-13
	1	·10285605788424D-01			
	2	·24020608122701D-02			
	3	·45692033702670D-03			
	4	·80334083235406D-04			
	5	·16219309230847D-04			
·500	6	·37491010486700D-05			
	7	·41731715285220D-06			
	8	·81438490401305D-07			
	9	·82636875703405D-07			
	10	·15318232771569D-08			
	11	·14663281865723D-09			
	12	·25311075338671D-10			

In the case $d=1/2, N=12$ we obtain the following scheme

T	N	$\|F(s)-\tilde{x}\|$	T	N	$\|F(s)-\tilde{x}\|$
	1	·14856590983454D-03		1	·34689138220770D-03
	2	·62858457583795D-05		2	·29416346951326D-04
	3	·20838899335990D-06		3	·16693987613439D-05
	4	·82708089989271D-08		4	·95529770615527D-07
	5	·28570151522711D-09		5	·55004900823389D-08
·125	6	·12869537928406D-10	·250	6	·36783971910316D-09
	7	·27241736671524D-12		7	·12465922917311D-10
	8	·11909852043922D-13		8	·69478527847280D-12
	9	·30634799812404D-15		9	·25155264985311D-13
	10	·13197919739244D-16		10	·13719857474412D-14
	11	·32243999060493D-18		11	·50446453999003D-16
	12	·12490710562027D-19		12	·24788808211120D-17
	1	·12857007235530D-02			
	2	·15012880076688D-03			
	3	·14278760532084D-04			
	4	·12552200505532D-05			
	5	·12671335336599D-06			
·500	6	·14644925971367D-07			
	7	·81507256416446D-09			
	8	·79529775782524D-10			
	9	·40350036967481D-11			
	10	·37398029227463D-12			
	11	·17899513996244D-13			
	12	·15448654381513D-14			

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PRIMENA JEDNE APROKSIMACIJE NA SPECIJALNI OPERATOR

Rezime

Sledeći ideje mojih ranijih radova [3] [4] ovde je data aproksimacija jednog specijalnog operatora

$$F(s)=\exp\left(-\sqrt{s+d-b^2(cs+b)^{-1}}\right)$$

$b>0, c>1, d>0, s$ je operator diferenciranja u polju M operatora Mikusinskog. Ovaj operator se javlja pri određivanju aksijalnih disperzionih koeficijenata u stabilnim gasnim sistemima [5]. Data je i greška ove aproksimacije.