Novi Sad J. Math.
Vol. 44, No. 1, 2014, 59-73

EXISTENCE OF SOLUTIONS FOR A SYSTEM OF INTEGRAL EQUATIONS VIA MEASURE OF NONCOMPACTNESS

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Abstract. Using the techniques of measures of noncompactness and Darbo fixed point theorem, we present some existence results for solutions of systems of nonlinear equations in Banach spaces. Also, as an application, we discuss the existence of solutions for a general system of nonlinear functional integral equations, which extends some previous results in the literature. An example is given to show the efficiency and usefulness of the results.

AMS Mathematics Subject Classification (2010): 47H09, 47H10

*Key words and phrases:*Measure of noncompactness, Darbo fixed point theorem, Coupled fixed point, System of integral equations.

1. Introduction

Recently, there have been several successful efforts to apply the concept of measure of noncompactness in the study of the existence and behavior of solutions of nonlinear differential and integral equations ([1, 2, 3, 4, 7, 8, 9, 10, 11, 13, 14, 15, 16, 19]). In this paper, we present and prove some new existence theorems for solutions of systems of nonlinear equations which are formulated in terms of condensing operators in Banach spaces (i.e. mappings under which the image of any set is in a certain sense more compact than the set itself [6]). Moreover, as an application, we study the problem of existence of solutions for the following system of nonlinear integral equation

$$(1.1)\quad \begin{cases} x(t) = f_1\Big(t, x(\xi_1(t)), y(\xi_1(t)), \int_0^{\beta_1(t)} g_1(t,s,x(\eta_1(s)), y(\eta_1(s)))ds\Big), \\ y(t) = f_2\Big(t, x(\xi_2(t)), y(\xi_2(t)), \int_0^{\beta_2(t)} g_2(t,s,x(\eta_2(s)), y(\eta_2(s)))ds\Big), \end{cases}$$

where f_i, g_i, ξ_i, η_i and β_i satisfy certain conditions.

The organization of this paper is as follows. In Section 2, some basic notations, definitions and auxiliary results are given. Section 3 is devoted to state and prove some existence theorems for systems of equations involving condensing operators using the Darbo fixed point theorem. Finally in Section 4, using the obtained results in Section 3, we investigate the problem of existence of solutions for the system of nonlinear integral equation (1.1).

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2. Preliminaries

The first measure of noncompactness was defined by Kuratowski [18]. In a metric space X and for a bounded subset S of X the Kuratowski measure of noncompactness is defined as

(2.1)

$$\alpha(S) := \inf\{\delta > 0 | S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } diam(S_i) \leq \delta \text{ for } 1 \leq i \leq n \leq \infty\}.$$

Here $diam(T)$ denotes the diameter of a set $T \subset X$, i.e.,

$$diam(T) := sup\{d(x, y) | x, y \in T\}.$$

Another important measure of noncompactness is the so-called Hausdorff (or ball) measure of noncompactness defined as

$$\chi(X) = \inf\{\varepsilon : X \ has \ a \ finite \ \varepsilon - net \ in \ E\}.$$

Since a ball of radius r has diameter at most $2r$, then the measures χ and α are equivalent i.e., for any bounded subset X of E the following estimate holds [6]

$$\chi(X) \leq \alpha(X) \leq 2\chi(X).$$

The two measures χ and α share many properties [6, 8]. Here, we recall some basic facts concerning measures of noncompactness from [8], which is defined axiomatically in terms of some natural conditions. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|.\|)$ be a Banach space. The symbol $\overline{X}$, $ConvX$ will denote the closure and closed convex hull of a subset X of E, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets.

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subseteq \mathfrak{N}_E$.

2° $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$.

3° $\mu(\overline{X}) = \mu(X)$.

4° $\mu(ConvX) = \mu(X)$.

5° $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \cdots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

Here we recall the well known fixed point theorem of Darbo [12].

Theorem 2.2. *[12] Let Ω be a nonempty, bounded, closed and convex subset of a space E and let $F : \Omega \longrightarrow \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property*

$$\mu(FX) \leq k\mu(X)$$

for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.

The following theorem and example are basic to prove all the results of this work.

Theorem 2.3. *[8] Suppose $\mu_1, \mu_2, \cdots, \mu_n$ are measures in $E_1, E_2, \cdots, E_n$, respectively. Moreover, assume that the function $F : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ is convex and $F(x_1, \cdots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \cdots, n$. Then*

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \cdots, \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$ where X_i denotes the natural projection of X into E_i for $i = 1, 2, \cdots, n$.

As results from Theorem 2.2 we present the following example.

Example 2.4. [5] Let μ be a measure of noncompactness, considering $F_1(x, y) = \max\{x, y\}$ and $F_2(x, y) = x + y$ for any $(x, y) \in \mathbb{R}^2_+$ then all the conditions of Theorem 2.2 are satisfied. Therefore, $\widetilde{\mu}_1 = \max\{\mu(X_1), \mu(X_2)\}$ and $\widetilde{\mu}_2 = \mu(X_1) + \mu(X_2)$ are measures of noncompactness in the space $E \times E$ where X_i, $i = 1, 2$ denote the natural projections of X.

3. Main results

In this section, we state and prove some existence results for solutions of systems of equations involving condensing operators in Banach spaces which will be used in Section 4.

Theorem 3.1. *Let C be a nonempty, bounded and closed subset of a Banach space E and μ an arbitrary measure of noncompactness on E. If $F_i : C \times C \longrightarrow C$ for $i = 1, 2$ are continuous operators and there exists a constant $k \in [0, 1)$ such that*

$$\mu(F_i(X_1 \times X_2)) \leq k \max\{\mu(X_1), \mu(X_2)\}, \tag{3.1}$$

for any subset X_1, X_2 of C, then there exist $x^*, y^* \in X$ such that

$$\begin{cases} F_1(x^*, y^*) = x^*, \\ F_2(x^*, y^*) = y^*. \end{cases} \tag{3.2}$$

Proof. Consider the operator $\widetilde{F} : C \times C \longrightarrow C \times C$ defined by

$$\widetilde{F}(x, y) = (F_1(x, y), F_2(x, y)).$$

Example 2.4 shows that $\widetilde{\mu}(X) := \max\{\mu(X_1), \mu(X_2)\}$ is a measure of noncompactness in the space $C \times C$, where X_i, $i = 1, 2$ denote the natural projections of X. Now let X be any nonempty subset of $C \times C$. Then by (2°) and (3.1) we obtain

$$\begin{aligned}
\widetilde{\mu}(\widetilde{F}(X)) &\leq \widetilde{\mu}(F_1(X_1 \times X_2) \times F_2(X_1 \times X_2)) \\
&= \max\{\mu(F_1(X_1 \times X_2)), \mu(F_2(X_1 \times X_2))\} \\
&\leq \max\{k \max\{\mu(X_1), \mu(X_2)\}, k \max\{\mu(X_2), \mu(X_1)\} \\
&\leq k\widetilde{\mu}(X).
\end{aligned}$$

Since $\widetilde{\mu}$ is also a measure of noncompactness, therefore all conditions of Theorem 2.3 are satisfied. Hence $\widetilde{F}$ has a fixed point, i.e., there exist $x^*, y^* \in X$ such that

$$(x^*, y^*) = \widetilde{F}(x^*, y^*) = (F_1(x^*, y^*), F_2(x^*, y^*)),$$

which means (x^*, y^*) solves (3.2). □

Corollary 3.2. *Let C be a nonempty, bounded and closed subset of a Banach space E and μ an arbitrary measure of noncompactness on E. If $F_i : C \times C \longrightarrow C$ for $i = 1, 2$ are continuous operators for which there exist nonnegative constants k_1, k_2 with $k_1 + k_2 < 1$ such that*

$$\mu(F_i(X_1 \times X_2)) \leq k_1\mu(X_1) + k_2\mu(X_2) \tag{3.3}$$

for any subsets X_1, X_2 of C, then there exist $x^*, y^* \in X$ such that

$$\begin{cases} F_1(x^*, y^*) = x^*, \\ F_2(x^*, y^*) = y^*. \end{cases}$$

Proof. It is enough to show that (3.1) holds. Let $X_1, X_2 \subseteq C$ be given, then

$$\begin{aligned}
\mu(F_i(X_1 \times X_2)) &\leq k_1\mu(X_1) + k_2\mu(X_2) \\
&\leq k_1 \max\{\mu(X_1), \mu(X_2)\} + k_2 \max\{\mu(X_1), \mu(X_2)\} \\
&\leq (k_1 + k_2) \max\{\mu(X_1), \mu(X_2)\}.
\end{aligned}$$

Now the conclusion follows from Theorem 3.1. □

Definition 3.3. [17] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Note that if $F : C \times C \longrightarrow C$ is a continuous operator and we define $F_1(x, y) = F(x, y)$ and $F_2(x, y) = F(y, x)$ then as a result of Theorem 3.1 and Corollary 3.2 we have the main results of [5].

Corollary 3.4. *Let C be a nonempty, bounded and closed subset of a Banach space E, μ an arbitrary measure of noncompactness on E and $F : C \times C \longrightarrow C$ a continuous operator. Suppose either:*

(I) *There exist nonnegative constants k_1, k_2 with $k_1 + k_2 < 1$ such that*

$$\mu(F(X_1 \times X_2)) \leq k_1\mu(X_1) + k_2\mu(X_2),$$

or

(*II*) *There exists a constant* $k \in [0,1)$ *such that*

$$\mu(F(X_1 \times X_2)) \leq k \max\{\mu(X_1), \mu(X_2)\}$$

for any subset X_1, X_2 *of* C*. Then* F *has a coupled fixed point.*

Proof. Take $F_1(x,y) = F_2(x,y) = F(y,x)$ in Theorem 3.1 and Corollary 3.2. □

Corollary 3.5. *Let* C *be a nonempty, bounded, closed and convex subset of a Banach space* E *and let* $F_i : C \times C \longrightarrow E$ *for* $i = 1,2$ *be operators such that*

$$\|F_i(x,y) - F_i(u,v)\| \leq k \max\{\|x-u\|, \|y-v\|\}, \tag{3.4}$$

where $k \in [0,1)$ *. Assume that* $G_i : C \times C \longrightarrow X$ *are compact and continuous operators and the operators* $T_i : C \times C \longrightarrow C$ *defined by*

$$\|T_i(x,y) - T_i(u,v)\| \leq \|F_i(x,y) - F_i(u,v)\| + \Phi(\|G_i(x,y) - G_i(u,v)\|) \tag{3.5}$$

for $i = 1,2$ *where* $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a nondecreasing continuous function and* $\Phi(0) = 0$*. Then there exist* $x^*, y^* \in C$ *such that*

$$\begin{cases} T_1(x^*, y^*) = x^*, \\ T_2(x^*, y^*) = y^*. \end{cases}$$

Proof. Let X_1 and X_2 be arbitrary subsets of C and fixed $1 \leq i \leq 2$. By the definition of Kuratowski measure of noncompactness for every $\varepsilon > 0$ there exist $S_1, \cdots, S_n$ such that $X_1 \times X_2 \subseteq \bigcup_{k=1}^{i=n} S_k$,

$$diam(F_i(S_k)) < \alpha(F_i(X_1 \times X_2) + \varepsilon$$

and

$$diam(G_i(S_k)) < \varepsilon.$$

Let us fix arbitrarily $1 \leq k \leq n$. Then for every $p, q \in S_k$ we have

$$\|T_i(p) - T_i(q)\| \leq \|F_i(p) - F_i(q)\| + \Phi(\|G_i(p) - G_i(q)\|).$$

Thus, by properties of Φ we obtain

$$diam(T_i(S_k)) \leq diam(F_i(S_k)) + \Phi(diam(G_i(S_k))),$$

$$diam(T_i(S_k)) \leq \alpha(F_i(X_1 \times X_2)) + \varepsilon + \Phi(\varepsilon)$$

and since ε was chosen arbitrarily and Φ is a nondecreasing continuous function, so

$$\alpha(T_i(X_1 \times X_2)) \leq \alpha(F_i(X_1 \times X_2)). \tag{3.6}$$

Now we show that T_i satifies (3.1). To do this fix arbitrary $x, y \in X_1$ and $u, v \in X_2$. Then we have

$$\begin{aligned} \|F_i(x,y) - F_i(u,v)\| &\leq k \max\{\|x-u\|, \|y-v\|\} \\ &\leq k \max\{diam X_1, diam X_2\} \end{aligned}$$

so

$$diam(F_i(X_1 \times X_2) \le k \max\{diam X_1, diam X_2\}$$

Therefore, by definition of Kuratowski measure of noncompactness we have

$$\alpha(F_i(X_1 \times X_2) \le k \max\{\alpha(X_1), \alpha(X_2)\}. \tag{3.7}$$

By (3.6) and (3.7) we deduce

$$\alpha(T_i(X_1 \times X_2)) \le k \max\{\alpha(X_1), \alpha(X_2)\}.$$

Also, by conditon (3.5), T_i $(i = 1, 2)$ are continuous operators and the application of Theorem 3.1 completes the proof. □

In the same way as the above proof, we can extend Theorem 3.1 for n-dimensional systems of equations.

Theorem 3.6. *Let C be a nonempty, bounded and closed subset of a Banach space E and μ an arbitrary measure of noncompactness on E. If $F_i : C^n \longrightarrow C$, $i = 1, \cdots, n$ are continuous operators for which there exists a constant $k \in [0, 1)$ such that*

$$\mu(F_i(X_1 \times \cdots \times X_2)) \le k \max\{\mu(X_1), \cdots, \mu(X_n)\}$$

for any subset $X_1, \cdots, X_n$ of C. Then there exist $x_1^*, \cdots, x_n^* \in X$ such that

$$\begin{cases} F_1(x_1^*, \cdots, x_n^*) = & x_1^* \\ \vdots & \vdots \\ F_n(x_1^*, \cdots, x_n^*) = & x_n^*. \end{cases}$$

Proof. Define $\widetilde{F}(x_1, \cdots, x_n) = (F_1(x_1, \cdots, x_n), \cdots, F_n(x_1, \cdots, x_n))$ and follow the proof of Theorem 3.1. □

4. Application

In this section, as an application of Theorem 3.1, we prove an existence result for solutions of system (1.1). We will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+)$ is furnished with the standard supremum norm i.e., the norm defined by the formula

$$\|x\| = \sup\{|x(t)| : \ t \ge 0\}.$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which is stated in ([8, 9]). In order to define this measure, let us fix a nonempty bounded subset of X of $BC(\mathbb{R}_+)$ and a positive number $L > 0$. For $x \in X$ and $\varepsilon \ge 0$ denote by $\omega^L(x, \varepsilon)$, the modulus of continuity of x on the interval $[0, L]$, i.e,

$$\omega^L(x, \varepsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0, L], \ |t - s| \le \varepsilon\}.$$

Moreover, let us put

$$\begin{aligned}\omega^L(X,\varepsilon) &= \sup\left\{\omega^L(x,\varepsilon):\ x\in X\right\},\\ \omega_0^L(X) &= \lim_{\varepsilon\to 0}\omega^L(X,\varepsilon),\\ \omega_0(X) &= \lim_{L\to\infty}\omega_0^L(X).\end{aligned}$$

If t is a fixed number from $\mathbb{R}_+$, let us denote

$$X(t)=\{x(t):\ x\in X\}.$$

Finally, consider the function μ defined on $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X)=\omega_0(X)+\limsup_{t\to\infty} diam X(t)$$

where

$$diam X(t)=\sup\{|x(t)-y(t)|:\ x,y\in X\}.$$

It can be shown (cf. [8, 9]) that the function $\mu(X)$ defines a measure of non-compactness on $BC(\mathbb{R}_+)$ in the sense of the above accepted definition.

Now, we are ready to state and prove the main result of this section on the existence of solutions for the system of integral equations (1.1).

Theorem 4.1. *Assume that the following conditions are satisfied:*

(*i*) $\xi_i,\eta_i,\beta_i:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ *(i=1,2) are continuous and* $\xi_i(t)\longrightarrow\infty$ *as* $t\longrightarrow\infty$ *for* $i=1,2$,

(*ii*) $f_i:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ *for* $i=1,2$ *are continuous. Moreover, there exist constant* $k\in[0,1)$ *and nondecreasing continuous functions* $\Phi_i:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ *with* $\Phi_i(0)=0$, $i=1,2$, *such that*
(4.1)
$$|f_i(t,x,y,z)-f_i(t,u,v,w)|\le k\max\{|x-u|,|y-v|\}+\Phi_i(m_i(t)|z-w|),$$
where $m_i(t):\mathbb{R}_+\longrightarrow\mathbb{R}_+$ *are continuous functions.*

(*iii*) *The functions* $|f_i(t,0,0,0)|$ *for* $i=1,2$ *are bounded on* $\mathbb{R}_+$, *i.e.*

$$M_i=\sup\{f_i(t,0,0,0):t\in\mathbb{R}_+\}<\infty. \tag{4.2}$$

(*iv*) $g_i:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ *for* $i=1,2$ *are continuous and there exists a positive constant* D *such that*
(4.3)
$$\sup\{m_i(t)\left|\int_0^{\beta_i(t)}g_i(t,s,x(\eta_i(s)),y(\eta_i(s)))ds\right|:\begin{matrix}t\in\mathbb{R}_+,\ x,y\in BC(\mathbb{R}_+),\\ 1\le i\le 2\end{matrix}\}<D.$$

Morever,
(4.4)
$$\lim_{t\longrightarrow\infty}m_i(t)\left|\int_0^{\beta_i(t)}[g_i(t,s,x(\eta_i(s)),y(\eta_i(s)))-g_i(t,s,u(\eta_i(s)),v(\eta_i(s)))]ds\right|=0,$$

uniformly with respect to $x,y,u,v\in BC(\mathbb{R}_+)$ *for* $i=1,2$.

Then the system of equations (1.1) *has at least one solution in the space* $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Proof. The proof is carried out in two steps.

Step 1: $G_i : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \longrightarrow BC(\mathbb{R}_+)$ *defined by*

$$G_i(x,y)(t) = m_i(t) \int_0^{\beta_i(t)} g_i(t,s,x(\eta_i(s)),y(\eta_i(s)))ds \tag{4.5}$$

for $i = 1,2$ *are compact and continuous operators.*

Let $1 \le i \le 2$ be fixed. Notice that the continuity of $G_i(x,y)(t)$ for any $x \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ is obvious. Moreover, by (4.3), G_i is an operator on $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ into $BC(\mathbb{R}_+)$. Now, we show that G_i is continuous. For this, take $x, y \in BC(\mathbb{R}_+)$ and $\varepsilon > 0$ arbitrarily, and consider $u, v \in BC(\mathbb{R}_+)$ with $\|x-u\| < \varepsilon$ and $\|v-y\| < \varepsilon$. Then we have

$$\begin{aligned} |G_i(x,y)(t) - G_i(u,v)(t)| &\le \Big| m_i(t) \int_0^{\beta_i(t)} g_i(t,s,x(\eta_i(s)),y(\eta_i(s)))ds \\ &\quad - m_i(t) \int_0^{\beta_i(t)} g_i(t,s,u(\eta_i(s)),v(\eta_i(s)))ds \Big| \\ &\le m_i(t) \Big| \int_0^{\beta_i(t)} [g_i(t,s,x(\eta_i(s)),y(\eta_i(s))) \\ &\quad - g_i(t,s,u(\eta_i(s)),v(\eta_i(s)))]ds \Big|. \end{aligned} \tag{4.6}$$

Furthermore, considering condition (iv), there exists $T > 0$ such that for $t > T$ we have

$$|G_i(x,y)(t) - G_i(u,v)(t)| \le \varepsilon.$$

Also, if $t \in [0,T]$, then from (4.6) it follows that

$$|G_i(x,y)(t) - G_i(u,v)(t)| \le m_T \beta_T \vartheta(\varepsilon),$$

where

$$\begin{aligned} &\beta_T = \sup\{\beta_i(t) : t \in [0,T],\ 1 \le i \le 2\}, \\ &m_T = \sup\{m_i(t) : t \in [0,T],\ 1 \le i \le 2\}, \\ &b = \max\{\|x\|, \|y\|\} + \varepsilon, \\ &\vartheta(\varepsilon) = \sup\{|g_i(t,s,x,y) - g_i(t,s,u,v)| : t \in [0,T],\ s \in [0,\beta_T], \\ &\qquad\qquad x,y,u,v \in [-b,b],\ |x-u| \le \varepsilon,\ |y-v| \le \varepsilon\}. \end{aligned}$$

By using the continuity of g_i on the compact set $[0,T] \times [0,\beta_T] \times [-b,b] \times [-b,b]$, we have $\vartheta(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$. Thus, G_i is a continuous function from $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ into $BC(\mathbb{R}_+)$.

Now, Let X_1, X_2 be two nonempty and bounded subsets of $BC(\mathbb{R}_+)$, and assume that $T > 0$ and $\varepsilon > 0$ are chosen arbitrarily. Let $t_1, t_2 \in [0,T]$, with

$|t_2 - t_1| \le \varepsilon$ and $x, y \in X$, we obtain

$$\begin{aligned}
&|G_i(x,y)(t_2) - G_i(x,y)(t_1)| \le \\
&\qquad \le \Big| m_i(t_1) \int_0^{\beta_i(t_2)} g_i(t_2, s, x(\eta_i(s)), y(\eta_i(s)))ds \\
&\qquad\quad - m_i(t_2) \int_0^{\beta_i(t_1)} g_i(t_1, s, x(\eta_i(s)), y(\eta_i(s)))ds \Big| \\
&\qquad \le m_T \Big| \int_0^{\beta_i(t_2)} [g_i(t_2, s, x(\eta_i(s)), y(\eta_i(s))) \\
&\qquad\quad - g_i(t_1, s, x(\eta_i(s)), y(\eta_i(s)))]ds \Big| \\
&\qquad\quad + m_T \Big| \int_{\beta_i(t_1)}^{\beta_i(t_2)} g_i(t_1, s, x(\eta_i(s)), y(\eta_i(s)))ds \Big| \\
&\qquad \le m_T \beta_T \ \omega_r^T(g_i, \varepsilon) + m_T U_r^T \ \omega^T(\beta_i, \varepsilon),
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
&r = \max\{\sup\{\|x\| : x \in X_1\}, \sup\{\|x\| : x \in X_2\}\}, \\
&\omega^T(\beta_i, \varepsilon) = \{|\beta_i(t_1) - \beta_i(t_2)| : t_1, t_2 \in [0, T], |t_1 - t_2| \le \varepsilon\}, \\
&\omega_r^T(g_i, \varepsilon) = \sup\{|g_i(t_2, s, x, y) - g_i(t_1, s, x, y)| : t_1, t_2 \in [0, T], |t_2 - t_1| \le \varepsilon, \\
&\qquad\qquad\qquad x, y \in [-r, r], s \in [0, \beta_T]\}, \\
&U_r^T = \sup\{|g_i(t, s, x, y)| : t \in [0, T], s \in [0, \beta_T], x, y \in [-r, r]\}.
\end{aligned}$$

Since (x, y) was an arbitrary element of $X_1 \times X_2$ in (4.7), so we obtain

$$\omega^T(G_i(X_1 \times X_2), \varepsilon) \ \le \ m_T \beta_T \ \omega_r^T(g_i, \varepsilon) + m_T U_r^T \omega^T(\beta, \varepsilon). \tag{4.8}$$

On the other hand by the uniform continuity of g_i on $[0, T] \times [0, \beta_T] \times [-r, r] \times [-r, r]$, we have $\omega_r^T(g_i, \varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$ and also because of the uniform continuity of β on $[0, T]$, we derive that $\omega^T(\beta, \varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Therefore we obtain

$$m_T \beta_T \ \omega_r^T(g_i, \varepsilon) + m_T U_r^T \omega^T(\beta, \varepsilon) \longrightarrow 0,$$

as $\varepsilon \longrightarrow 0$ and

$$\omega_0^T(G_i(X_1 \times X_2)) = 0,$$

therefore

$$\omega_0(G_i(X_1 \times X_2)) = 0. \tag{4.9}$$

Finally, for arbitrary $(x, y), (u, v) \in X_1 \times X_2$ and $t \in \mathbb{R}_+$ we get

$$\begin{aligned}
&\Big| G_i(x,y)(t) - G_i(u,v)(t) \Big| \le \\
&\qquad \le m_i(t) \Big| \int_0^{\beta_i(t)} [g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) \\
&\qquad\quad - g_i(t, s, u(\eta_i(s)), v(\eta_i(s))]ds \Big| \\
&\qquad \le m_i(t) \theta_i(t),
\end{aligned} \tag{4.10}$$

where

$$\theta_i(t) = \sup\Big\{\Big| \int_0^{\beta_i(t)} [g_i(t,s,x(\eta_i(s)), y(\eta_i(s))ds) - g_i(t,s,u(\eta_i(s)), v(\eta_i(s))]ds\Big| : \\ x,y,u,v \in BC(\mathbb{R}_+)\Big\}.$$

Since (x,y), (u,v) and t were chosen arbitrarily in (4.10), we conclude that

$$diamG_i(X_1 \times X_2)(t) \quad \leq \quad m(t)\theta(t). \tag{4.11}$$

Taking $t \longrightarrow \infty$ in the inequality (4.11), then using (iv) we deduce that

$$\limsup_{t \longrightarrow \infty} diamG_i(X_1 \times X_2)(t) = 0. \tag{4.12}$$

Further, combining (4.9) and (4.12) we get

$$\limsup_{t,s \longrightarrow \infty} diamG_i(X_1 \times X_2)(t) + \omega_0(G_i(X_1 \times X_2)) = 0$$

or, equivalently

$$\mu(G_i(X_1 \times X_2)) = 0.$$

Thus, G_i is a compact and continuous operator.

Step 2: There exists $r_0 \in \mathbb{R}_+$ such that the operators $T_i : \bar{B}_{r_0} \times \bar{B}_{r_0} \longrightarrow \bar{B}_{r_0}$ $(i = 1, 2)$ defined by

$$T_i(x,y)(t) = f_i\Big(t, x(\xi(t)), y(\xi(t)), \int_0^{\beta(t)} g_i(t,s,x(\eta(s)), y(\eta(s))ds)\Big) \tag{4.13}$$

are well defined and satisfy condition (3.5) *where G_i is given by* (4.5) *and*

$$F_i(x,y)(t) = k \max\{x(t), y(t)\},$$

for $i = 1, 2$.

Using conditions (i)-(iv), for arbitrarily fixed $t \in \mathbb{R}_+$ and $i = 1, 2$ we get

(4.14)

$$\begin{aligned}
|T_i(x,y)(t)| \leq& \\
&\leq \Big|f_i(t, x(\xi_i(t)), y(\xi_i(t)), \int_0^{\beta_i(t)} g_i(t,s,x(\eta_i(s)), y(\eta_i(s))ds) - f_i(t,0,0,0)\Big| \\
&\quad + |f_i(t,0,0,0)| \\
&\leq k \max\{|x(\xi_i(t))|, |y(\xi_i(t))|\} + |f_i(t,0,0,0)| \\
&\quad + \Phi_i\Big(m_i(t)\Big| \int_0^{\beta_i(t)} g_i(t,s,x(\eta_i(s)), y(\eta_i(s)))ds\Big|\Big) \\
&\leq k \max\{\|x\|, \|y\|\} + M_i + \Phi_i(D),
\end{aligned}$$

therefore,

$$\|T_i(x,y)\| \leq k \max\{\|x\|, \|y\|\} + M_i + \Phi_i(D). \tag{4.15}$$

Thus, from the estimate (4.15) we have $T_i(\bar{B}_{r_0} \times \bar{B}_{r_0}) \subseteq \bar{B}_{r_0}$ for

$$r_0 = \max\{\frac{M_1 + \Phi_1(D)}{1-k}, \frac{M_2 + \Phi_2(D)}{1-k}\}.$$

Next, by condition (ii) of Theorem 4.1, it is obvious that F_i and $F_i(x)$ for $x \in BC(\mathbb{R}_+)$ are continuous functions on $BC(\mathbb{R}_+)$ and $\mathbb{R}_+$, respectively, and for $i = 1, 2$, $x, y, u, v \in BC(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
|T_i(x,y)(t) - T_i(u,v)(t)| &= \\
&= \Big| f_i\Big(t, x(\xi_i(t)), y(\xi_i(t)), \int_0^{\beta_i(t)} g_i(t,s,x(\eta_i(s)), y(\eta_i(s))ds)\Big) \\
&\quad - f_i\Big(t, u(\xi_i(t)), v(\xi_i(t)), \int_0^{\beta_i(t)} g_i(t,s,u(\eta_i(s)), v(\eta_i(s))ds)\Big)\Big| \\
&\le k\max\{|x(t)-u(t)|, |y(t)-v(t)|\} \\
&\quad + \Phi(m_i(t)\Big|\int_0^{\beta_i(t)} g_i(t,s,x(\eta_i(s)), y(\eta_i(s))ds \\
&\qquad - \int_0^{\beta_i(t)} g_i(t,s,u(\eta_i(s)), v(\eta_i(s))ds\Big|) \\
&\le |F_i(x,y)(t) - F_i(u,v)(t)| + \Phi(|G_i(x,y)(t) - G_i(u,v)(t)|) \\
&\le \|F_i(x,y) - F_i(u,v)\| + \Phi(\|G_i(x,y) - G_i(u,v)\|),
\end{aligned}$$

therefore,

$$\|T_i(x,y) - T_i(u,v)\| \le \|F_i(x,y) - F_i(u,v)\| + \Phi(\|G_i(x,y) - G_i(u,v)\|).$$

Obviously, F_i satisfies condition (3.4) and thus by Corollary 3.4, there exist $x_0, y_0 \in BC(\mathbb{R}_+)$ that are solutions of the system of integral equations (1.1), and the proof is complete. □

In the same way as the above proof, we can extend Theorem 4.1 for finite system of nonlinear integral equation

$$x_i(t) = f_i\Big(t, x_1(\xi_i(t)), \cdots, x_n(\xi_i(t)), \int_0^{\beta_i(t)} g_i(t,s,x_1(\eta_i(s)), \cdots, x_n(\eta_i(s)))ds\Big)$$

where f_i, g_i, ξ_i, η_i and β_i satisfy certain conditions. As a corollary of Theorem 4.1 we have the main results of [5].

Corollary 4.2. *[5] Suppose that (i) $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to f(t,0,0)$ is a member of the space $BC(\mathbb{R}_+)$;*

(ii) there exists $k \in [0,1)$ such that

$$|f(t,x,y) - f(t,u,v)| \le \frac{k}{2}(|x-u| + |y-v|), \tag{4.16}$$

for any $t \ge 0$ and for all $x, y, u, v \in \mathbb{R}$;

(iii) the functions ξ, η, $q : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\xi(t) \to \infty$ as $t \to \infty$.

(iv) $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist $x_0, y_0 \in \mathbb{R}$ and a positive constant d such that

$$\int_0^{q(t)} |h(t, s, x_0, y_0)|\, ds \leq d \tag{4.17}$$

for all $t \in \mathbb{R}_+$. In addiition,

$$\lim_{t\to\infty} \int_0^{q(t)} |h(t, s, x(\eta(s)), y(\eta(s))) - h(t, s, u(\eta(s)), v(\eta(s))|\, ds = 0, \tag{4.18}$$

$$\int_0^{q(t)} |h(t, s, x(\eta(s)), y(\eta(s))) - h(t, s, u(\eta(s)), v(\eta(s))|\, ds \leq \infty \tag{4.19}$$

for any $t \in \mathbb{R}_+$ and uniformly respect to $x, y, u, v \in BC(\mathbb{R}_+)$.

Then the system of equations

$$\begin{cases} x(t) = f(t, x(\xi(t)), y(\xi(t))) + \int_0^{q(t)} h(t, s, x(\eta(s)), y(\eta(s)))ds, \\ y(t) = f(t, y(\xi(t)), x(\xi(t))) + \int_0^{q(t)} h(t, s, y(\eta(s)), x(\eta(s)))ds, \end{cases} \tag{4.20}$$

has at least one solution in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

Proof. Take

$$\begin{aligned} f_1(t, x, y, z) &= f(t, x, y) + z, \\ f_2(t, x, y, z) &= f(t, y, x) + z, \\ g_1(t, s, x, y) &= h(t, s, x, y), \\ g_1(t, s, x, y) &= h(t, s, y, x) \end{aligned}$$

in Theorem 4.1. □

Now, we give an example where Theorem 4.1 can be applied but the previous results [5] are not applicable.

Example 4.3. Consider the system of integral equations

(4.21)
$$\begin{cases} x(t) = \frac{t^2(x(t)+y(t))}{2(1+t^4)} + \int_0^{t^2} \frac{s^3\cos(sx(\sqrt{s}))+e^s(2+\sin(x^4(\sqrt{s})+y^4(\sqrt{s})))}{e^{t^2}(2+\sin(x^4(\sqrt{s})+y^4(\sqrt{s})))} ds, \\ \\ y(t) = \frac{\sin(t^2(x(t)+y(t)))}{2(1+t^4)} + \arctan \int_0^{\sqrt{t}} \frac{\sqrt[4]{1+sy(s)}+ts^{11}(1+x^4(s)+y^4(s))}{(1+t^7)(1+x^4(t)+y^4(t))} ds, \end{cases}$$

where $t \in [0, \infty)$.

Eq. (4.21) is a special case of Eq. (1.1) where

$$\begin{aligned}
&\xi_1(t) = \xi_2(t) = \eta_2(t) = t,\ \beta_1(t) = t^2,\ \beta_2(t) = \eta_1(t) = \sqrt{s}\\
&f_1(t,x,y,z) = \frac{t^2(x+y)}{2(1+t^4)} + z,\\
&f_2(t,x,y,z) = \frac{\sin(t^2(x+y))}{2(1+t^4)} + \arctan z,\\
&g_1(t,s,x,y) = \frac{s^3\cos(sx) + e^s(2+\sin(x^4+y^4))}{e^{t^2}(2+\sin(x^4+y^4))},\\
&g_1(t,s,x,y) = \frac{\sqrt[4]{1+sy} + ts^{11}(1+x^4+y^4)}{(1+t^7)(1+x^4+y^4)}.
\end{aligned}$$

Now we check all conditions of Theorem 4.1. It is clear that condition (i) is satisfied. Assume that $t \in \mathbb{R}_+$ and $x, y, z, u, v, w \in \mathbb{R}$. Then we get

$$\begin{aligned}
|f_1(t,x,y,z) - f_1(t,u,v,w)| &\leq \frac{t^2}{1+t^4}\frac{|x-u|+|y-v|}{2} + |z-w|\\
&\leq \frac{1}{2}\max\{|x-u|,|y-v|\} + |z-w|
\end{aligned}$$

and

$$\begin{aligned}
|f_2(t,x,y,z) - f_2(t,u,v,w)| &\leq \frac{|\sin(t^2(x-u+y-v))|}{2(1+t^4)}\\
&\quad +|\arctan(z) - \arctan(w)|\\
&\leq \frac{1}{2}\max\{|x-u|,|y-v|\} + |z-w|.
\end{aligned}$$

Therefore f_1 and f_2 satisfy condition (ii) of Theorem 4.1 with $k = \frac{1}{2}$. Also it is clear that f_i and g_i are continuous and by simple calculation we obtain that

$$\begin{aligned}
&M_1 = \sup\{\frac{t^2(0+0)}{2(1+t^4)} + 0 : t \in \mathbb{R}_+\} = 0,\\
&M_2 = \sup\{\frac{\sin(t^2(0+0))}{2(1+t^4)} + 0 : t \in \mathbb{R}_+\} = 0,\\
&\Big|\frac{s^3\cos(sx(\sqrt{s})) + e^s(2+\sin(x^4(\sqrt{s})+y^4(\sqrt{s})))}{e^{t^2}(2+\sin(x^4(\sqrt{s})+y^4(\sqrt{s})))}\Big| \leq \Big|\frac{s^3+2e^s}{e^{t^2}}\Big|,\\
&\lim_{t\longrightarrow\infty}\Big|\int_0^{\sqrt{t}} \frac{\sqrt[4]{1+sx(s)} + ts^{11}(1+x^4(t)+y^4(s))}{(1+t^7)(1+x^4(t)+y^4(t))}ds\Big| = \frac{1}{12},\\
&|g_1(t,s,x(\eta_1(s)),y(\eta_1(s))) - g_1(t,s,u(\eta_1(s)),v(\eta_1(s)))| \leq \frac{2s^3}{e^{t^2}},\\
&|g_2(t,s,x(\eta_2(s)),y(\eta_2(s))) - g_2(t,s,u(\eta_2(s)),v(\eta_2(s)))| \leq \frac{2(1+s)}{1+t^7}.
\end{aligned}$$

Thus, $D \leq \infty$ and we have

$$\lim_{t\longrightarrow\infty}\Big|\int_0^{\beta_i(t)} g_i(t,s,x(\eta_i(s)),y(\eta_i(s))) - g_i(t,s,u(\eta_i(s)),v(\eta_i(s)))ds\Big| = 0$$

Therefore, as a result of Theorem 4.1, the system of integral equations (4.21) has at least one solution in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

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Received by the editors October 10, 2012