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SOME COMMON FIXED POINT THEOREMS IN METRIC SPACES UNDER A DIFFERENT SET OF CONDITIONS

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Abstract. The purpose of this paper is to prove some new common fixed point theorems in metric spaces for weakly compatible mapping satisfying an implicit relation under a different set of conditions, which unify and generalize most of the existing relevant fixed point theorems. While proving our results, we utilize an implicit function due to Popa et al. [Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces. Bull. Malays. Math. Sci. Soc. (2) 33 (1) (2010), 105-120] keeping in view their unifying power. Our results improve some recent results contained in Imdad and Ali [Jungck's common fixed point theorem and (E.A) property. Acta Math. Sinica, Eng. Ser. 24(1) (2008), 87-94]. Some related results and illustrative examples to highlight the realized improvements are also furnished.

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1. Introduction and Preliminaries

In nonlinear functional analysis, fixed point theory is indispensable due to its wide application to nonlinear sciences besides various research fields in mathematics. One of the essential and initial result in this direction was proved by Stefan Banach [3] in 1922. The classical results of Banach [3] and Edelstein [11] continue to be the source of inspiration for many researchers working in the area of metric fixed point theory. A metrical common fixed point theorem generally involves conditions on commutativity, continuity, completeness and suitable containment of ranges of the involved mappings besides an appropriate contraction condition and researchers in this domain are aimed at weakening one or more of these conditions.

In 1976, Jungck [22] generalized the Banach contraction principle by using the notion of commuting mappings and settled the open problem that a pair of commuting and continuous self-mappings on the unit interval $[0, 1]$ need not have a common fixed point [4, 13]. He also generalized the idea of weakly commuting mappings due to Sessa [42] and showed that the compatible pair

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of mappings commutes on the set of coincidence points. After this definition, there came a host of definitions which are not relevant in the present context.

Jungck and Rhoades [27] (also Dhage [10]) termed a pair of self-mappings to be coincidentally commuting (or weakly compatible) if they merely commute at their coincidence points. The study of common fixed points for non-compatible mappings is equally interesting (cf. Pant [35]). Consequently, the recent literature of fixed point theory has witnessed the evolution of several weak conditions of commutativity such as; Compatible mappings of type (A) [24], Compatible mappings of type (B), Compatible mappings of type (P), Compatible mappings of type (C), Biased mappings [25], R-weakly commuting mappings and several others whose lucid survey and illustration are available in Murthy [31]. For more details on systematic comparisons and illustrations of these described notions, we also refer to Singh and Tomar [44], Murthy [31] and Kadelburg et al. [29]. In what follows, we choose to utilize the most natural and weak condition amongst all the commutativity conditions, namely, 'weak compatibility' due to Jungck [26]. Thereafter, many authors established a host of classical metrical common fixed point theorems (e.g. [5, 6, 7, 8, 18, 19, 20, 28, 45]).

The tradition of improving contraction conditions in fixed and common fixed point theorems is still in fashion. For an extensive collection of contraction conditions one can refer to Rhaoades [39, 40] and references cited therein. Most recently, with a view to accommodate many contraction conditions, Popa [37, 38] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions.

The object of this paper is to prove some common fixed point theorems in metric spaces for weakly compatible mapping satisfying an implicit relation under a different set of conditions. We generalize the result of Imdad and Ali [16] (Theorem 2.10, mentioned in Section 2) without using the property (E.A). Also the completeness requirement of the space is replaced with a relatively more natural condition (given in Section 3). We utilize implicit functions to prove our results because of their versatility of deducing several contraction conditions at the same time. Also, we furnish some illustrative examples to highlight the superiority of our results over several results existing in the literature.

The following relevant known definitions will be needed in our subsequent discussion.

Definition 1.1. A pair (A, S) of self-mappings defined on a metric space (X, d) is said to be

1. compatible [23] if $\lim\limits_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = t$ for some $t \in X$,

2. non-compatible [34] if there exists at least one sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = t$ for some $t \in X$, but $\lim\limits_{n\to\infty} d(ASx_n, SAx_n)$ is either non-zero or non-existent,

3. weakly compatible [33] if the mappings commute at their coincidence points, that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$,

4. tangential (or satisfying the property (E.A)) ([1, 41]) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 1.2 ([17])**.** Two finite families of self-mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ of a non-empty set X are said to be pairwise commuting if

1. $A_iA_j = A_jA_i$, $i, j \in \{1, 2, \ldots, m\}$,
2. $S_kS_l = S_lS_k$, $k, l \in \{1, 2, \ldots, p\}$,
3. $A_iS_k = S_kA_i$, $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, p\}$.

2. Implicit relations

Popa [36, 37] proved several fixed point theorems satisfying suitable implicit relations. For proving such results, Popa [36, 37] considered Ψ to be the set of all continuous functions $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ satisfying the following conditions:

(ψ_1) ψ is non-increasing in variables t_5 and t_6,

(ψ_2) there exists $k \in (0, 1)$ such that for $u, v \geq 0$ with

(ψ_{2a}) $\psi(u, v, v, u, u + v, 0) \leq 0$ or

(ψ_{2b}) $\psi(u, v, u, v, 0, u + v) \leq 0$ implies $u \leq kv$,

(ψ_3) $\psi(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

Some of the following examples of such functions ψ satisfying ψ_1, ψ_2 and ψ_3 are taken from Popa [37] and Imdad and Ali [16].

Example 2.1. Define $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ as:

$$\psi(t_1, t_2, \ldots, t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}, \quad k \in (0, 1). \tag{2.1}$$

$$\psi(t_1, t_2, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - d' t_5 t_6, \tag{2.2}$$

where $a > 0$, $b, c, d' \geq 0$, $a + b + c < 1$ and $a + d' < 1$.

$$\psi(t_1, t_2, \ldots, t_6) = t_1^3 - at_1^2 t_2 - bt_1 t_3 t_4 - ct_5^2 t_6 - d' t_5 t_6^2, \tag{2.3}$$

where $a > 0$, $b, c, d' \geq 0$, $a + b < 1$ and $a + c + d' < 1$.

$$\psi(t_1, t_2, \ldots, t_6) = t_1^3 - c\frac{t_3^2 t_4^2 + t_5^2 t_6^2}{t_2 + t_3 + t_4 + 1}, \quad c \in (0, 1). \tag{2.4}$$

(2.5) $$\psi(t_1,t_2,\ldots,t_6)=t_1^2-at_2^2-b\frac{t_5t_6}{t_3^2+t_4^2+1},$$

where $a>0$, $b\geq 0$ and $a+b<1$.

(2.6) $$\psi(t_1,t_2,\ldots,t_6) = t_1^2-a\max\left\{t_2^2,t_3^2,t_4^2\right\}-b\max\left\{t_3t_5,t_4t_6\right\}-ct_5t_6,$$

where $a>0$, $b,c\geq 0$, $a+2b<1$ and $a+c<1$.

(2.7) $$\psi(t_1,t_2,\ldots,t_6)=t_1-k\max\left\{t_2,t_3,t_4,\frac{t_5}{2},\frac{t_6}{2}\right\},\quad k\in(0,1).$$

(2.8) $$\psi(t_1,t_2,\ldots,t_6)=t_1-k\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\},\quad k\in(0,1).$$

(2.9) $$\psi(t_1,t_2,\ldots,t_6)=t_1-(at_2+bt_3+ct_4+d^{'}t_5+et_6),$$

where $a+b+c+d^{'}+e<1$ and $d^{'},e\geq 0$.

(2.10) $$\psi(t_1,t_2,\ldots,t_6)=t_1-\frac{k}{2}\max\left\{t_2,t_3,t_4,t_5,t_6\right\},\quad k\in(0,1).$$

(2.11) $$\psi(t_1,t_2,\ldots,t_6)=t_1-\left[at_2+bt_3+ct_4+d^{'}(t_5+t_6)\right],$$

where $a+b+c+2d^{'}<1$ and $d^{'}\geq 0$.

Since verifications of requirements (ψ_1), (ψ_2) and (ψ_3) for Examples (2.1)-(2.11) are straightforward, hence details are omitted. Here one may further notice that some other well known contraction conditions ([12], [15], [21]) can also be deduced as particular cases of implicit relation of Popa [37]. In order to strengthen this viewpoint, we add some more examples to this effect and utilize them to demonstrate how this implicit relation can cover several other known contractive conditions and is also good enough to yield further unknown natural contractive conditions as well.

Example 2.2. Define $\psi(t_1,t_2,\ldots,t_6):\mathcal{R}_+^6\to\mathcal{R}$ as:

(2.12)
$$\psi(t_1,t_2,\ldots,t_6)=\begin{cases}t_1-a_1\dfrac{t_3^2+t_4^2}{t_3+t_4}-a_2t_2-a_3(t_5+t_6), & \text{if } t_3+t_4\neq 0;\\ t_1, & \text{if } t_3+t_4=0,\end{cases}$$

where $a_i\geq 0$ $(i=1,2,3)$ with at least one a_i non-zero and $a_1+a_2+2a_3<1$.

(ψ_1) Obvious.

(ψ_{2a}) Let $u > 0$. Then $\psi(u, v, v, u, u+v, 0) = u - a_1 \dfrac{v^2 + u^2}{(v+u)} - a_2 v - a_3(u+v) \leq 0$. If $u \geq v$, then $u \leq (a_1 + a_2 + 2a_3)u < u$ which is a contradiction. Hence $u < v$ and $u \leq kv$ where $k \in (0, 1)$.

(ψ_{2b}) Similar argument as in (ψ_{2a}).

(ψ_3) $\psi(u, u, 0, 0, u, u) = u > 0$, for all $u > 0$.

Example 2.3. Define $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ as:
(2.13)

$$\psi(t_1, t_2, \ldots, t_6) = \begin{cases} t_1 - a_1 t_2 - \dfrac{a_2 t_3 t_4 + a_3 t_5 t_6}{t_3 + t_4}, & \text{if } \; t_3 + t_4 \neq 0; \\ t_1, & \text{if } t_3 + t_4 = 0, \end{cases}$$

where $a_1, a_2, a_3 \geq 0$ such that $1 < 2a_1 + a_2 < 2$.

Example 2.4. Define $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ as:

$$\begin{aligned} (2.14)\ \psi(t_1, t_2, \ldots, t_6) \quad &= \quad t_1 - a_1 \Bigg[a_2 \max \left\{ t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6) \right\} \\ &\quad + (1 - a_2) \left[\max \left\{ t_2^2, t_3 t_4, t_5 t_6, \frac{t_3 t_6}{2}, \frac{t_4 t_5}{2} \right\} \right]^{\frac{1}{2}} \Bigg], \end{aligned}$$

where $a_1 \in (0, 1)$ and $0 \leq a_2 \leq 1$.

Example 2.5. Define $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ as:

$$(2.15)\ \ \psi(t_1, t_2, \ldots, t_6) = t_1^2 - a_1 \max\{t_2^2, t_3^2, t_4^2\} - a_2 \max \left\{ \frac{t_3 t_5}{2}, \frac{t_4 t_6}{2} \right\} - a_3 t_5 t_6,$$

where $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 < 1$.

Very recently, Popa et al. [38] proved several fixed point theorems satisfying suitable implicit relations in which Husain and Sehgal [14] type contraction conditions ([9], [30], [32], [43]) can be deduced from similar implicit relations in addition to all earlier ones if there is a slight modification in condition (ψ_1) as follows:

($\psi_1^{'}$) ψ is decreasing in variables $t_2, \ldots, t_6$.

Hereafter, let $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ be a continuous function which satisfies the conditions ($\psi_1^{'}$), (ψ_2) and (ψ_3) and $\mathcal{F}$ be the family of such functions ψ. In this paper, we employ such implicit relation to prove our results. But before we proceed further, let us furnish some examples to highlight the utility of the modifications instrumented herein.

Example 2.6. Define $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ as:

$$(2.16) \qquad \psi(t_1, t_2, \ldots, t_6) = t_1 - \phi\left(\max \left\{ t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6) \right\} \right),$$

where $\phi : \mathcal{R}^+ \to \mathcal{R}^+$ is an increasing upper semi-continuous function with $\phi(0) = 0$ and $\phi(t) < t$ for each $t > 0$.

(ψ_1') Obvious.

(ψ_{2a}) Let $u > 0$. Then $\psi(u, v, v, u, u+v, 0) = u - \phi(\max\{v, v, u, \frac{1}{2}(u+v)\}) < 0$. If $u \geq v$, then $u \leq \phi(u) < u$, which is a contradiction. Hence $u < v$ and $u \leq kv$ where $k \in (0, 1)$.

(ψ_{2b}) Similar argument as in (ψ_{2a}).

(ψ_3) $\psi(u, u, 0, 0, u, u) = u - \phi(\max\{u, 0, 0, \frac{1}{2}(u+v)\}) = u - \phi(u) > 0$, for all $u > 0$.

Example 2.7. Define $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ as:

$$\psi(t_1, t_2, \ldots, t_6) = t_1 - \phi(t_2, t_3, \ldots, t_6), \tag{2.17}$$

where $\phi : \mathcal{R}_+^5 \to \mathcal{R}^+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\phi(t, t, at, bt, ct) < t$ for each $t > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Example 2.8. Define $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}_+^6 \to \mathcal{R}$ as:

$$\psi(t_1, t_2, \ldots, t_6) = t_1^2 - \phi(t_2^2, t_3t_4, t_5t_6, t_3t_6, t_4t_5), \tag{2.18}$$

where $\phi : \mathcal{R}_+^5 \to \mathcal{R}^+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\phi(t, t, at, bt, ct) < t$ for each $t > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Here it may be noticed that all earlier mentioned examples continue to enjoy the format of modified implicit relation as adopted herein.

Motivated by the fact that a fixed point of any map on metric spaces can always be viewed as a common fixed point of that map and identity map on the space. Jungck [22] proved the following interesting generalization of celebrated Banach contraction principle. While proving his result, Jungck [22] replaced identity map with a continuous mapping. The result is as follows:

Theorem 2.9 ([22])**.** *Let T be a continuous mapping of a complete metric space (X, d) into itself. Then T has a fixed point in X if there exists $\alpha \in (0, 1)$ and a mapping $S : X \to X$ which commutes with T and satisfies $S(X) \subset T(X)$ and $d(Sx, Sy) \leq \alpha d(Tx, Ty)$, for all $x, y \in X$.*

In [16], Imdad and Ali established a general common fixed point theorem for a pair of mappings using a suitable implicit function without the requirement of the containment of ranges.

Theorem 2.10 ([16])**.** *Let T and I be self-mappings of a metric space (X, d) such that:*

1. *T and I satisfy the property (E.A),*
2. *for all $x, y \in X$ and $\psi \in \Psi$,*

$$\psi(d(Tx, Ty), d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)) \leq 0,$$

3. *$I(X)$ is a complete subspace of X.*

   Then

   (a) *the pair (T, I) has a point of coincidence,*

   (b) *the pair (T, I) has a common fixed point provided it is weakly compatible.*

Remark 2.11. Theorem 2.10 is a generalized and improved form of the result of Jungck [22] (Theorem 2.9 above) without any continuity requirement. Also the commutativity requirement is reduced to points of coincidence along with replacement of the completeness of the space with a natural condition.

3. Common fixed point theorems for two self-mappings

In this paper, we prove a general fixed point theorem for a pair of mappings under a different set of conditions for weakly compatible mapping satisfying an implicit relation which unify and generalize most of the existing relevant fixed point theorems. While proving our results, we utilize the idea of implicit functions due to Popa et al. [38]. We also utilize our main theorem to demonstrate how several fixed point theorems can be unified by using an implicit function.

Theorem 3.1. *Let A and S be two self-mappings of a metric space (X, d) such that*

1. *$\overline{A(X)} \subseteq S(X)$,*

2. *for all $x, y \in X$ and some $\psi \in \Psi$,*

$$\psi\begin{pmatrix} d(Ax, Ay), d(Sx, Sy), d(Sx, Ax), \\ d(Sy, Ay), d(Sx, Ay), d(Sy, Ax) \end{pmatrix} \leq 0, \tag{3.1}$$

3. *$\overline{A(X)}$ is a complete subspace of X.*

   Then the pair (A, S) has a point of coincidence.

Moreover, the mappings A and S have a unique common fixed point in X provided the pair (A, S) is weakly compatible.

Proof. Let x_0 be an arbitrary element in X. Then due to (1), i.e., $\overline{A(X)} \subseteq S(X)$ which implies $A(X) \subseteq \overline{A(X)} \subseteq S(X)$, hence one can inductively define a sequence

$$\{Ax_0, Ax_1, Ax_2, \ldots, Ax_n, Ax_{n+1}, \ldots\}, \tag{3.2}$$

such that $Ax_n = Sx_{n+1}$ for $n = 0, 1, 2, \ldots$. Now, we show that the sequence defined by (3.2) is Cauchy. Using (3.1) with $x = x_n$ and $y = x_{n+1}$, we have

$$\psi\begin{pmatrix} d(Ax_n, Ax_{n+1}), d(Sx_n, Sx_{n+1}), d(Sx_n, Ax_n), \\ d(Sx_{n+1}, Ax_{n+1}), d(Sx_n, Ax_{n+1}), d(Sx_{n+1}, Ax_n) \end{pmatrix} \leq 0.$$

As $Ax_n = Sx_{n+1}$ for $n = 0, 1, 2, \ldots$, we have

$$\psi\left(\begin{array}{c} d(Ax_n, Ax_{n+1}), d(Sx_n, Ax_n), d(Sx_n, Ax_n), \\ d(Ax_n, Ax_{n+1}), d(Sx_n, Ax_{n+1}), 0 \end{array}\right) \leq 0.$$

Since ψ is non-decreasing in variable t_5, we have

$$\psi\left(\begin{array}{c} d(Ax_n, Ax_{n+1}), d(Sx_n, Ax_n), d(Sx_n, Ax_n), \\ d(Ax_n, Ax_{n+1}), d(Sx_n, Ax_n) + d(Ax_n, Ax_{n+1}), 0 \end{array}\right) \leq 0.$$

Now, using property (ψ_{2a}), we have

$$d(Ax_n, Ax_{n+1}) \leq kd(Sx_n, Ax_n) = kd(Ax_{n-1}, Ax_n),$$

and so

$$d(Ax_n, Ax_{n+1}) \leq k^n d(Ax_0, Ax_1), \quad \text{for all } n \geq 0.$$

Hence by a routine calculation, it follows that $\{Ax_n\}$ is a Cauchy sequence. Since $\overline{A(X)}$ is a complete subspace of X, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{n+1} \in \overline{A(X)} \subseteq S(X) \subset X,$$

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{n+1} = t \in S(X).$$

Hence there exists $u \in X$ such that $Su = t$. We assert that $Su = Au$. If not, then $d(Au, Su) > 0$. Using (3.1) with $x = u$ and $y = x_n$, we get

$$\psi\left(\begin{array}{c} d(Ax, Ax_n), d(Su, Sx_n), d(Su, Au), \\ d(Sx_n, Ax_n), d(Su, Ax_n), d(Sx_n, Au) \end{array}\right) \leq 0.$$

Taking limit as $n \to \infty$ we get

$$\psi(d(Au, t), d(Su, t), d(Su, Au), d(t, t), d(Su, t), d(t, Au)) \leq 0,$$

or,

$$\psi(d(Au, Su), 0, d(Su, Au), 0, 0, d(Su, Au)) \leq 0,$$

yielding thereby (due to (ψ_{2b})) $d(Au, Su) \leq 0$ which is a contradiction. Then we have $Au = Su$, which shows that u is a coincidence point of A and S. Since the pair (A, S) is weakly compatible, we have $St = SAu = ASu = At$. Now, we show that t is a common fixed point of mappings A and S. We assert that $At = t$. If not then $d(At, t) > 0$. Again using (3.1) with $x = t$ and $y = u$, we have

$$\psi(d(At, Au), d(St, Su), d(St, At), d(Su, Au), d(St, Au), d(Su, At)) \leq 0,$$

or,

$$\psi(d(At, t), d(At, t), 0, 0, d(At, t), d(t, At)) \leq 0,$$

which contradicts (ψ_3). Hence $At = t$ or $St = At = t$. This shows that t is a common fixed point of A and S. The uniqueness of common fixed point is an easy consequence of implicit relation (3.1) in view of (ψ_3). This completes the proof. □

Remark 3.2. 1. Theorem 3.1 is a generalized and improved form of Theorem 2.10 (due to Imdad and Ali [16]) in which a relatively more natural condition is used in place of the property (E.A).

2. In 1976, Jungck[22] used the continuity and commutativity of the mapping but in our result continuity requirement is replaced by the closure of the range of one mapping into the range of the other, i.e., $\overline{A(X)} \subseteq S(X)$ and commutativity requirement is reduced to points of coincidence along with the completeness of the subspace $\overline{A(X)}$. Thus Theorem 3.1 is a generalized form of the results of Imdad and Ali [16] and Jungck [22].

From Theorem 3.1, we can deduce a host of corollaries which are embodied in the following:

Corollary 3.3. *The conclusions of Theorem 3.1 remain true if for all* $x, y \in X$, $(x \neq y)$, *the implicit relation* (3.1) *is replaced by one of the following:*

$$d(Ax, Ay) \leq k \max\Big\{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), \frac{1}{2}[d(Sx, Ay) + d(Sy, Ax)]\Big\}, \quad k \in (0, 1). \tag{3.3}$$

$$d^2(Ax, Ay) \leq d(Ax, Ay)[ad(Sx, Sy) + bd(Sx, Ax) + cd(Sy, Ay)] + d' d(Sx, Ay)d(Sy, Ax), \tag{3.4}$$

where $a > 0$, $b, c, d' \geq 0$, $a + b + c < 1$ *and* $a + d' < 1$.

$$d^3(Ax, Ay) \leq ad^2(Ax, Ay)d(Sx, Sy) + bd(Ax, Ay)d(Sx, Ax)d(Sy, Ay) + cd^2(Sx, Ay)d(Sy, Ax) + d' d(Sx, Ay)d^2(Sy, Ax), \tag{3.5}$$

where $a > 0$, $b, c, d' \geq 0$, $a + b < 1$ *and* $a + c + d' < 1$.

$$d^3(Ax, Ay) \leq c\frac{d^2(Sx, Ax)d^2(Sy, Ay) + d^2(Sx, Ay)d^2(Sy, Ax)}{d(Sx, Sy) + d(Sx, Ax) + d(Sy, Ay) + 1}, \tag{3.6}$$

where $c \in (0, 1)$.

$$d^2(Ax, Ay) \leq ad^2(Sx, Sy) + b\frac{d(Sx, Ay)d(Sy, Ax)}{d^2(Sx, Ax) + d^2(Sy, Ay) + 1}, \tag{3.7}$$

where $a > 0$, $b \geq 0$ *and* $a + b < 1$.

$$d^2(Ax, Ay) \leq a \max\{d^2(Sx, Sy), d^2(Sx, Ax), d^2(Sy, Ay)\} + b \max\{d(Sx, Ax)d(Sx, Ay), d(Sy, Ay)d(Sy, Ax)\} + cd(Sx, Ay)d(Sy, Ax), \tag{3.8}$$

where $a > 0,\ b, c \geq 0,\ a + 2b < 1$ *and* $a + c < 1$.

$$\text{(3.9)} \quad d(Ax, Ay) \leq k \max\Big\{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), \frac{1}{2}d(Sx, Ay), \frac{1}{2}d(Sy, Ax)\Big\}, \quad k \in (0, 1).$$

$$\text{(3.10)} \quad d(Ax, Ay) \leq k \max\Big\{d(Sx, Sy), \frac{1}{2}[d(Sx, Ax) + d(Sy, Ay)], \frac{1}{2}[d(Sx, Ay) + d(Sy, Ax)]\Big\}, \quad k \in (0, 1).$$

$$\text{(3.11)} \quad d(Ax, Ay) \leq ad(Sx, Sy) + bd(Sx, Ax) + cd(Sy, Ay) + d^{'} d(Sx, Ay) + ed(Sy, Ax),$$

where $a + b + c + d^{'} + e < 1$ *and* $d^{'}, e \geq 0$.

$$\text{(3.12)} \quad d(Ax, Ay) \leq \frac{k}{2} \max\Big\{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), d(Sx, Ay), d(Sy, Ax)\Big\}, \quad k \in (0, 1).$$

$$\text{(3.13)} \quad d(Ax, Ay) \leq ad(Sx, Sy) + bd(Sx, Ax) + cd(Sy, Ay) + d^{'}[d(Sx, Ay) + d(Sy, Ax)],$$

where $a + b + c + 2d^{'} < 1$ *and* $d^{'} \geq 0$.

$$\text{(3.14)} \quad d(Ax, Ay) \leq a_1 \frac{d^2(Sx, Ax) + d^2(Sy, Ay)}{d(Sx, Ax) + d(Sy, Ay)} + a_2 d(Sx, Sy) + a_3[d(Sx, Ay) + d(Sy, Ax)],$$

where $a_i \geq 0$ $(i = 1, 2, 3)$ *with at least one* a_i *non-zero and* $a_1 + a_2 + 2a_3 < 1$.

$$\text{(3.15)} \quad d(Ax, Ay) \leq a_1 d(Sx, Sy) + \frac{a_2 d(Sx, Ax)d(Sy, Ay) + a_3 d(Sx, Ay)d(Sy, Ax)}{d(Sx, Ax) + d(Sy, Ay)},$$

where $a_1, a_2, a_3 \geq 0$ *such that* $1 < 2a_1 + a_2 < 2$.

$$(3.16)\ d(Ax,Ay) \leq a_1\Big[a_2 \max\Big\{d(Sx,Sy), d(Sx,Ax), d(Sy,Ay), \frac{1}{2}(d(Sx,Ay)+d(Sy,Ax))\Big\} + (1-a_2)\Big[\max\Big\{d^2(Sx,Sy), d(Sx,Ax)d(Sy,Ay), d(Sx,Ay)d(Sy,Ax), \frac{d(Sx,Ax)d(Sy,Ax)}{2}, \frac{d(Sy,Ay)d(Sx,Ay)}{2}\Big\}\Big]^{\frac{1}{2}}\Big],$$

where $a_1 \in (0,1)$ *and* $0 \leq a_2 \leq 1$.

$$(3.17)\ d^2(Ax,Ay) \leq a_1 \max\{d^2(Sx,Sy), d^2(Sx,Ax), d^2(Sy,Ay)\} - a_2 \max\Big\{\frac{d(Sx,Ax)d(Sx,Ay)}{2}, \frac{d(Sy,Ay),d(Sy,Ax)}{2}\Big\} - a_3 d(Sx,Ay)d(Sy,Ax),$$

where $a_1, a_2, a_3 \geq 0$ *and* $a_1 + a_2 + a_3 < 1$.

$$(3.18)\qquad d(Ax,Ay) \leq \phi\Big(\max\Big\{d(Sx,Sy), d(Sx,Ax), d(Sy,Ay), \frac{1}{2}(d(Sx,Ay)+d(Sy,Ax))\Big\}\Big),$$

where $\phi : \mathcal{R}^+ \to \mathcal{R}^+$ *is an increasing upper semi-continuous function with* $\phi(0) = 0$ *and* $\phi(t) < t$ *for each* $t > 0$.

$$(3.19)\qquad d(Ax,Ay) \leq \phi\Big(d(Sx,Sy), d(Sx,Ax), d(Sy,Ay), d(Sx,Ay), d(Sy,Ax)\Big),$$

where $\phi : \mathcal{R}_+^5 \to \mathcal{R}^+$ *is an upper semi-continuous and non-decreasing function in each coordinate variable such that* $\phi(t,t,at,bt,ct) < t$ *for each* $t > 0$ *and* $a,b,c \geq 0$ *with* $a+b+c \leq 3$.

$$(3.20)\qquad d^2(Ax,Ay) \leq \phi\Big(d^2(Sx,Sy), d(Sx,Ax)d(Sy,Ay), d(Sx,Ay)d(Sy,Ax), d(Sx,Ax)d(Sy,Ax), d(Sy,Ay)d(Sx,Ay)\Big),$$

where $\phi : \mathcal{R}_+^5 \to \mathcal{R}^+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\phi(t, t, at, bt, ct) < t$ for each $t > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Proof. The proof follows from Theorem 3.1 and Examples 2.1-2.8. □

Setting $S = I$ (the identity mapping on X) in Theorem 3.1, we get the following corresponding fixed point theorem.

Corollary 3.4. *Let A be a self-mappings of a metric space (X, d) such that*

1. *for all $x, y \in X$ and some $\psi \in \Psi$,*

$$\psi\begin{pmatrix} d(Ax, Ay), d(x, y), d(x, Ax), \\ d(y, Ay), d(x, Ay), d(y, Ax) \end{pmatrix} \leq 0, \tag{3.21}$$

2. *$\overline{A(X)}$ is a complete subspace of X.*

Then A has a unique fixed point in X.

Remark 3.5. A corollary similar to Corollary 3.4 can be outlined in respect of Corollary 3.3 yielding thereby a host of fixed point theorems.

4. A common fixed point theorem for finite families of self-mappings

As an application of Theorem 3.1, we prove a common fixed point theorem for two finite families of mappings which runs as follows:

Theorem 4.1. *Let $\{A_1, A_2, \ldots, A_m\}$ and $\{S_1, S_2, \ldots, S_p\}$ be two finite families of self-mappings of a metric space (X, d) with $A = A_1A_2 \ldots A_m$ and $S = S_1S_2 \ldots S_p$ satisfying condition* (3.1) *of Theorem 3.1. Suppose that $\overline{A(X)} \subseteq S(X)$, wherein $\overline{A(X)}$ is a complete subspace of X. Then (A, S) has a point of coincidence.*

Moreover, if $A_iA_j = A_jA_i$, $S_kS_l = S_lS_k$ and $A_iS_k = S_kA_i$ for all $i, j \in I_1 = \{1, 2, \ldots, m\}$ and $k, l \in I_2 = \{1, 2, \ldots, p\}$, then (for all $i \in I_1$ and $k \in I_2$) A_i and S_k have a common fixed point in X.

Proof. The conclusion '(A, S) has a point of coincidence' is immediate as A and S satisfy all the conditions of Theorem 3.1. Now appealing to componentwise commutativity of various pairs, one can immediately assert that $AS = SA$ and hence, obviously the pair (A, S) is weakly compatible. Note that all the conditions of Theorem 3.1 (for mappings A and S) are satisfied ensuring the existence of unique common fixed point, say t. Now one needs to show that t remains the fixed point of all the component mappings. For this consider

$$\begin{aligned} A(A_it) = ((A_1A_2 \ldots A_m)A_i)t &= (A_1A_2 \ldots A_{m-1})((A_mA_i)t) \\ &= (A_1 \ldots A_{m-1})(A_iA_mt) \\ &\ \ \vdots \\ &= A_1A_i(A_2A_3A_4 \ldots A_mt) \\ &= A_iA_1(A_2A_3 \ldots A_mt) = A_i(At) = A_it. \end{aligned}$$

Similarly, one can show that,

$$A(S_k t) = S_k(At) = S_k t, \quad S(S_k t) = S_k(St) = S_k t,$$

and

$$S(A_i t) = A_i(St) = A_i t,$$

which shows that (for all i and k) $A_i t$ and $S_k t$ are other fixed points of the pair (A, S). Now appealing to the uniqueness of common fixed points of the pair separately, we get

$$t = A_i t = S_k t,$$

which shows that t is a common fixed point of A_i and S_k for all i and k. □

By setting $A_1 = A_2 = \ldots = A_m = A$ and $S_1 = S_2 = \ldots = S_p = S$ in Theorem 4.1, we deduce the following corollary:

Corollary 4.2. *Let A and S be self-mappings of a metric space (X, d) satisfying inequality (3.1) of Theorem 3.1 for all distinct $x, y \in X$. If $\overline{A^m(X)} \subseteq S^p(X)$, then A and S have a unique common fixed point in X provided $AS = SA$.*

5. Illustrative examples

Now we furnish examples to demonstrate the validity of the hypotheses and degree of generality of Theorem 3.1 over earlier result due to Popa et al. [38] besides establishing its utility over earlier results due to Imdad and Ali [16] and others.

Example 5.1. Consider $X = [2, 25]$ with usual metric. Define self-mappings A and S on X as

$$Ax = \begin{cases} 2, & \text{if } x \in \{2\} \cup (5, 25]; \\ 4, & \text{if } 2 < x \leq 5. \end{cases} \quad Sx = \begin{cases} 2, & \text{if } x = 2; \\ 8, & \text{if } 2 < x \leq 5; \\ x - 3, & \text{if } x > 5. \end{cases}$$

We can see that the mappings A and S commute at 2 which is their coincidence point. Also $A(X) = \{2, 4\}$ and $S(X) = [2, 22]$. Clearly, $\overline{A(X)} = \{2, 4\} \subset [2, 22] = S(X)$.

Now define $\psi(t_1, t_2, \ldots, t_6) : \mathcal{R}^6_+ \to \mathcal{R}$ as:

$$\psi(t_1, t_2, \ldots, t_6) = t_1 - a_1 \frac{t_3^2 + t_4^2}{t_3 + t_4} - a_2 t_2 - a_3(t_5 + t_6),$$

where $a_i \geq 0$ with at least one a_i non-zero and $a_1 + a_2 + 2a_3 < 1$.

By a routine calculation one can verify that contraction condition (3.1) is satisfied for $a_1 = \frac{1}{5}$ and $a_2 = a_3 = \frac{1}{4}$. If $x, y \in \{2\} \cup (5, 25]$, then $d(Ax, Ay) = 0$

and verification is trivial. If $x \in (2,5]$ and $y > 5$, then

$$\begin{aligned}
\Rightarrow \quad & a_1 \frac{d^2(Sx,Ax)+d^2(Sy,Ay)}{d(Sx,Ax)+d(Sy,Ay)} + a_2 d(Sx,Sy) + a_3(d(Sx,Ay)+d(Sy,Ax)) \\
= \quad & \frac{1}{5}\frac{4^2+|y-5|^2}{4+|y-5|} + \frac{1}{4}|y-11| + \frac{1}{4}[6+|y-7|] \\
\geq \quad & \begin{cases} \frac{4}{5}+\frac{1}{4}(24-2y) > 2 = d(Ax,Ay), & \text{if} \quad y \in (5,7], \\ \frac{4}{5}+\frac{10}{4} = \frac{33}{10} > 2 = d(Ax,Ay), & \text{if} \quad y \in (7,11], \\ \frac{4}{5}+\frac{1}{4}(2y-12) > 2 = d(Ax,Ay), & \text{if} \quad y > 11. \end{cases}
\end{aligned}$$

Similarly, one can verify the other cases. Thus all the conditions of Theorem 3.1 are satisfied and 2 is the unique common fixed point of the mappings A and S, which is their coincidence point also.

Finally, we have one example which illustrates the hypotheses of Theorem 3.1 except the condition of weak compatibility.

Example 5.2. Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$ be a metric space with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. Define the mappings $A, S : X \to X$ by $A(0) = \frac{1}{2^2}$, $A(\frac{1}{2^n}) = \frac{1}{2^{n+2}}$ and $S(0) = 0$, $S(\frac{1}{2^n}) = \frac{1}{2^{n+1}}$ for $n = 0, 1, 2, \ldots$ respectively. Clearly, $A(X) = \left\{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\right\}$ and $S(X) = \left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$. Then clearly,

$$\overline{A(X)} = \left\{0, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\} \subset \left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\} = S(X).$$

Considering the same implicit function as in Example 5.2, one can verify that the contraction condition (3.1) is satisfied for $a_1 = \frac{1}{8}$, $a_2 = \frac{1}{6}$ and $a_3 = \frac{1}{4}$. For example, choose $x = 0$ and $y = \frac{1}{2}$, then

$$\begin{aligned}
\Rightarrow \quad & a_1 \frac{d^2(Sx,Ax)+d^2(Sy,Ay)}{d(Sx,Ax)+d(Sy,Ay)} + a_2 d(Sx,Sy) + a_3(d(Sx,Ay)+d(Sy,Ax)) \\
= \quad & \frac{1}{8}\left\{\frac{\left|\frac{1}{2}-\frac{1}{2^2}\right|^2 + \left|\frac{1}{2^2}-\frac{1}{2^3}\right|^2}{\left|\frac{1}{2}-\frac{1}{2^2}\right| + \left|\frac{1}{2^2}-\frac{1}{2^3}\right|}\right\} + \frac{1}{6}\left|\frac{1}{2}-\frac{1}{2^2}\right| + \frac{1}{4}\left\{\left|\frac{1}{2}-\frac{1}{2^3}\right| + \left|\frac{1}{2^2}-\frac{1}{2^2}\right|\right\} \\
= \quad & \frac{1}{8}\left(\frac{5}{24}\right) + \frac{1}{6}\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{3}{8}\right) = \frac{31}{192} > \frac{1}{8} = d(Ax,Ay).
\end{aligned}$$

Clearly, all the conditions of Theorem 3.1 are satisfied except the condition of weak compatibility. Therefore, the mappings A and S do not have any point of coincidence.

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