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THE HOMOMORPHISMS OF TOPOLOGICAL GROUPOIDS

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Abstract. The main purpose of this paper is to study topological groupoid homomorphisms and to give some kinds of special topological groupoid homomorphisms. Finally, some characterizations of these homomorphisms are given.

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1. Introduction

The concept of groupoid was first introduced by Brandt[1] in 1926. The groupoids can be thought as generalizations of groups. Namely; a group is a groupoid with only one object. After introducing of topological and differentiable groupoids by Ehresmann[5] in the 1950's, it has been widely studied by many mathematicians in [2, 3, 4, 6, 7, 8, 10].

We know from group theory that there exist special morphisms between groups, such as homomorphism, epimorphism, monomorphism, etc.. Since the groupoids are generalizations of groups, there are also special homomorphisms between groupoids. In [9], Ivan studied some kind of special morphisms of groupoids in the algebraic sense.

In this work, we deal with the special homomorphisms of topological groupoids. We give the relations between them. Further, in the case of induced topological groupoid, we obtain some characterizations of the special homomorphisms.

2. Topological Groupoids and their Properties

In this section we present the concept of topological groupoid and their characteristic properties. Let us start with the definition of the groupoid.

Definition 2.1. ([2], pp.193) A groupoid is a small category in which every arrow is invertible.

To clarify concepts and to fix notation, we briefly describe the groupoid structure. A groupoid G over G_0 consists of a set of arrows G and a set of objects G_0, together with the following five structure maps:

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$\alpha : G \to G_0$, called the source map,
$\beta : G \to G_0$, called the target map,
$\epsilon : G_0 \to G, x \mapsto \widetilde{x}$, called the object map,
$i : G \to G, a \longmapsto a^{-1}$, called the inverse map,
$\mu : G_2 \to G, (a,b) \mapsto \mu(a,b) = ab$, called the composition map, where $G_2 = \{(a,b) \in G \times G \mid \alpha(a) = \beta(b)\}$ denotes the set of composable arrows.

These maps must satisfy the following conditions:
i) $\alpha(ab) = \alpha(a)$ and $\beta(ab) = \beta(b)$ for all $(a,b) \in G_2$,
ii) $a(bc) = (ab)c$ for all $a,b,c \in G$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$,
iii) $\alpha(\widetilde{x}) = \beta(\widetilde{x}) = x$ for all $x \in G_0$,
iv) $a\widetilde{\beta(a)} = a$ and $\widetilde{\alpha(a)}a = a$ for all $a \in G$,
v) each $a \in G$ has a (two-sided) inverse a^{-1} such that $\alpha(a^{-1}) = \beta(a)$, $\beta(a^{-1}) = \alpha(a)$ and $aa^{-1} = \widetilde{\alpha(a)}$, $a^{-1}a = \widetilde{\beta(a)}$.

We usually denote a groupoid G over G_0 by (G, G_0). Elements of G_0 are called objects of the groupoid G and elements of G are called arrows. The arrow $\widetilde{x}$ corresponding to an object $x \in G_0$ is called the *unity* or *identity corresponding to* x. We have also following notations about the groupoids. We denote the set of arrows started at any object $x \in G_0$ by G_x (or $St_G x$), and the set of arrows ended at any object $y \in G_0$ by G^y (or $Cost_G y$) in a groupoid (G, G_0). Also we denote the set of arrows from x to y by $G(x,y)$.

Specially, given a groupoid (G, G_0), the *vertex groups* or *isotropy groups* in G are subsets of the form $G(x,x)$, where x is any object of G. It follows easily from the axioms above that these are indeed groups, as every pair of elements is composable and inverses are in the same vertex group. If we consider the union of all isotropy groups in a groupoid, we reach the concept of isotropy group bundle. That is, if (G, G_0) is a groupoid, then $Is(G) = \{a \mid \alpha(a) = \beta(a)\}$ is a group bundle called the *isotropy group bundle* associated to G.

Definition 2.2. ([10], pp. 8) Let (G, G_0) be a groupoid. A subgroupoid of (G, G_0) is a pair of subsets $G^{'} \subseteq G$, $G^{'}_0 \subseteq G_0$ such that $\alpha(G^{'}) \subseteq G^{'}_0$, $\beta(G^{'}) \subseteq G^{'}_0$, $\widetilde{x} \in G^{'}$ for every $x \in G^{'}_0$, and $G^{'}$ is closed under the composition map and inversion in (G, G_0). A subgroupoid $(G^{'}, G^{'}_0)$ of (G, G_0) is called wide if $G^{'}_0 = G_0$ and is called full if $G^{'}(x,y) = G(x,y)$ for every $x, y \in G^{'}_0$.

Definition 2.3. ([10], pp. 8) Let (G, G_0) be a groupoid. A normal subgroupoid of (G, G_0) is a wide subgroupoid (N, G_0) such that for any $n \in N$ and any $a \in G$ with $\beta(a) = \alpha(n) = \beta(n)$, we have $ana^{-1} \in N$.

Definition 2.4. ([9]) Let (G, G_0) and $(G^{'}, G^{'}_0)$ be two groupoids. A groupoid homomorphism is a pair $(f, f_0) : (G, G_0) \to (G^{'}, G^{'}_0)$ of maps $f_0 : G_0 \to G^{'}_0$ and $f : G \to G^{'}$ which send each object x of G to an object $f_0(x)$ of $G^{'}$ and each arrow $a \in G(x,y)$ to an arrow $f(a) \in G^{'}(f_0(x), f_0(y))$, respectively, such that
M1) $f(\mu(a,b)) = \mu^{'}(f(a), f(b))$ for every $(a,b) \in G_2$
M2) $\alpha^{'} \circ f = f_0 \circ \alpha$ and $\beta^{'} \circ f = f_0 \circ \beta$.

These conditions mean that the following diagrams are commutative:

$$
\begin{array}{ccc}
\begin{tikzcd} G_2 \arrow[r, "f\times f"] \arrow[d, "\mu"'] & G_2^{'} \arrow[d, "\mu^{'}"] \\ G \arrow[r, "f"'] & G^{'} \end{tikzcd}
&
\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\alpha"'] & G^{'} \arrow[d, "\alpha^{'}"] \\ G_0 \arrow[r, "f_0"'] & G_0^{'} \end{tikzcd}
&
\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\beta"'] & G^{'} \arrow[d, "\beta^{'}"] \\ G_0 \arrow[r, "f_0"'] & G_0^{'} \end{tikzcd}
\end{array}
$$

Remark 2.5. If $G_0 = G_0^{'}$ and $f_0 = Id_{G_0}$, we say that f is a G_0-*homomorphism.*

We can now give the definition of topological groupoid.

Definition 2.6. ([10], pp.17) A topological groupoid is a groupoid (G, G_0) together with topologies on G and G_0 such that the structure maps are continuous.

Here are some basic examples of topological groupoids that are well known.

Example 2.7. ([4]) A topological group can be regarded as a topological groupoid with only one object.

Example 2.8. ([10], pp. 22) If G is a topological group and B is a topological space, then $B \times G \times B$ is a topological groupoid with the product topology, called the trivial topological groupoid over B with topological group G. More precisely, we have

$$
\begin{aligned}
&\alpha : B \times G \times B \to B\ ,\ \alpha(x, g, y) = x,\\
&\beta : B \times G \times B \to B\ ,\ \beta(x, g, y) = y,\\
&\epsilon : B \to B \times G \times B\ ,\ \epsilon(x) = (x, e, x),\\
&i : B \times G \times B \to B \times G \times B\ ,\ i(x, g, y) = (y, g^{-1}, x),
\end{aligned}
$$

$$\mu\Big((x, g, y), (y^{'}, h, z)\Big) = (x, gh, z) \Leftrightarrow y = y^{'}.$$

Definition 2.9. ([4]) A topological groupoid homomorphism $(f, f_0) : (G, G_0) \to (G^{'}, G_0^{'})$ is a groupoid homomorphism which is continuous on both objects and arrows.

Proposition 2.10. ([10], pp. 7) *Let* $(f, f_0) : (G, G_0) \to (G^{'}, G_0^{'})$ *be topological groupoid homomorphism. Then* (f, f_0) *satisfies the following properties:*

$$
\begin{aligned}
f(\widetilde{x}) &= \widetilde{f_0(x)}\ ,\ \forall x \in G_0\\
f(a^{-1}) &= (f(a))^{-1},\ \forall a \in G.
\end{aligned}
$$

That is, we have the equalities $f \circ \epsilon = \epsilon^{'} \circ f_0$ *and* $f \circ i = i^{'} \circ f$*. These equalities correspond to the following commutative diagrams.*

$$
\begin{array}{cc}
\begin{tikzcd} G_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & G \arrow[d, "f"] \\ G_0^{'} \arrow[r, "\epsilon^{'}"'] & G_0^{'} \end{tikzcd}
&
\begin{tikzcd} G \arrow[r, "i"] \arrow[d, "f"'] & G \arrow[d, "f"] \\ G^{'} \arrow[r, "i^{'}"'] & G^{'} \end{tikzcd}
\end{array}
$$

Proposition 2.11. *Let $f : G \to G^{'}$ is a continuous map. Then the pair $(f, f_0) : (G, G_0) \to (G^{'}, G_0^{'})$ is a topological groupoid homomorphism if and only if the condition*

$$(2.1) \quad (f(a), f(b)) \in G_2^{'} \text{ and } f(\mu(a, b)) = \mu^{'}(f(a), f(b)) \ , \ for \ \forall (a, b) \in G_2$$

holds.

Proof. Let (f, f_0) be a topological groupoid homomorphism. Then the condition (2.1) follows from the definition of topological groupoid homomorphism.

Conversely, we suppose that $f : G \to G^{'}$ satisfies (2.1) and let us define the map $f_0 : G_0 \to G_0^{'}$ by $f_0(x) = \alpha^{'}(f(\epsilon(x)))$, for all $x \in G_0$. f_0 is continuous, because it is a composition of continuous maps. To see that (f, f_0) is a topological groupoid homomorphism, we must show that $\alpha^{'} \circ f = f_0 \circ \alpha$ and $\beta^{'} \circ f = f_0 \circ \beta$.

Since $(a, \epsilon(\beta(a))) \in G_2$, it follows that $(f(a), f(\epsilon(\beta(a)))) \in G_2^{'}$. Thus we have

$$f(a)f(\epsilon(\beta(a))) = f(a\epsilon(\beta(a))) = f(a),$$

but $f(a)\epsilon^{'}(\beta^{'}(f(a))) = f(a)$. So

$$\epsilon^{'}(\beta^{'}(f(a))) = f(\epsilon(\beta(a))) \implies \alpha^{'}(\epsilon^{'}(\beta^{'}(f(a)))) = \alpha^{'}(f(\epsilon(\beta(a))))$$

and if we apply Definition 2.1, then we obtain $\beta^{'}(f(a)) = f_0(\beta(a))$. Hence it follows that $\beta^{'} \circ f = f_0 \circ \beta$.

Similarly, we can show that $\alpha^{'} \circ f = f_0 \circ \alpha$. Thus (f, f_0) is a topological groupoid homomorphism. □

Definition 2.12. ([10], pp. 8) Let $(f, f_0) : (G, G_0) \to (G^{'}, G_0^{'})$ be a topological groupoid homomorphism. The set $Kerf = \{a \in G \mid f(a) \in \epsilon^{'}(G_0^{'})\}$ is called the kernel of f, which has the subspace topology. We say that (f, f_0) has discrete kernel, if $Kerf = \epsilon(G_0)$.

Here we give two trivial examples about the homomorphisms of the topological groupoids, which are well-known.

Example 2.13. i) Let (G, G_0) be a topological groupoid. Then $(Id_G, Id_{G_0}) : (G, G_0) \to (G, G_0)$ is a topological groupoid homomorphism.
ii) Let $(f, f_0) : (G, G_0) \to (G^{'}, G_0^{'})$ and $(g, g_0) : (G^{'}, G_0^{'}) \to (G^{''}, G_0^{''})$ be two topological groupoid homomorphisms. Then the composition $(g, g_0) \circ (f, f_0) : (G, G_0) \to (G^{''}, G_0^{''})$ is a topological groupoid homomorphism. Namely, it is defined by $(g, g_0) \circ (f, f_0) = (g \circ f, g_0 \circ f_0)$.

If $(f, f_0) : (G, G_0) \to (G^{'}, G_0^{'})$ is a topological groupoid homomorphism, then we have

$$f(G_x) \subseteq G^{'}_{f_0(x)} \quad , \quad f(G^y) \subseteq (G^{'})^{f_0(y)} \quad , \quad f(G(x, y)) \subseteq (G^{'})(f_0(x), f_0(y))$$

for every $x, y \in G_0$. Then the restrictions of f to $G_x,\ G^y,\ G(x,y)$ resp., define the continuous maps

$$f_x : G_x \to G'_{f_0(x)} \quad , \quad f^y : G^y \to (G')^{f_0(y)} \quad , \quad f_x^y : G_x^y \to (G')(f_0(x), f_0(y)),$$

but these maps are not topological groupoid homomorphisms.

Proposition 2.14. ([11], Proposition 2.3.3) *Let $(f, f_0) : (G, G_0) \to (G', G'_0)$ be a homomorphism of groupoids of G onto G', where G' is a topological groupoid. Then f induces a topology on G compatible with the groupoid structure of G and f is then a homomorphism of topological groupoids.*

For the readers convenience we rewrite the proof from [11].

Proof. Define a set $U \subset G$ to be open if and only if $U = f^{-1}(V)$ for some open set $V \subset G'$, and define $\overline{U} \subset G_0$ to be open if and only if $\overline{U} = f_0^{-1}(\overline{V})$ for some open set $\overline{V} \subset G'_0$. This defines a topology on G and on G_0. In addition, f and f_0, being surjective, are both continuous and open maps.

Since the diagram

$$\begin{tikzcd} G_2 \arrow[r, "f \times f"] \arrow[d, "\mu"'] & G'_2 \arrow[d, "\mu'"] \\ G \arrow[r, "f"'] & G' \end{tikzcd}$$

is commutative, it follows that μ is continuous. Because if $U \subset G$ is open, $U = f^{-1}(V)$ for some open $V \subset G'$, and then $\mu^{-1}(U) = \mu^{-1}(f^{-1}(V)) = (f \times f)^{-1}((\mu')^{-1}(V))$, which is open in G_2. Similarly, the continuity of the inverse map and the commutativity of the diagrams

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\alpha"'] & G' \arrow[d, "\alpha'"] \\ G_0 \arrow[r, "f_0"'] & G'_0 \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\beta"'] & G' \arrow[d, "\beta'"] \\ G_0 \arrow[r, "f_0"'] & G'_0 \end{tikzcd} \qquad \begin{tikzcd} G_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & G \arrow[d, "f"] \\ G'_0 \arrow[r, "\epsilon'"'] & G' \end{tikzcd}$$

are proved. The commutativity of these diagrams provides the continuity of α, β and ϵ. Thus the result is established. $\square$

Definition 2.15. ([10], pp. 21) Let (G, G_0) and (G', G'_0) be topological groupoids. An isomorphism of topological groupoids $(f, f_0) : (G, G_0) \to (G', G'_0)$ is a homomorphism of topological groupoids such that f (and hence f_0) is a homeomorphism.

Now let us give the characterizations of topological groupoid homomorphisms by a proposition.

Proposition 2.16. *Let $(f, f_0) : (G, G_0) \to (G^{'}, G_0^{'})$ be a topological groupoid homomorphism. Then the following assertions hold:*
i) if f is injective (or surjective), then f_0 is also injective (or surjective),
ii) (f, f_0) is an isomorphism if and only if the map f is bijection,
iii) $f(Is(G)) \subseteq Is(G^{'})$,
iv) a topological groupoid homomorphism (f, f_0) such that f is surjective and f_0 is injective (in particular, every surjective G_0-homomorphism of topological groupoids) preserves the isotropy group bundles, i.e. $f(Is(G)) = Is(G^{'})$.

Proof. i) We prove that f_0 is injective when f is injective. Let f be injective. Then,

$$a = b \ \ whenever \ \ f(a) = f(b) \quad for\ every\ a, b \in G.$$

Hence if $f(a) = f(b)$, then we have $\alpha^{'}(f(a)) = \alpha^{'}(f(b))$ and $\beta^{'}(f(a)) = \beta^{'}(f(b))$, and if $a = b$ then we have $\alpha(a) = \alpha(b)$ and $\beta(a) = \beta(b)$. Now let us $f_0(\alpha(a)) = f_0(\alpha(b))$. Since the second condition in definition of topological groupoid homomorphism, we have $\alpha^{'}(f(a)) = \alpha^{'}(f(b))$, and $\alpha(a) = \alpha(b)$ because $a = b$. Similar considerations are also said for β. Thus it follows that f_0 is injective.

The proof of surjectivity is straightforward.

ii) The proof follows easily from the Definition 2.15.

iii) Let $a^{'} \in f(Is(G))$. Then $a^{'} = f(a)$ with $a \in Is(G)$ and we have

$$\alpha^{'}(a^{'}) = \alpha^{'}(f(a)) = f_0(\alpha(a)) = f_0(\beta(a)) = \beta^{'}(f(a)) = \beta^{'}(a^{'}),$$

since $\alpha(a) = \beta(a)$, hence $a^{'} \in Is(G^{'})$. Consequently, $f(Is(G)) \subseteq Is(G^{'})$.

iv) We know that $f(Is(G)) \subseteq Is(G^{'})$. For this reason, it sufficies to prove $Is(G^{'}) \subseteq f(Is(G))$. Let us take any $a^{'} \in Is(G^{'})$. Hence $\alpha^{'}(a^{'}) = \beta^{'}(a^{'})$. Since f is surjective, for $a^{'} \in G^{'}$ there exists an arrow $a \in G$ such that $a^{'} = f(a)$. Then $\alpha^{'}(f(a)) = \beta^{'}(f(a))$ and hence we have $f_0(\alpha(a)) = f_0(\beta(a))$, because f is a topological groupoid homomorphism. Further, it follows that $\alpha(a) = \beta(a)$, since f_0 is injective. Therefore $a \in Is(G)$ and $a^{'} \in f(Is(G))$. Consequently, it follows that $Is(G^{'}) \subseteq f(Is(G))$. □

3. The Induced Topological Groupoid

There are various ways of producing topological groupoids from existing ones. In this section we present the concept of induced topological groupoid. Then we give the characterizations concerning it.

Definition 3.1. ([10], pp. 19) Let (G, G_0) be a topological groupoid and let $h : X \to G_0$ be a continuous map, where X is any topological space. Then the induced topological groupoid of (G, G_0) under h is a set

$$h^*(G) = \{(x, y, a) \in X \times X \times G \mid h(x) = \alpha(a), h(y) = \beta(a)\}$$

together with the groupoid structure consisting of the projections $\alpha^*(x, y, a) = x$, $\beta^*(x, y, a) = y$, the object map $\epsilon^*(x) = (x, x, \epsilon(h(x)))$ and the composition $\mu^*((x, y, a), (y, z, b)) = (x, z, ab)$. The inversion is defined by $i^*(x, y, a) = (y, x, i(a))$. We denote it by $(h^*(G), X)$.

Let $h^*(G)$ be induced topological groupoid of G under $h : X \to G_0$. Then there exists a topological groupoid homomorphism $(h^*_G, h) : (h^*(G), X) \to (G, G_0)$ defined by $h^*(x, y, a) = a$. It is called the canonical homomorphism of an induced topological groupoid.

$$\begin{tikzcd} h^*(G) \arrow[r, "h^*_G"] \arrow[d] & G \arrow[d] \\ X \arrow[r, "h"'] & G_0 \end{tikzcd}$$

Let us give the universal property of an induced topological groupoid homomorphism. The proof of the following theorem is a special case of the Corollary 6. given by Brown and Hardy in [4].

Theorem 3.2. *Let (G, G_0) be a topological groupoid and let $(h^*(G), X)$ be induced topological groupoid for it, where $h : X \to G_0$ is a continuous map. Then the homomorphism $(h^*_G, h) : (h^*(G), X) \to (G, G_0)$ satisfies the following property:*

> *for every topological groupoid homomorphism $(g, h) : (G^{'}, X) \to (G, G_0)$ there exists a unique $X-$homomorphism $g^{'} : G^{'} \to h^*(G)$ of topological groupoids such that $h^*_G \circ g^{'} = g$.*

Let (G, X) and $(G^{'}, X)$ be two topological groupoids. Then we can give the relation between the induced topological groupoids of them as follows:

Let $f : (G, X) \to (G^{'}, X)$ be a topological groupoid homomorphism over the same base X. Let us consider a continuous map $h : Y \to X$ of topological spaces. Then there is a homomorphism $h^*(f) : (h^*(G), Y) \to (h^*(G^{'}), Y)$ of the induced topological groupoids associated to (G, X) and $(G^{'}, X)$, which is defined by for all $(y_1, y_2, a) \in h^*(G)$

$$h^*(f)(y_1, y_2, a) = (y_1, y_2, f(a)) \in h^*(G^{'}).$$

It is easy to show that $h^*(f)$ is well-defined and $h^*(f)$ is a topological groupoid homomorphism over the same base Y.

It is clear that f^* is a functor. Namely, $h^*(id_G) = id_{h^*(G)}$ and if $g : (G^{'}, X) \to (G^{''}, X)$ is another topological groupoid homomorphism over X, then $h^*(g \circ f) : h^*(G) \to h^*(G^{''})$, defined for every $(y_1, y_2, a) \in h^*(G)$ by

$$h^*(g \circ f)(y_1, y_2, a) = (y_1, y_2, (g \circ f)(a))$$

is a topological groupoid homomorphism over Y such that $h^*(g \circ f) = h^*(g) \circ h^*(f)$.

Now let us state these facts in the following proposition.

Proposition 3.3. *Let $h : Y \to X$ be a continuous map of topological spaces. Then there exists a functor h^* from the category $\mathcal{G}(X)$ of topological groupoids over same base X to the category $\mathcal{G}(Y)$ of induced topological groupoids of them over same base Y.*

Proposition 3.4. *Let $h : Z \to Y$ and $k : Y \to X$ be two continuous maps of topological spaces and let (G, X) be a topological groupoid. Then the topological groupoids $h^*(k^*(G))$ and $(k \circ h)^*(G)$ are isomorphic.*

Proof. Let us first constitute the induced topological groupoids $k^*(G)$, $h^*(k^*(G))$ and $(k \circ h)^*(G)$. Clearly, we have

$$k^*(G) = \{(y_1, y_2, a) \in Y \times Y \times G \mid k(y_1) = \alpha(a),\ k(y_2) = \beta(a)\},$$

$$h^*(k^*(G)) = \{(z_1, z_2, (y_1, y_2, a)) \in Z \times Z \times k^*(G) \mid h(z_1) = y_1,\ h(z_2) = y_2\},$$

$$(k \circ h)^*(G) = \{(z_1, z_2, a) \in Z \times Z \times G \mid (k \circ h)(z_1) = \alpha(a),\ (k \circ h)(z_2) = \beta(a)\}.$$

It is enough to prove that there exists an isomorphism between the induced topological groupoids $h^*(k^*(G))$ and $(k \circ h)^*(G)$. Now let us consider the maps

$$\begin{aligned} \varphi &: (k \circ h)^*(G) \to h^*(k^*(G)) \\ (z_1, z_2, a) &\mapsto \varphi(z_1, z_2, a) = (z_1, z_2, (h(z_1), h(z_2), a)) \end{aligned}$$

and

$$\begin{aligned} \psi &: h^*(k^*(G)) \to (k \circ h)^*(G) \\ (z_1, z_2, (y_1, y_2, a)) &\mapsto \psi(z_1, z_2, (y_1, y_2, a)) = (z_1, z_2, a). \end{aligned}$$

It is easy to see that φ and ψ are topological groupoid homomorphisms over base Z such that $\psi \circ \varphi = id_{(k \circ h)^*(G)}$ and $\varphi \circ \psi = id_{h^*(k^*(G))}$. Therefore φ is an isomorphism of topological groupoids over the base Z . □

Proposition 3.5. *Let (G, X) be a topological groupoid. Then the topological groupoids $id_X^*(G)$ and G are $X-$isomorphic.*

Proof. Firstly, let us write precisely the induced topological groupoid $id_X^*(G)$.

$$id_X^*(G) = \{(x_1, x_2, a) \in X \times X \times G \mid \alpha(a) = x_1, \beta(a) = x_2\}.$$

It is clear that

$$\begin{aligned} \varphi &: G \to id_X^*(G) \\ a &\mapsto \varphi(a) = (\alpha(a), \beta(a), a) \end{aligned}$$

is an $X-isomorphism$. □

Proposition 3.6. *Let (G, X) be a transitive topological groupoid and let $f : Y \to X$ be a continuous map of topological spaces. Then the induced topological groupoid $f^*(G)$ is also a transitive topological groupoid over Y.*

Proof. Since (G, X) is transitive, the continuous map $\alpha \times \beta : G \to X \times X$, $(\alpha \times \beta)(a) = (\alpha(a), \beta(a))$ is surjective. Hence there exists an arrow $a \in G$ such that $(\alpha \times \beta)(a) = (f(y_1), f(y_2))$ whenever $(y_1, y_2) \in Y \times Y$, i.e. $\alpha(a) = f(y_1)$ and $\beta(a) = f(y_2)$. Thus $(y_1, y_2, a) \in f^*(G)$ and $(\alpha^* \times \beta^*)(y_1, y_2, a) = (\alpha^*(y_1, y_2, a), \beta^*(y_1, y_2, a)) = (y_1, y_2)$. That is, $\alpha^* \times \beta^* : f^*(G) \to Y \times Y$ is surjective. Further, the map $\alpha^* \times \beta^*$ is continuous, because α^* and β^* are projections onto the first and second factors, respectively. This means that $f^*(G)$ is a transitive topological groupoid. □

4. Special Homomorphisms of Topological Groupoids

In this section we obtain some characterizations which deal with the special homomorphisms of the topological groupoids. Also we present the relation between the induced topological groupoids and these special homomorphisms.

Let us begin this section with the following definition.

Definition 4.1. Let $(f, f_0) : (G, G_0) \to (G', G_0')$ be topological groupoid homomorphism. Then
(i) If f_0 is injective (resp., surjective, bijective), then (f, f_0) is called base injective (resp., surjective, bijective).
(ii) If $f_x : G_x \to G'_{f_0(x)}$ is injective (resp., surjective, bijective), for all $x \in G_0$, then (f, f_0) is called fibrewise injective (resp., surjective, bijective).
(iii) If $f_x^y : G(x, y) \to G'(f_0(x), f_0(y))$ is injective (resp.,surjective,bijective), for all $x, y \in G_0$, then (f, f_0) is called piecewise injective (resp.,surjective ,bijective).

Definition 4.2. [10], p.19) Let $(f, f_0) : (G, G_0) \to (G', G_0')$ be a topological groupoid homomorphism. Then (f, f_0) is called a pullback if for every topological groupoid homomorphism $(g, g_0 = f_0) : (G_1, G_0) \to (G', G_0')$ there exists a unique topological groupoid homomorphism $\overline{g} : (G_1, G_0) \to (G, G_0)$ such that $f \circ \overline{g} = g$.

This definition means that every topological groupoid homomorphism $(g, g_0 = f_0) : (G_1, G_0) \to (G', G_0')$ can be factored uniquely into $G_1 \xrightarrow{\overline{\tau}} G \xrightarrow{\varphi} G'$ so that the diagram

$$\begin{tikzcd} G_1 \arrow[d, "\bar{g}"'] \arrow[dr, "g"] & \\ G \arrow[r, "f"'] & G' \end{tikzcd}$$

is commutative.

The following proposition gives the relation between the induced topological groupoid and the notion of pullback.

Proposition 4.3. ([10], p.19) *The canonical homomorphism (h_G^*, h) of the induced topological groupoid $h^*(G)$ of G by the continuous map $h : X \to G_0$ is a pullback.*

Proof. The proof is a direct consequent of the Theorem 3.2. □

Proposition 4.4. *(i) The canonical homomorphism (h_G^*, h) of the induced topological groupoid $h^*(G)$ of G by the continuous map $h : X \to G_0$ is piecewise bijective.*
(ii) If $(f, f_0) : (G, G_0) \to (G', G_0')$ is base surjective and piecewise surjective then f is surjective.
(iii) A topological groupoid homomorphism $(f, f_0) : (G, G_0) \to (G', G_0')$ has discrete kernel iff $Ker\ f_x^x = \{\epsilon(x)\}$, for all $x \in G_0$.

Proof. (i) To see that (h^*_G, h) is piecewise bijective, we will prove that $(h^*_G)^y_x : h^*(G)(x,y) \to G(h(x), h(y))$ is bijective for all $x, y \in G_0$. For injectivity, let us take $(x_1, y_1, a_1), (x_2, y_2, a_2) \in h^*(G)(x,y)$ so that

$$(h^*_G)^y_x\,(x_1, y_1, a_1) = (h^*_G)^y_x\,(x_2, y_2, a_2) \Rightarrow \left\{ \begin{array}{c} \alpha^*(x_1, y_1, a_1) = \alpha^*(x_2, y_2, a_2) = x \\ \beta^*(x_1, y_1, a_1) = \beta^*(x_2, y_2, a_2) = y \\ a_1 = a_2. \end{array} \right.$$

Hence $(x_1, y_1, a_1) = (x_2, y_2, a_2)$. That is, $(h^*_G)^y_x$ is injective. For surjectivity, let $b \in G(h(x), h(y))$. Then $\alpha(b) = h(x)$ and $\beta(b) = h(y)$, and hence $(x, y, b) \in h^*(G)$ and $(x, y, b) \in h^*(G)(x, y)$. Since the definition of h^*_G, $h^*_G(x, y, b) = b$. Thus $(h^*_G)^y_x$ is surjective.
(ii) Let $b^{'} \in G^{'}$. We suppose that $\alpha^{'}(b^{'}) = x^{'}$, $\beta^{'}(b^{'}) = y^{'}$. Since (f, f_0) is base surjective, f_0 is surjective. Hence there exist objects $x, y \in G_0$ such that $x^{'} = f_0(x)$ and $y^{'} = f_0(y)$. Since $b^{'} \in (G^{'})^{y^{'}}_{x^{'}}$ and (f, f_0) is piecewise surjective, there exists $a \in G^y_x$ such that $f^y_x(a) = b^{'}$. Thus f is surjective, because $f(a) = b^{'}$.
(iii) The proof is straightforward. □

Theorem 4.5. *Let $(f, f_0) : (G, G_0) \to (G^{'}, G^{'}_0)$ be a topological groupoid homomorphism. Then, we have*
(i) if (f, f_0) is a pullback, then it is piecewise bijective.
(ii) f is injective if and only if it is base injective and piecewise injective.
(iii) (f, f_0) is fibrewise injective if and only if it has discrete kernel.

Proof. (i) Let (f, f_0) be a pullback and $f^*_0(G^{'})$ the induced topological groupoid of $G^{'}$ by $f_0 : G_0 \to G^{'}_0$. By Proposition 4.3, $(f_0)^*_{G^{'}}$ is a pullback. Hence (f, f_0) can be factored uniquely into $f = (f_0)^*_{G^{'}} \circ g$, where $g : G \to f^*_0(G^{'})$ is a topological groupoid homomorphism over G_0.

Furthermore, since (f, f_0) is a pullback, the topological groupoid homomorphism $(f_0)^*_{G^{'}} : f^*_0(G^{'}) \to G^{'}$ can be factored uniquely into $(f_0)^*_{G^{'}} = f \circ \bar{g}$, where $\bar{g} : f^*_0(G^{'}) \to G$ is a topological groupoid homomorphism over G_0.

From $f = (f_0)^*_{G^{'}} \circ g$ and $(f_0)^*_{G^{'}} = f \circ \bar{g}$, we obtain that $f = f \circ (\bar{g} \circ g) = f \circ Id$ and $(f_0)^*_{G^{'}} = (f_0)^*_{G^{'}} \circ (g \circ \bar{g}) = (f_0)^*_{G^{'}} \circ Id$. Since f and $(f_0)^*_{G^{'}}$ are pullbacks, $\bar{g} \circ g = Id$, and $g \circ \bar{g} = Id$. Hence $g : G \to f^*_0(G^{'})$ is an isomorphism of topological groupoids such that $f = (f_0)^*_{G^{'}} \circ g$. Since $(f_0)^*_{G^{'}}$ is piecewise bijective and g is bijective, we obtain that f is piecewise bijective.
(ii) We assume that f is injective. Then $f \circ \epsilon$ is injective, because the object map ϵ is injective. Further, since $f \circ \epsilon = \epsilon^{'} \circ f_0$, $\epsilon^{'} \circ f_0$ is also injective. Therefore f_0 is injective.

Conversely, let f_0 and f^y_x be injective, for all $x, y \in G_0$. We will prove that f is injective. Let us take $a, b \in G$ such that $f(a) = f(b)$. Hence,

$$\begin{aligned} f(a) = f(b) &\Longrightarrow (\alpha^{'} \circ f)(a) = (\alpha^{'} \circ f)(b) \text{ and } (\beta^{'} \circ f)(a) = (\beta^{'} \circ f)(b) \\ &\Longrightarrow (f_0 \circ \alpha)(a) = (f_0 \circ \alpha)(b) \text{ and } (f_0 \circ \beta)(a) = (f_0 \circ \beta)(b) \end{aligned}$$

$$\Longrightarrow \alpha(a) = \alpha(b) \text{ and } \beta(a) = \beta(b).$$

If we denote $\alpha(a) = x$ and $\beta(b) = y$, then $a, b \in G_x^y$ and $f_x^y(a) = f_x^y(b)$. Since f_x^y is injective, we obtain $a = b$. Therefore f is injective.

(iii) Let (f, f_0) be fibrewise injective. Hence $f_x : G_x \to G'_{f_0(x)}$ is injective. For the proof we need only show that $Ker\ f = \epsilon(G_0)$. It is obvious that $\epsilon(G_0) \subseteq Ker\ f$. Therefore, it is enough to prove only that $Ker\ f \subseteq \epsilon(G_0)$. For this, let us take any $a \in Ker\ f$. Let us denote $\alpha(a)$, $f_0(x)$ by x and x', respectively. If $f(a) \in \epsilon'(G'_0)$, then $f(a) = \epsilon'(x')$ with $x' \in G'_0$. Further

$$\begin{aligned} f(\epsilon(x)) = \epsilon'(f_0(x)) = \epsilon'(x') &\Longrightarrow f(a) = f(\epsilon(x)) \\ &\Longrightarrow f_x(a) = f_x(\epsilon(x))\ ,\ f_x \textit{ is injective} \\ &\Longrightarrow a = \epsilon(x). \end{aligned}$$

Hence $a \in \epsilon(G_0)$. This means that $Ker\ f \subseteq \epsilon(G_0)$. Thus $Ker\ f = \epsilon(G_0)$. Consequently, $Ker\ f$ is discrete.

Conversely, assume that f has discrete kernel. Let us take $a, b \in G_x$ such that $f_x(a) = f_x(b)$. Then $\alpha(a) = \alpha(b) = x$ and $f(a) = f(b)$. Hence we obtain $f(a)(f(b))^{-1} = \epsilon'(x')$ with $x' \in G'_0$. Then

$$\begin{aligned} f(ab^{-1}) = \epsilon'(x') &\Longrightarrow ab^{-1} \in Ker\ f = \epsilon(G_0) \\ &\Longrightarrow ab^{-1} = \epsilon(x)\ ,\ x \in G_0 \\ &\Longrightarrow a = \epsilon(x)b \\ &\Longrightarrow \epsilon(x)a = \epsilon(x)\epsilon(x)b \\ &\Longrightarrow \epsilon(x)a = \epsilon(x)b \\ &\Longrightarrow a = b. \end{aligned}$$

Therefore (f, f_0) is fibrewise injective. □

Proposition 4.6. *Let $(f, f_0) : (G, G_0) \to (G', G'_0)$ be a fibrewise surjective homomorphism of topological groupoids. Then (f, f_0) is a fibrewise bijective homomorphism if and only if $Ker\ f$ is discrete.*

Proof. The proof is a direct consequence of Theorem 4.5(iii). □

Theorem 4.7. *Let $(f, f_0) : (G, G_0) \to (G', G'_0)$ and $(g, g_0) : (G', G'_0) \to (G'', G''_0)$ be topological groupoid homomorphisms. Then*
(i) if f, g are fibrewise surjective (resp.,bijective) homomorphisms, then so is $g \circ f$,
(ii) if $g \circ f$ and f are fibrewise surjective homomorphisms and $f_0 : G_0 \to G'_0$ is surjective, then g is a fibrewise surjective homomorphism,
(iii) if $g \circ f$ and g are fibrewise bijective homomorphisms, then so is f.

Proof. (i) The proof follows from a direct computation.
(ii) We need show that $g_{x'} : St_{G'}(x') \to St_{G''}(g_0 x')$, $x' \in G'_0$ is surjective. If we take any $b'' \in St_{G''}(g_0 x')$, then $(\alpha'')(b'') = g_0(x')$ with $x' \in G'_0$. There

exists $x \in G_0$ such that $f_0(x) = x^{'}$, because f_0 is surjective. Then $(g \circ f)_x : St_G(x) \to St_{G^{''}}(g \circ f)_0(x)$ is surjective. Hence $(\alpha^{''})(b^{''}) = (g_0 \circ f_0)(x)$ and therefore there exists some $a \in St_G(x)$ such that $\alpha(a) = x$ and $(g \circ f)(a) = b^{''}$. If we say that $a^{'} = f(a)$, $a^{'} \in St_{G^{'}} x^{'}$ since

$$\alpha^{'}(a^{'}) = \alpha^{'}(f(a)) = f_0(\alpha(a)) = f_0(x) = x^{'}.$$

Furthermore, from $(g \circ f)(a) = b^{''}$, we obtain $g_{x^{'}}(a^{'}) = b^{''}$. Thus $g_{x^{'}}$ is surjective. So g is a fibrewise surjective homomorphism.
(iii) We show that f_x is bijective for all $x \in St_G x$. For any $a, b \in St_G(x)$, we suppose that $f_x(a) = f_x(b)$. Then

$$\begin{aligned} f(a) = f(b) &\implies (g \circ f)(a) = (g \circ f)(b) \\ &\implies (g \circ f)_x(a) = (g \circ f)_x(b), (g \circ f)_x \text{ is injective} \\ &\implies a = b. \end{aligned}$$

Hence f is fibrewise injective.

It is clear that $(g \circ f)_x = g_{f_0(x)} \circ f_x$ and $(g \circ f)_0 = g_0 \circ f_0$. Let us take any $b^{'} \in St_{G^{''}}(y)$, where $y = f_0(x)$. Then

$$g_y(b^{'}) \in St_{G^{''}}(g_0(y)) \Longrightarrow g_{f_0(x)}(b^{'}) \in St_{G^{''}}(g_0 \circ f_0)(x).$$

Because of the surjectivity of $(g \circ f)_x$, for $g_{f_0(x)}(b^{'}) \in St_{G^{''}}(g_0 \circ f_0)(x)$ there exists an arrow $a \in St_G(x)$ such that $(g \circ f)(a) = g_{f_0(x)}(b^{'})$. It follows that $g_{f_0(x)}(f_x(a)) = g_{f_0(x)}(b^{'})$. We obtain $f_x(a) = b^{'}$ because of the injectivity of $g_{f_0(x)}$. Hence f_x is surjective. Consequently, f is a fibrewise bijective homomorphism. This completes the proof. $\square$

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