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ON THE W_2-CURVATURE TENSOR OF THE SEMI-SYMMETRIC NON-METRIC CONNECTION IN A KENMOTSU MANIFOLD

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Abstract. The objective of the present paper is to study the W_2-curvature tensor of the semi-symmetric non-metric connection in a Kenmotsu manifold. It is shown that if in M^n, $W_2^* = 0$, then M^n is isometric to the hyperbolic space $H^n(-1)$, where W_2^* is the W_2-curvature tensor of the semi-symmetric non-metric connection. Also, locally W_2-ϕ-symmetric Kenmotsu manifold and W_2-ϕ-recurrent Kenmotsu manifold with respect to the semi-symmetric non-metric connection have been studied.

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1. Introduction

In 1924, A. Friedmann and J.A. Schouten [6] introduced the notion of semi-symmetric linear connection on a differentiable manifold. In 1930, Bartolotti [4] gave a geometrical meaning of such a connection. In 1932, H.A. Hayden [7] defined and studied semi-symmetric metric connection. In 1970, K. Yano [19], started a systematic study of the semi-symmetric metric connection in a Riemannian manifold, and this was further studied by various authors.

A linear connection ∇^* on a Riemannian manifold M^n is called semi-symmetric if its torsion tensor

$$T^*(X,Y) = \nabla^*_X Y - \nabla^*_Y X - [X,Y]$$

satisfies

$$T^*(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a non-zero 1-form associated with a vector field ξ defined by $\eta(X) = g(X,\xi)$. A semi-symmetric connection ∇^* is called semi-symmetric metric connection [7] if it further satisfies $\nabla^*_X g = 0$; otherwise it is non-metric.

In 1975, Prvanović [14] introduced the concept of semi-symmetric non-metric connection with the name pseudo-metric, which was further studied by Andonie ([2], [3]). The study of semi-symmetric non-metric connection

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is much older than the nomenclature 'non-metric' was introduced. In 1992, Agashe and Chafle [1] introduced a semi-symmetric connection ∇^* satisfying $\nabla^*_X g \neq 0$, and called such a connection as *semi-symmetric non-metric connection*. Semi-symmetric connections were further studied by several authors such as Sengupta, De and Binh [15], Pathak and De [11], Singh and Pandey [16], Singh, Pandey and Pandey [17], and many others.

Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jaruselam or Mekka or the North pole, then this displacement is semi-symmetric and metric [6].

On the other hand, in 1972, K. Kenmotsu [9] studied a class of contact Riemannian manifolds satisfying some special conditions. We call it Kenmotsu manifold. Kenmotsu proved that if in a Kenmotsu manifold the condition $R(X,Y).R = 0$ holds, then the manifold is of negative curvature -1, where R is the curvature tensor of type (1, 3) and $R(X,Y)$ denotes the derivations of the tensor algebra at each point of the the tangent space. A Riemannian manifold satisfying the condition $R(X,Y).R = 0$ is called semi-symmetric [18]. In [8], Jun, De and Pathak have studied some relations about semi-symmetric, Ricci semi-symmetric or Weyl semi-symmetric conditions in Riemannian manifolds. In [20], Yildiz and De have studied W_2-semi-symmetric Kenmotsu manifolds. They have classified Kenmotsu manifolds which satisfy $P.W_2 = 0$, $I.W_2 = 0$, $C.W_2 = 0$ and $\tilde{C}.W_2 = 0$, where P, I, C and $\tilde{C}$ are the projective curvature tensor, concircular curvature tensor, conformal curvature tensor and quasi-conformal curvature tensor respectively.

In 1970, Pokhariyal and Mishra [13] have introduced new tensor fields, called W_2 and E-tensor fields in a Riemannian manifold and studied their properties. Again, Pokhariyal [12] have studied some properties of these tensor fields in a Sasakian manifolds. Recently, Matsumoto, Ianus and Mihai [10] have studied P-Sasakian manifolds admitting W_2 and E-tensor fields. Also, De and Sarkar [5], Yildiz and De [20] have studied W_2-curvature tensor. The curvature tensor $'W_2$ is defined by

(1.1)

$$'W_2(X,Y,Z,U) = {}'R(X,Y,Z,U) + \frac{1}{n-1}\{g(X,Z)Ric(Y,U) - g(Y,Z)Ric(X,U)\},$$

where Ric is the Ricci tensor of type (0, 2) and

$$'W_2(X,Y,Z,U) = g(W_2(X,Y)Z,U)$$

and

$$'R(X,Y,Z,U) = g(R(X,Y)Z,U),$$

for the arbitrary vector fields X, Y, Z and U.

Motivated by the above studies, in the present paper, we consider the W_2-curvature tensor of a semi-symmetric non-metric connection and study some curvature conditions. The present paper is organized as follows: In Section 2, some preliminary results regarding Kenmotsu manifold are recalled. In Section 3, we obtain the curvature tensor, Ricci tensor and scalar curvature of the semi-symmetric non-metric connection. Section 4 is devoted to the study of the W_2-curvature tensor of semi-symmetric non-metric connection in the Kenmotsu manifold. In this section is shown that, if $W_2^* = 0$ in M^n then M^n is isomorphic to hyperbolic space $H^n(-1)$, where W_2^* is the W_2-curvature tensor of the semi-symmetric non-metric connection ∇^* . Also, $R^*(\xi, X).W_2^* = 0$, $W_2^*(\xi, X).R^* = 0$ and $W_2^*(\xi, X).Ric^* = 0$ have been studied and obtained in each case that M^n is an Einstein manifold, where R^* and Ric^* are the curvature tensor and Ricci tensor respectively of the semi-symmetric non-metric connection ∇^*. In Section 5, a locally W_2-ϕ-symmetric Kenmotsu manifold with respect to semi-symmetric non-metric connection have been studied. The last section is devoted to the study of the W_2-ϕ-recurrent Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

2. Preliminaries

If on an odd-dimensional differentiable manifold M^n of differentiability class C^{r+1} , there exists a vector valued real linear function ϕ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

(2.1) $$\phi^2 X = -X + \eta(X)\xi,$$

(2.2) $$\eta(\phi X) = 0,$$

(2.3) $$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for arbitrary vector fields X and Y, then the structure (ϕ, ξ, η, g) is called an almost contact metric structure and the manifold M^n with this structure is called an almost contact metric manifold. In view of equations (2.1), (2.2) and (2.3), we have

(2.4) $$\eta(\xi) = 1, g(X, \xi) = \eta(X), \phi\xi = 0.$$

An almost contact metric manifold is called Kenmotsu manifold ([9]) if

(2.5) $$(\nabla_X \phi) = -\eta(Y)\phi X - g(X, \phi Y)\xi,$$

(2.6) $$\nabla_X \xi = X - \eta(X)\xi,$$

where ∇ is the Levi-Civita connection. Also the following relations hold in the Kenmotsu manifolds

(2.7) $$(\nabla_X \eta)(Y) = g(X, Y) + \eta(X)\eta(Y),$$

(2.8) $$R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

(2.9) $$R(\xi, X)Y = -R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi,$$

(2.10) $$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X),$$

(2.11) $$Ric(X,\xi) = -(n-1)\eta(X),$$

(2.12) $$Q\xi = -(n-1)\xi,\ r = -n(n-1),$$

where Q is the Ricci operator, i.e.

$$g(QX,Y) = Ric(X,Y),$$

and r is the scalar curvature of the connection ∇,

(2.13) $$Ric(\phi X,\phi Y) = Ric(X,Y) + (n-1)\eta(X)\eta(Y),$$

for the arbitrary vector fields X,Y,Z on M^n.
A Kenmotsu manifold is said to be an η-Einstein manifold if its Ricci tensor *Ric* is of the form

(2.14) $$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for the vector fields X and Y, where a and b are functions on M^n.

Let M^n be an n-dimensional Kenmotsu manifold and ∇ be the Levi-Civita connection on M^n. A linear connection ∇^* [19] on M^n is given by

(2.15) $$\nabla^*_X Y = \nabla_X Y + \eta(Y)X.$$

Using equation (2.15), the torsion tensor T^* of M^n with respect to the connection ∇^* is given by

(2.16) $$T^*(X,Y) = \nabla^*_X Y - \nabla^*_Y X - [X,Y] = \eta(Y)X - \eta(X)Y,$$

which shows that the linear connection defined in equation (2.15) is a semi-symmetric connection.
Moreover, using equation (2.15) we have, for all vector fields X, Y, Z

(2.17) $$\begin{aligned}(\nabla^*_X g)(Y,Z) &= \nabla^*_X g(Y,Z) - g(\nabla^*_X Y,Z) - g(Y,\nabla^*_X Z)\\ &= -\eta(Y)g(X,Z) - \eta(Z)g(X,Y).\end{aligned}$$

A linear connection ∇^* defined in equation (2.15) satisfies equations (2.16) and (2.17), and therefore we call ∇^* a semi-symmetric non-metric connection.

3. Curvature tensor of semi-symmetric non-metric connection in a Kenmotsu manifold

The curvature tensor R^* of the semi-symmetric non-metric connection ∇^* in M^n is defined by

(3.1) $$R^*(X,Y)Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z.$$

Using equations (2.7) and (2.15) in the above equation, we get

$$\begin{aligned} R^*(X,Y)Z = R(X,Y)Z + \{g(X,Z)Y - g(Y,Z)X\} \\ + 2\eta(Z)\{\eta(Y)X - \eta(X)Y\}, \end{aligned} \tag{3.2}$$

where R is the Riemannian curvature tensor of ∇. From the above equation, we have

$$\begin{aligned} 'R^*(X,Y,Z,U) = 'R(X,Y,Z,U) + \{g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\} \\ + 2\eta(Z)\{\eta(Y)g(X,U) - \eta(X)g(Y,U)\}, \end{aligned} \tag{3.3}$$

where $'R^*(X,Y,Z,U) = g(R^*(X,Y)Z,U)$.
Putting $X = U = e_i$ in the above equation and taking summation over i, $1 \le i \le n$, we get

$$Ric^*(Y,Z) = Ric(Y,Z) - (n-1)g(Y,Z) + 2(n-1)\eta(Y)\eta(Z), \tag{3.4}$$

where Ric^* and Ric are the Ricci tensors of the connections ∇^* and ∇ respectively.
This gives

$$Q^*Y = QY - (n-1)Y + 2(n-1)\eta(Y)\xi. \tag{3.5}$$

Contracting the above equation, we get

$$r^* = r - n^2 + 3n - 2, \tag{3.6}$$

where r^* and r are the scalar curvatures of the connections ∇^* and ∇ respectively. Putting $X = \xi$ in equation (3.2) and using equations (2.4) and (2.9), we get

$$R^*(\xi,Y)Z = -R^*(Y,\xi)Z = 2\{\eta(Y)\eta(Z) - g(Y,Z)\}\xi. \tag{3.7}$$

4. W_2-Curvature Tensor of Semi-Symmetric Non-Metric Connection in a Kenmotsu Manifold

From equation (1.1), we have

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\{g(X,Z)QY - g(Y,Z)QX\}. \tag{4.1}$$

The W_2- curvature tensor of the semi-symmetric non-metric connection ∇^* in a Kenmotsu manifold M^n is given by

$$W_2^*(X,Y)Z = R^*(X,Y)Z + \frac{1}{n-1}\{g(X,Z)Q^*Y - g(Y,Z)Q^*X\}. \tag{4.2}$$

Using equations (3.2) and (3.5) in the above equation, we get

(4.3)
$$\begin{aligned} W_2^*(X,Y)Z = R(X,Y)Z + 2\eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ + 2\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi + \frac{1}{n-1}\{g(X,Z)QY - g(Y,Z)QX\}, \end{aligned}$$

which on using equation (4.1), gives

$$W_2^*(X,Y)Z = W_2(X,Y)Z + 2\{\eta(Y)X - \eta(X)Y\}\eta(Z) + 2\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi. \tag{4.4}$$

Putting $Z = \xi$ in the above equation and using equations (2.4), (2.8) and (4.1), we get

$$W_2^*(X,Y)\xi = \{\eta(Y)X - \eta(X)Y\} + \frac{1}{n-1}\{\eta(X)QY - \eta(Y)QX\}, \tag{4.5}$$

which gives

$$\eta(W_2^*(X,Y)\xi) = 0. \tag{4.6}$$

Again, putting $X = \xi$ in equation (4.4) and using equations (2.4), (2.9), (3.5) and (4.1), we get

$$\begin{aligned} W_2^*(\xi,Y)Z =& - W_2^*(Y,\xi)Z \\ =& - \eta(Z)Y + \frac{1}{(n-1)}\eta(Z)QY + 4\eta(Y)\eta(Z)\xi - 2g(Y,Z)\xi. \end{aligned} \tag{4.7}$$

Lemma 4.1. *An η-Einstein Kenmotsu manifold of the form*

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

is an Einstein manifold, where a or b are constants [8].

Theorem 4.2. *In a Kenmotsu manifold M^n, if W_2-curvature tensor of the semi-symmetric non-metric connection vanishes, then it is isomorphic to the hyperbolic space $H^n(-1)$.*

Proof. Let $W_2^* = 0$. In view of equation (4.3), we have

$$\begin{aligned} R(X,Y)Z =& 2\{\eta(X)Y - \eta(Y)X\}\eta(Z) - 2\{g(X,Z)\eta(Y) \\ & - g(Y,Z)\eta(X)\}\xi - \frac{1}{n-1}\{g(X,Z)QY - g(Y,Z)QX\}. \end{aligned} \tag{4.8}$$

Taking the inner product of the above equation with ξ and using equation (2.4), we get

$$g(R(X,Y)Z,\xi) = -[g(X,Z)g(Y,\xi) - g(Y,Z)g(X,\xi)], \tag{4.9}$$

which gives

$$R(X,Y,Z,U) = -[g(X,Z)g(Y,U) - g(Y,Z)g(X,U)]. \tag{4.10}$$

This shows that M^n is isomorphic to the hyperbolic space $H^n(-1)$. □

Theorem 4.3. *A Kenmotsu manifold M^n with the semi-symmetric non-metric connection ∇^* satisfying $R^*(\xi,X).W_2^* = 0$, is an η-Einstein manifold.*

Proof. Let $(R^*(\xi, Z).W_2^*)(X,Y)U = 0$. Then, we have

$$\begin{aligned} R^*(\xi, Z)W_2^*(X,Y)U - W_2^*(R^*(\xi, Z)X, Y)U \\ - W_2^*(X, R^*(\xi, Z)Y)U - W_2^*(X,Y)R^*(\xi, Z)U = 0, \end{aligned} \tag{4.11}$$

which on using equation (3.7), gives

$$\begin{aligned} &\eta(Z)\eta(W_2^*(X,Y)U)\xi) - g(Z, W^*{}_2(X,Y)U)\xi - \eta(X)\eta(Z)W_2^*(\xi, Y)U \\ &+ g(X,Z)W_2^*(\xi, Y)U - \eta(Y)\eta(Z)W_2^*(\xi, X)U + g(Y,Z)W_2^*(X,\xi)U \\ &- \eta(Z)\eta(U)W_2^*(X,Y)\xi + g(Z,U)W_2^*(Y,Z)\xi = 0. \end{aligned} \tag{4.12}$$

Now, using equations (4.4), (4.5) and (4.7) in the above equation, we get

$$\begin{aligned} &\eta(Z)\eta(W_2(X,Y)U)\xi) - g(Z, W_2(X,Y)U)\xi - 2g(Z,Y)\eta(X)\eta(U)\xi \\ &+ 2g(Z,X)\eta(Y)\eta(U)\xi + 2\eta(X)\eta(Z)\eta(U)Y - \frac{2}{n-1}\eta(X)\eta(Z)\eta(U)QY \\ &+ 2g(Y,U)\eta(X)\eta(Z)\xi - g(Z,X)\eta(U)Y + \frac{1}{n-1}g(Z,X)\eta(U)QY \\ &- 2g(Z,X)g(Y,U)\xi - 2\eta(Y)\eta(Z)\eta(U)X + \frac{2}{n-1}\eta(Y)\eta(Z)\eta(U)QX \\ &- 2g(X,U)\eta(Y)\eta(Z)\xi + g(Z,Y)\eta(U)X - \frac{1}{n-1}g(Z,Y)\eta(U)QX \\ &+ 2g(Z,Y)g(X,U)\xi + g(Z,U)\eta(Y)X - g(Z,U)\eta(X)Y \\ &+ \frac{1}{n-1}g(Z,U)[\eta(X)QY - \eta(Y)QX] = 0. \end{aligned} \tag{4.13}$$

Taking the inner product of the above equation with ξ, we get

$$\begin{aligned} &\eta(Z)\eta(W_2(X,Y)U) - g(Z, W_2(X,Y)U) - 2g(X,U)\eta(Y)\eta(Z) \\ &+ 2g(Y,U)\eta(X)\eta(Z) - 2g(Z,X)g(Y,U) + 2g(Z,Y)g(X,U) = 0. \end{aligned} \tag{4.14}$$

Using equation (4.2) in the above equation, we get

$$\begin{aligned} &{}'R(X,Y,U,Z) = 2[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(Z) \\ &- \frac{1}{n-1}[Ric(Y,Z)g(X,U) - Ric(X,Z)g(Y,U)] \\ &- 2[g(Z,X)g(Y,U) - g(Y,Z)g(X,U)]. \end{aligned} \tag{4.15}$$

Putting $X = Z = e_i$ in the above equation and taking summation over i, $1 \leq i \leq n$, we get

$$Ric(Z,U) = ag(Y,U) + b\eta(Y)\eta(U), \tag{4.16}$$

where $a = \frac{(4-3n)(n-1)}{n}$ and $b = \frac{-2(n-1)}{n}$. This shows that M^n is an η-Einstein manifold.
This completes the proof. □

Now, in view of Lemma 4.1, we can state as follows

Corollary 4.4. *A Kenmotsu manifold M^n with the semi-symmetric non-metric connection ∇^* satisfying $R^*(\xi, X).W_2^* = 0$, is an Einstein manifold.*

Theorem 4.5. *A Kenmotsu manifold M^n with the semi-symmetric non-metric connection ∇^* satisfying $W_2^*(\xi, Z).R^* = 0$, is an η-Einstein manifold.*

Proof. Consider $(W_2^*(\xi, Z).R^*)(X, Y)\xi = 0$. Then, we have

$$\begin{aligned} &W^*{}_2(\xi, Z)R^*(X, Y)\xi - R^*(W^*{}_2(\xi, Z)X, Y)\xi \\ &- R^*(X, W^*{}_2(\xi, Z)Y)\xi - R^*(X, Y)W^*{}_2(\xi, Z)\xi = 0, \end{aligned} \tag{4.17}$$

which on using equation (4.7), gives

(4.18)
$$\begin{aligned} &- \eta(R^*(X, Y, U)Z) + \frac{1}{n-1}\eta(R^*(X, Y, U)QZ) + 4\eta(Z)\eta(R^*(X, Y, U)\xi) \\ &- 2g(Z, R^*(X, Y, U))\xi + \eta(X)R^*(Z, Y, U) - \frac{1}{n-1}\eta(X)R^*(QZ, Y, U) \\ &- 4\eta(X)\eta(Z)R^*(\xi, Y, U) + 2g(X, Z)R^*(\xi, Y, U) + \eta(Y)R^*(X, Z, U) \\ &- \frac{1}{n-1}\eta(Y)R^*(X, QZ, U) - 4\eta(Y)\eta(Z)R^*(X, \xi, U) + 2g(Y, Z)R^*(X, \xi, U) \\ &+ \eta(U)R^*(X, Y, Z) - \frac{1}{n-1}\eta(U)R^*(X, Y, QZ) - 4\eta(Z)\eta(U)R^*(X, Y)\xi \\ &+ 2g(Z, U)R^*(X, Y)\xi = 0. \end{aligned}$$

Now, using equation (3.2) in the above equation, we get

(4.19)
$$\begin{aligned} &2'R(X, Y, U, Z)\xi = -\eta(R(X, Y, U))Z + \frac{1}{n-1}\eta(R(X, Y, U))QZ \\ &+ 4\eta(Z)\eta(R(X, Y, U))\xi - 2g(Y, U)g(Z, X)\xi - 4\eta(Y)\eta(U)g(Z, X)\xi \\ &+ \eta(X)R(Z, Y, U) + g(Z, U)\eta(X)Y - 4\eta(X)\eta(Z)\eta(U)Y \\ &- \frac{1}{n-1}\eta(X)R(QZ, Y, U) - \frac{1}{n-1}\eta(X)Ric(Z, U)Y + 4\eta(X)\eta(Z)g(Y, U)\xi \\ &- 4g(X, Z)g(Y, U)\xi + 2g(X, Z)\eta(U)Y + 4g(X, Z)\eta(Y)\eta(U)\xi \\ &+ \eta(Y)R(X, Z, U) - g(Z, U)\eta(Y)X - \frac{1}{n-1}\eta(Y)R(X, QZ, U) \\ &+ \frac{1}{n-1}\eta(Y)Ric(Z, U)X - 4\eta(Y)\eta(Z)g(X, U)\xi + 2g(Y, Z)g(X, U)\xi \\ &+ \eta(U)R(X, Y, Z) + 4\eta(Y)\eta(Z)\eta(U)X - \frac{1}{n-1}\eta(U)R(X, Y, QZ) \\ &- \frac{1}{n-1}Ric(X, Z)\eta(U)Y + \frac{1}{n-1}Ric(Y, Z)\eta(U)X. \end{aligned}$$

Taking the inner product of the above equation with ξ, we get

(4.20)

$$\begin{aligned}&2'R(X,Y,U,Z) = 2\eta(Z)\eta(R(X,Y,U)) - 6g(X,Z)g(Y,U)\\&+ 2\eta(X)\eta(R(Z,Y,U)) - \frac{1}{n-1}Ric(Z,U)\eta(X)\eta(Y) + 4\eta(X)\eta(Z)g(Y,U)\\&+ 2g(X,Z)\eta(Y)\eta(U) + 2\eta(Y)\eta(R(X,Z,U)) + \frac{1}{n-1}\eta(X)\eta(Y)Ric(Z,U)\\&- 4\eta(Y)\eta(Z)g(X,U) + 2g(Y,Z)g(X,U) + 2\eta(U)\eta(R(X,Y,Z))\\&- \frac{1}{n-1}Ric(X,Z)\eta(Y)\eta(U) + \frac{1}{n-1}Ric(Y,Z)\eta(X)\eta(U).\end{aligned}$$

Putting $X = Z = e_i$ in the above equation and taking summation over i, $1 \le i \le n$, we get

$$Ric(Y,U) = ag(Y,U) + b\eta(Y)\eta(U), \tag{4.21}$$

where $a = -(3n-1)$ and $b = \frac{3n-1}{2}$. This shows that M^n is an η-Einstein manifold.
This completes the proof. □

Now by Lemma 4.1, we can state as follows

Corollary 4.6. *A Kenmotsu manifold M^n with the semi-symmetric non-metric connection ∇^* satisfying*
$W_2^*(\xi,Z).R^* = 0$, is an Einstein manifold.

Theorem 4.7. *A Kenmotsu manifold M^n with the semi-symmetric non-metric connection ∇^* satisfying*
$(W_2^*(\xi,Z).Ric^*)(X,Y) = 0$, is an η-Einstein manifold.

Proof. Consider $(W_2^*(\xi,Z).Ric^*)(X,Y) = 0$. Then, we have

$$Ric^*(W_2^*(\xi,Z)X,Y) + Ric^*(X,W_2^*(\xi,Z)Y) = 0, \tag{4.22}$$

which on using equations (3.4) and (4.7), gives

$$\begin{aligned}&-3Ric(Y,Z)\eta(X) - 3Ric(X,Z)\eta(Y)\\&+(n-1)g(Y,Z)\eta(X) + (n-1)g(X,Z)\eta(Y) = 0.\end{aligned} \tag{4.23}$$

Now, putting $X = \xi$ in the above equation and using equations (2.4) and (2.11), we get

$$Ric(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{4.24}$$

where $a = \frac{(n-1)}{3}$ and $b = \frac{4(n-1)}{3}$.
This shows that M^n is an η-Einstein manifold.
This completes the proof. □

Now by Lemma 4.1, we can state as follows

Corollary 4.8. *A Kenmotsu manifold M^n with the semi-symmetric non-metric connection ∇^* satisfying ${W^*}_2(\xi,Z).Ric^*(X,Y) = 0$, is an Einstein manifold.*

5. Locally W_2-ϕ-symmetric Kenmotsu manifold with semi-symmetric non-metric connection

Definition 5.1. An n-dimensional Kenmotsu manifold M^n is said to be locally W_2-ϕ-symmetric, if

$$\phi^2((\nabla_U W_2)(X,Y)Z) = 0, \tag{5.1}$$

for all vector fields X, Y, Z and U orthogonal to ξ.

Definition 5.2. An n-dimensional Kenmotsu manifold M^n is said to be locally W_2-ϕ-symmetric with respect to the semi-symmetric non-metric connection if

$$\phi^2((\nabla^*_U W^*_2)(X,Y)Z) = 0, \tag{5.2}$$

for all vector fields X, Y, Z and U orthogonal to ξ, where W^*_2 is the W_2-curvature tensor of the semi-symmetric non-metric connection ∇^*.

Theorem 5.3. *A Kenmotsu manifold M^n is locally W_2-ϕ-symmetric with respect to the semi-symmetric non-metric connection ∇^* if and only if it is so with respect to the Levi-Civita connection ∇.*

Proof. From equation (2.15), we have

$$(\nabla^*_U W^*_2)(X,Y)Z = (\nabla_U W^*_2)(X,Y)Z + \eta(W^*_2(X,Y)Z)U. \tag{5.3}$$

Now, differentiating equation (4.4) covariantly with respect to U, we get

$$\begin{aligned}(\nabla_U W^*_2)(X,Y)Z &= (\nabla_U W_2)(X,Y)Z - 2(\nabla_U\eta)(X)\eta(Z)Y\\ &- 2(\nabla_U\eta)(Z)\eta(X)Y + 2(\nabla_U\eta)(Y)\eta(Z)X + 2(\nabla_U\eta)(Z)\eta(Y)X\\ &+ 2(\nabla_U\eta)(Y)g(X,Z)\xi - 2(\nabla_U\eta)(X)g(Y,Z)\xi.\end{aligned} \tag{5.4}$$

Now, using equation (5.4) in equation (5.3), we get

$$\begin{aligned}(\nabla^*_U W^*_2)(X,Y)Z &= (\nabla_U W_2)(X,Y)Z - 2(\nabla_U\eta)(X)\eta(Z)Y\\ &- 2(\nabla_U\eta)(Z)\eta(X)Y + 2(\nabla_U\eta)(Y)\eta(Z)X + 2(\nabla_U\eta)(Z)\eta(Y)X\\ &+ 2(\nabla_U\eta)(Y)g(X,Z)\xi - 2(\nabla_U\eta)(X)g(Y,Z)\xi\\ &+ 2g(X,Z)\eta(Y)U - 2g(Y,Z)\eta(X)U.\end{aligned} \tag{5.5}$$

Using equation (2.7) in the above equation, we get

$$\begin{aligned}(\nabla^*_U W^*_2)(X,Y)Z &= (\nabla_U W_2)(X,Y)Z - 2g(U,X)\eta(Z)Y\\ &+ 2g(U,Y)\eta(Z)X + 2g(U,Z)\eta(Y)X - 2g(U,Z)\eta(X)Y\\ &+ 2g(X,Z)\eta(Y)U - 2g(Y,Z)\eta(X)U + 2g(X,Z)g(Y,U)\xi\\ &- 2g(Y,Z)g(U,X)\xi + 2g(Y,Z)\eta(X)\eta(U)\xi - 2g(X,Z)\eta(Y)\eta(U)\xi\\ &+ 4\eta(X)\eta(Z)\eta(U)Y - 4\eta(Y)\eta(Z)\eta(U)X.\end{aligned} \tag{5.6}$$

Applying ϕ^2 on both sides of the above equation and using equations (2.1) and (2.2), we get

$$\begin{aligned}\phi^2((\nabla^*_U W^*_2)(X,Y)Z) &= \phi^2((\nabla_U W_2)(X,Y)Z) + 2g(U,X)\eta(Z)Y \\ &- 2g(U,X)\eta(Y)\eta(Z)\xi - 2g(U,Y)\eta(Z)X + 2g(U,Y)\eta(X)\eta(Z)\xi \\ &- 2g(U,Z)\eta(Y)X - 2g(U,Z)\eta(X)Y - 2g(X,Z)\eta(Y)U \\ &+ 2g(X,Z)\eta(Y)\eta(U)\xi + 2g(Y,Z)\eta(X)U - 2g(Y,Z)\eta(X)\eta(U)\xi \\ &- 4\eta(X)\eta(Z)\eta(U)Y + 4\eta(Y)\eta(Z)\eta(U)X.\end{aligned} \tag{5.7}$$

Now, if X, Y, Z, U are orthogonal to ξ , then the above equation reduces to

$$\phi^2((\nabla^*_U W^*_2)(X,Y)Z) = \phi^2((\nabla_U W_2)(X,Y)Z). \tag{5.8}$$

This completes the proof. □

6. W_2-ϕ-recurrent Kenmotsu manifold with semi-symmetric non-metric connection

Definition 6.1. An n-dimensional Kenmotsu manifold M^n is said to be W_2-ϕ-recurrent, if

$$\phi^2((\nabla_W W_2)(X,Y)Z) = A(W)W_2(X,Y)Z, \tag{6.1}$$

for the arbitrary vector fields X, Y, Z and W, where A is non-zero 1-form.

Definition 6.2. An n-dimensional Kenmotsu manifold M^n is said to be W_2-ϕ-recurrent with respect to the semi-symmetric non-metric connection if

$$\phi^2((\nabla^*_W W^*_2)(X,Y)Z) = A(W)W^*_2(X,Y)Z, \tag{6.2}$$

for arbitrary vector fields X, Y, Z and W.

Theorem 6.3. *A W_2- ϕ-recurrent Kenmotsu manifold with respect to a semi-symmetric non-metric connection is an η-Einstein manifold.*

Proof. From equations (2.1) and (6.2), we have

$$-((\nabla^*_W W^*_2)(X,Y)Z) + \eta((\nabla^*_W W^*_2)(X,Y)Z)\xi = A(W)W^*_2(X,Y)Z), \tag{6.3}$$

which reduces to

$$-g((\nabla^*_W W^*_2)(X,Y)Z,U)+\eta((\nabla^*_W W^*_2)(X,Y)Z)\eta(U) = A(W)g(W^*_2(X,Y)Z,U). \tag{6.4}$$

Using equations (4.4) and (5.6) in the above equation, we get

(6.5)

$$
\begin{aligned}
&- g((\nabla_W R)(X,Y)Z,U) - \frac{1}{n-1}[(\nabla_W Ric)(Y,U)g(X,Z) - (\nabla_W Ric)(X,U)g(Y,Z)]\\
&+ 2g(W,X)g(Y,U)\eta(Z) - 2g(W,Y)g(X,U)\eta(Z) - 2g(W,Z)g(X,U)\eta(Y)\\
&+ 2g(W,Z)g(Y,U)\eta(X) - 2g(X,Z)g(W,U)\eta(Y) + 2g(Y,Z)g(W,U)\eta(X)\\
&- 4g(Y,U)\eta(X)\eta(Z)\eta(W) + 4g(X,U)\eta(Y)\eta(Z)\eta(W) + 2g(X,Z)\eta(Y)\eta(W)\eta(U)\\
&- 2g(Y,Z)\eta(X)\eta(W)\eta(U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) - (\nabla_W \eta)(Y)\eta(U)g(X,Z)\\
&+ (\nabla_W \eta)(X)\eta(U)g(Y,Z) - 2g(W,X)\eta(Y)\eta(Z)\eta(U) + 2g(W,Y)\eta(X)\eta(Z)\eta(U)\\
&= A(W)g(W_2(X,Y)Z,U) - 2A(W)\{\eta(X)g(Y,U) - \eta(Y)g(X,U)\}\eta(Z)\\
&+ 2A(W)\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}\eta(U).
\end{aligned}
$$

Let $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point of the manifold. Then, putting $X = U = e_i$ in equation (6.5) and taking summation over i, $1 \le i \le n$, we get

(6.6)

$$
\begin{aligned}
&- \frac{n}{n-1}(\nabla_W Ric)(Y,Z) + \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) + \frac{1}{n-1}g(Y,Z)(\nabla_W r)\\
&+ (4n-4)\eta(Y)\eta(Z)\eta(W) - 2ng(W,Z)\eta(Y) - (\nabla_W \eta)(Y)\eta(Z)\\
&+ (4-2n)g(W,Y)\eta(Z) = \frac{n}{n-1}A(W)Ric(Y,Z) - \frac{(r+2n-2)}{n-1}A(W)g(Y,Z)\\
&+ 2nA(W)\eta(Y)\eta(Z).
\end{aligned}
$$

Putting $Z = \xi$ in the above equation, we get

$$
\begin{aligned}
&- \frac{n}{n-1}(\nabla_W Ric)(Y,\xi) + \eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) + \frac{1}{n-1}\eta(Y)(\nabla_W r)\\
&+ (2n-4)\eta(Y)\eta(W) - (\nabla_W \eta)(Y) + (4-2n)g(W,Y)\\
&= \frac{n^2-3n-r+2}{n-1}A(W)\eta(Y).
\end{aligned}
\tag{6.7}
$$

The second term on L.H.S. of equation (6.7) takes the form

$$E = \eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi), \tag{6.8}$$

which is denoted by λ. In this case λ vanishes. Namely, we have

$$
\begin{aligned}
&g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi)\\
&- g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi).
\end{aligned}
\tag{6.9}
$$

at p $\in M^n$. In the local coordinates $\nabla_W e_i = W^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore, $\nabla_W e_i = 0$. Also, we have

$$g(R(e_i,\nabla_W Y)\xi,\xi) = 0, \tag{6.10}$$

since R is skew-symmetric. Using equation (6.10) and $\nabla_W e_i = 0$ in equation (6.9), we get

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \tag{6.11}$$

In view of $g(R(e_i, Y)\xi, \xi) = -g(R(\xi, \xi)Y, e_i) = 0$ and $\nabla_W g = 0$, we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0, \tag{6.12}$$

which implies

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

Since R is skew-symmetric, we have

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = 0. \tag{6.13}$$

Using equation (6.13) in equation (6.7), we get

(6.14)

$$(\nabla_W Ric)(Y, \xi) = n\eta(Y)(\nabla_W r) + \frac{(2n-4)(n-1)}{n}\eta(Y)\eta(W)$$
$$- \frac{n-1}{n}(\nabla_W \eta)(Y) - \frac{(2n-4)(n-1)}{n}g(W, Y) + \frac{(n^2-3n-r+2)}{n}A(W)\eta(Y).$$

Now, we have

$$(\nabla_W Ric)(Y, \xi) = \nabla_W Ric(Y, \xi) - Ric(\nabla_W Y, \xi) - Ric(Y, \nabla_W \xi), \tag{6.15}$$

which on using equations (2.6) and (2.11) takes the form

$$(\nabla_W Ric)(Y, \xi) = -(n-1)g(Y, W) - Ric(Y, W). \tag{6.16}$$

Form equations (6.14) and (6.16), we have

$$\begin{aligned} Ric(Y, W) &= \Big(\frac{n^2-5n+4}{n}\Big)g(Y, W) - \Big(\frac{2n^2-6n+4}{n}\Big)\eta(Y)\eta(W) \\ &- \Big(\frac{n^2-3n-r+2}{n}\Big)A(W)\eta(Y) - n\eta(Y)(\nabla_W r) + \frac{n-1}{n}(\nabla_W \eta)(Y). \end{aligned} \tag{6.17}$$

Replacing Y and W by ϕY and ϕW respectively in the above equation and using equations (2.2), (2.3) and (2.13), we get

$$Ric(Y, W) = \frac{n^2-5n+4}{n}g(Y, W) - \frac{2n^2-6n+4}{n}\eta(Y)\eta(W), \tag{6.18}$$

which shows that M^n is an $\eta-$Einstein manifold. □

Theorem 6.4. *In a W_2-ϕ-recurrent Kenmotsu manifold M^n admitting semi-symmetric non-metric connection, the characteristic vector field ξ and the vector field ρ associated with 1-form A are co-directional and the 1-form A is given by*

$$A(W) = \eta(\rho)\eta(W).$$

Proof. By virtue of equations (2.1) and (6.2), we have

$$(\nabla^*_W W^*_2)(X,Y)Z = \eta((\nabla^*_W W^*_2)(X,Y)Z)\xi - A(W)W^*_2(X,Y)Z. \tag{6.19}$$

Using equations (4.4) and (5.6) in the above equation, we get

$$\begin{aligned}
&(\nabla_W W_2)(X,Y)Z - 2g(W,X)\eta(Z)Y + 2g(W,Y)\eta(Z)X \\
&+ 2g(W,Z)\eta(Y)X - 2g(W,Z)\eta(X)Y + 2g(X,Z)\eta(Y)W \\
&- 2g(Y,Z)\eta(X)W + 2g(X,Z)g(Y,W)\xi - 2g(Y,Z)g(W,X)\xi \\
&+ 2g(Y,Z)\eta(X)\eta(W)\xi - 2g(X,Z)\eta(Y)\eta(W)\xi + 4\eta(X)\eta(Z)\eta(W)Y \\
&- 4\eta(Y)\eta(Z)\eta(W)X = \eta((\nabla_W W_2)(X,Y)Z)\xi - 2g(W,X)\eta(Y)\eta(Z)\xi \\
&+ 2g(W,Y)\eta(X)\eta(Z)\xi + 2g(X,Z)g(Y,W)\xi - 2g(Y,Z)g(W,X)\xi \\
&- A(W)W_2(X,Y)Z - 2A(W)\{\eta(Y)X - \eta(X)Y\}\eta(Z) \\
&- 2A(W)\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi.
\end{aligned} \tag{6.20}$$

Taking the inner product of the above equation with ξ and using equation (4.2), we get

$$A(W)\eta(R(X,Y)Z) = A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{6.21}$$

Writing two more equations by the cyclic permutations of W, X and Y from equation (6.21) and adding them to equation (6.21), we get

$$\begin{aligned}
&A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) \\
&= A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + A(X)[g(W,Z)\eta(Y) \\
&- g(Y,Z)\eta(W)] + A(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)].
\end{aligned} \tag{6.22}$$

Using equation (2.10) in the above equation,we get

$$\begin{aligned}
&A(W)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] + A(X)[g(Y,Z)\eta(W) - g(W,Z)\eta(Y)] \\
&+ A(Y)[g(W,Z)\eta(X) - g(X,Z)\eta(W)] = 0.
\end{aligned} \tag{6.23}$$

Putting Y=Z=e_i in the above equation and taking summation over i, $1 \le i \le n$, we get

$$A(W)\eta(X) = A(X)\eta(W), \tag{6.24}$$

for all vector fields X and W. Replacing X by ξ in the above equation, we get

$$A(W) = \eta(\rho)\eta(W), \tag{6.25}$$

for all vector fields W, where $A(\xi) = g(\xi,\rho) = \eta(\rho)$, ρ being the vector field associated to the 1-form A, i.e.

$$g(X,\rho) = A(X). \tag{6.26}$$

This completes the proof. □

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