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COEFFICIENT ESTIMATES FOR A CLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

Serap Bulut[1]

Abstract. In this paper, we introduce and investigate an interesting subclass $\mathcal{B}_{\Sigma}^{h,p}$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{B}_{\Sigma}^{h,p}$ we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The results presented in this paper would generalize and improve some recent work of Brannan and Taha [1].

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1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \backslash \{0\}$$

be the set of positive integers.

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [2] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\; r_0(f) \geq \frac{1}{4}\right).$$

[1]Kocaeli University, Civil Aviation College, Arslanbey Campus, 41285 İzmit-Kocaeli, Turkey, e-mail: serap.bulut@kocaeli.edu.tr

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$.

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions in the class Σ, see [3].

Brannan and Taha [1] introduced the following two subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

Definition 1. (see [1]) A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{S}_\Sigma^*[\alpha]$ $(0 < \alpha \leq 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{1.2}$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}), \tag{1.3}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.4}$$

We call $\mathcal{S}_\Sigma^*[\alpha]$ the class of strongly bi-starlike functions of order α.

Theorem 1.1. *(see [1]) Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the class $\mathcal{S}_\Sigma^*[\alpha]$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}} \tag{1.5}$$

and

$$|a_3| \leq 2\alpha. \tag{1.6}$$

Definition 2. (see [1]) A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{S}_\Sigma^*(\beta)$ $(0 \leq \beta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \qquad (z \in \mathbb{U}) \tag{1.7}$$

and

$$\Re\left\{\frac{wg'(w)}{g(w)}\right\} > \beta \qquad (w \in \mathbb{U}), \tag{1.8}$$

where the function g is defined by (1.4). We call $\mathcal{S}_\Sigma^*(\beta)$ the class of bi-starlike functions of order β.

Theorem 1.2. (*see [1]*) *Let the function* $f(z)$, *given by the Taylor-Maclaurin series expansion* (1.1), *be in the class* $\mathcal{S}_{\Sigma}^{*}(\beta)$ $(0 \leq \beta < 1)$. *Then*

$$|a_2| \leq \sqrt{2(1-\beta)} \tag{1.9}$$

and

$$|a_3| \leq 2(1-\beta). \tag{1.10}$$

Here, in our present sequel to some of the aforecited works (especially [1]), we introduce the following subclass of the analytic function class $\mathcal{A}$, analogously to the definition given by Xu et al. [4].

Definition 3. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min\{\Re(h(z)), \Re(p(z))\} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad h(0) = p(0) = 1.$$

Also let the function f, defined by (1.1), be in the analytic function class $\mathcal{A}$. We say that $f \in \mathcal{B}_{\Sigma}^{h,p}$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \frac{zf'(z)}{f(z)} \in h(\mathbb{U}) \quad (z \in \mathbb{U}) \tag{1.11}$$

and

$$\frac{wg'(w)}{g(w)} \in p(\mathbb{U}) \quad (w \in \mathbb{U}), \tag{1.12}$$

where the function g is defined by (1.4).

Remark 1. There are many choices of the functions h and p which would provide interesting subclasses of the analytic function class $\mathcal{A}$. For example, if we let

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad \text{and} \quad p(z) = \left(\frac{1-z}{1+z}\right)^{\alpha} \qquad (0 < \alpha \leq 1,\ z \in \mathbb{U})$$

or

$$h(z) = \frac{1+(1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1-(1-2\beta)z}{1+z} \qquad (0 \leq \beta < 1,\ z \in \mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in \mathcal{B}_{\Sigma}^{h,p}$, then

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ z \in \mathbb{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ w \in \mathbb{U})$$

or

$$f \in \Sigma \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (0 \leq \beta < 1,\ z \in \mathbb{U})$$

and

$$\Re\left(\frac{wg'(w)}{g(w)}\right) > \beta \qquad (0 \leq \beta < 1,\, w \in \mathbb{U}),$$

where the function g is defined by (1.4). This means that

$$f \in \mathcal{S}^*_\Sigma [\alpha] \qquad (0 < \alpha \leq 1)$$

or

$$f \in \mathcal{S}^*_\Sigma (\beta) \qquad (0 \leq \beta < 1).$$

Motivated and stimulated especially by the work of Brannan and Taha [1], we propose to investigate the bi-univalent function class $\mathcal{B}_\Sigma^{h,p}$ introduced here in Definition 3 and derive coefficient estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for a function $f \in \mathcal{B}_\Sigma^{h,p}$ given by (1.1). Our results for the bi-univalent function class $\mathcal{B}_\Sigma^{h,p}$ would generalize and improve the related work of Brannan and Taha [1].

2. A Set of General Coefficient Estimates

In this section we state and prove our general results involving the bi-univalent function class $\mathcal{B}_\Sigma^{h,p}$ given by Definition 3.

Theorem 2.1. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1.1)*, be in the bi-univalent function class $\mathcal{B}_\Sigma^{h,p}$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{4}}\right\} \tag{2.1}$$

and

$$|a_3| \leq \min\left\{\frac{|h'(0)|^2 + |p'(0)|^2}{2} + \frac{|h''(0)| + |p''(0)|}{8}, \frac{3\,|h''(0)| + |p''(0)|}{8}\right\}. \tag{2.2}$$

Proof. First of all, we write the argument inequalities in (1.11) and (1.12) in their equivalent forms as follows:

$$\frac{zf'(z)}{f(z)} = h(z) \quad (z \in \mathbb{U}),$$

and

$$\frac{wg'(w)}{g(w)} = p(w) \quad (w \in \mathbb{U}),$$

respectively, where $h(z)$ and $p(w)$ satisfy the conditions of Definition 3. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expensions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \cdots ,$$

respectively. Now, upon equating the coefficients of $\frac{zf'(z)}{f(z)}$ with those of $h(z)$ and the coefficients of $\frac{wg'(w)}{g(w)}$ with those of $p(w)$, we get

$$a_2 = h_1, \tag{2.3}$$

$$2a_3 - a_2^2 = h_2, \tag{2.4}$$

$$-a_2 = p_1 \tag{2.5}$$

and

$$3a_2^2 - 2a_3 = p_2. \tag{2.6}$$

From (2.3) and (2.5), we obtain

$$h_1 = -p_1 \tag{2.7}$$

and

$$2a_2^2 = h_1^2 + p_1^2. \tag{2.8}$$

Also, from (2.4) and (2.6), we find that

$$2a_2^2 = h_2 + p_2. \tag{2.9}$$

Therefore, we find from the equations (2.8) and (2.9) that

$$|a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2}$$

and

$$|a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{4},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.6) from (2.4). We thus get

$$4a_3 - 4a_2^2 = h_2 - p_2. \tag{2.10}$$

Upon substituting the value of a_2^2 from (2.8) into (2.10), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{2} + \frac{h_2 - p_2}{4}.$$

We thus find that

$$|a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2} + \frac{|h''(0)| + |p''(0)|}{8}.$$

On the other hand, upon substituting the value of a_2^2 from (2.9) into (2.10), it follows that

$$a_3 = \frac{3h_2 + p_2}{4}.$$

We thus obtain

$$|a_3| \leq \frac{3\left|h''(0)\right| + \left|p''(0)\right|}{8}.$$

This evidently completes the proof of Theorem 2.1. □

3. Corollaries and Consequences

If we set

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad \text{and} \quad p(z) = \left(\frac{1-z}{1+z}\right)^{\alpha} \qquad (0 < \alpha \leq 1,\ z \in \mathbb{U})$$

in Theorem 2.1, we can readily deduce the following corollary.

Corollary 3.1. *Let the function $f(z)$, given by the Taylor-Maclaurin series expansion* (1.1)*, be in the bi-univalent function class $\mathcal{S}_{\Sigma}^{*}[\alpha]$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \sqrt{2}\alpha$$

and

$$|a_3| \leq 2\alpha^2.$$

Remark 2. It is easy to see that

$$\sqrt{2}\alpha \leq \frac{2\alpha}{\sqrt{1+\alpha}} \qquad (0 < \alpha \leq 1)$$

and

$$2\alpha^2 \leq 2\alpha \qquad (0 < \alpha \leq 1),$$

which, in conjunction with Corollary 3.1, would obviously yield an improvement of Theorem 1.1.

If we set

$$h(z) = \frac{1 + (1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1 - (1-2\beta)z}{1+z} \qquad (0 \leq \beta < 1,\ z \in \mathbb{U})$$

in Theorem 2.1, we can readily deduce the following corollary.

Corollary 3.2. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the bi-univalent function class $\mathcal{S}_{\Sigma}^{*}(\beta)$ $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq \begin{cases} 2(1-\beta) & , \quad 0 \leq \beta \leq \frac{3}{4} \\ 4(1-\beta)^2 + (1-\beta) & , \quad \frac{3}{4} \leq \beta < 1 \end{cases}.$$

Remark 3. It is easy to see that

(**i**) if $0 \leq \beta \leq \frac{3}{4}$, then

$$|a_3| \leq 2\,(1-\beta)\,;$$

(**ii**) if $\frac{3}{4} \leq \beta < 1$, then

$$|a_3| \leq 4\,(1-\beta)^2 + (1-\beta) \leq 2\,(1-\beta)\,.$$

Thus, clearly, Corollary 3.2 is an improvement of Theorem 1.2.

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