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CHARACTERIZATION OF H^p−SPACES WITH BOUNDARY VALUES IN SOME SPACES OF BEURLING TEMPERED ULTRADISTRIBUTIONS

Ljupčo Nastovski[1]

Abstract. In this paper we give necessary and sufficient conditions for a function f belonging to H^p space with the convergence in the sense of ultradistribution $S'^{(s)}, s > 1$.

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1. Introduction

In [11], the next theorem is proved:

Theorem 1.1. *Let f be an analytic function in the upper half-plane $Imz \geq 0$ and suppose that there exists $n \in \mathbb{N}$ such that in every half-plane $Imz \geq \delta > 0$, there exists $C_\delta > 0$ such that*

$$|f(z)| \leq C_\delta(1 + |z|)^n$$

Then, f is in $H^p(\Pi^+)$ $(1 \leq p \leq \infty)$ if and only if $f(x + iy)$ converges to $f(x) \in L^p(-\infty, \infty)$ in the sense of converges in $(S^1)'$, as $y \to 0$.

S. Pilipović in 2004 posed the following problem: Let $s > 1$ and let $f(z)$ be an analytic function in the upper half plain Π^+ and let for every $\delta > 0$ there exist $C_\delta > 0$ and $K_\delta > 0$ such that:

$$|f(z)| \leq C_\delta e^{K_\delta |z|^{1/s}}, Imz \geq \delta.$$

Is it true that $f \in H^p(\Pi^+)$ $(1 \leq p \leq \infty)$ if and only if $f(z)$ converges to $f(x) \in L^p(R)$ in the sense of ultradistribution $S^{'(s)}, s > 1$?

We refer to Section 2 for generalized function spaces $(S^1)'$ and $S'^{(s)}$.

The aim of this paper is to give the positive answer to this question (Theorem 3.2).

Boundary values in ultradistributions spaces were studied by many authors, for example, [5], [7], [9], [10], [12]. (see references therein).

[1]Department of Mathematics, Faculty of Science, University of Skopje,
e-mail: ljupcona@iunona.pmf.ukim.edu.mk

2. Notation and notions

The following definitions and results are given in [1]. By $(M_p) = (M_p)_{p\in\mathbb{N}_0}$ we will denote a sequence of positive numbers which satisfies some of the following conditions:

$$M_p^2 \le M_{p-1}M_{p+1}, \quad p \in \mathbb{N}; \tag{2.1}$$

there are positive constants A and H such that

$$M_p \le AH^p \min_{0\le q\le p} M_q M_{p-q} \quad p \in \mathbb{N}_0; \tag{2.2}$$

there is a constant $A > 0$ such that

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \le ApM_p/M_{p+1}, p \in \mathbb{N}; \tag{2.3}$$

Sometimes (2.2) and (2.3) will be replaced by the following weaker conditions:

(2.4) there are constants A and H such that $M_{p+1} \le AH^p M_p, \quad p \in \mathbb{N}_0$

$$\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty. \tag{2.5}$$

If $s > 1$ the Gevrey sequence (M_p) given by $M_p = (p!)^s$, $M_p = p^{ps}$ and $M_p = \Gamma(1+ps)$, where Γ denotes the gamma function, are basic examples of sequences satisfying some of the above stated conditions.

For a sequence (M_p), the associated functions M and M^* of Komatsu, are defined by

$$M(\rho) = \sup_{p\in\mathbb{N}_0} \log(\rho^p M_0/M_p), \quad 0 < \rho < \infty,$$

$$M^*(\rho) = \sup_{p\in\mathbb{N}_0} \log(\rho^p p! M_0/M_p), \quad 0 < \rho < \infty.$$

The formal series

$$\sum_{j=0}^{\infty} a_j z^j, \quad j \in \mathbb{C}$$

is an ultrapolynomial of class (M_p) (resp. of class $\{M_p\}$) if there are constants $A > 0$, $h > 0$ (resp. for every $h > 0$ there is an $A > 0$) such that

$$|a_j| \le Ah^j/M_j, \quad j \in \mathbb{N}_0.$$

Let (M_p), $p \in \mathbb{N}_0$, be a sequence of positive numbers. We define $D((M_p), \Omega)$ (resp. $D(\{M_p\}, \Omega)$), where Ω is an open set in $\mathbb{R}^n$ to be the set of all complex valued infinitely differentiable functions φ with compact support in Ω such that there exists an $N > 0$ for which

$$\sup_{t \in \mathbb{R}^n} |D_t^\alpha \varphi(t)| \leq N H^\alpha M_\alpha, \quad \alpha \in \mathbb{N}_0^n \tag{2.6}$$

for all $h > 0$ (resp. for some $h > 0$). Here the positive constants N and h depend only on φ: they do not depend on α.

The topologies of $D((M_p), \Omega)$ and $D(\{M_p\}, \Omega)$ are given in Komatsu [4]. Let $D(h, K)$ denote the space of smooth functions supported by a compact set K for which (2.6) holds and $D((M_p), K)$ and $D(\{M_p\}, K)$ denote subspaces of $D((M_p), \Omega)$ and $D(\{M_p\}, \Omega)$ consisting of the elements supported by K, respectively. Recall that

$$\begin{aligned} D^{(M_p)}(\Omega) = D((M_p), \Omega) &= \mathit{ind} \lim\nolimits_{K \subset \Omega} \mathit{proj} \lim\nolimits_{h \to 0} D(h, k) \\ &= \mathit{ind} \lim\nolimits_{K \subset \Omega} D((M_p), K); \\ D^{\{M_p\}}(\Omega) = D((M_p), \Omega) &= \mathit{ind} \lim\nolimits_{K \subset \Omega} \mathit{ind} \lim\nolimits_{h \to 0} D(h, k) \\ &= \mathit{ind} \lim\nolimits_{K \subset \Omega} D(\{M_p\}, K). \end{aligned}$$

The dual space of $D^{(M_p)}(\Omega)$ equipped with strong topology will be denoted with $D'^{(M_P)}(\Omega)$, and will be called a space of ultradistribution of Beurling type. Respectively, with $D'^{\{M_p\}}(\Omega)$ will be denoted the dual space of $D^{\{M_P\}, \Omega}$ and will be called ultradistribution of Roumieu type.

Let the sequence (M_p) satisfies the conditions (2.1) and (2.5). The spaces of the ultradifferentiable functions which has an ultrapolynomial growth are test spaces for the spaces of tempered ultradistributions.

Let $S_r^{(M_p),m} = S_r^{(M_p),m}(\mathbb{R}^n)$ and $S_\infty^{(M_p),m} = S_\infty^{(M_p),m}(\mathbb{R}^n)$ be the space of smooth functions φ on $\mathbb{R}^n$ such that

$$\begin{aligned} \sigma_{m,r}(\varphi) &= \left[\sum\nolimits_{\alpha,\beta \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} \left| \tfrac{m^{\alpha+\beta}}{M_\alpha M_\beta} < x >^\beta \varphi^{(\alpha)}(x) \right|^r dx \right]^{1/r} \\ &= \left[\sum\nolimits_{\alpha,\beta \in \mathbb{N}_0^n} \left(\| \tfrac{m^{\alpha+\beta}}{M_\alpha M_\beta} < x >^\beta \varphi^{(\alpha)} \|_r \right)^r \right]^{1/r} < \infty, \end{aligned}$$

and

$$\sigma_{m,\infty}(\varphi) = \sup_{\alpha,\beta \in \mathbb{N}_0^n} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} \| < x >^\beta \varphi^{(\alpha)} \|_\infty,$$

equipped with the topology induced by the norms $\sigma_{m,r}$ and $\sigma_{m,\infty}$, respectively, where $< x >= (1 + |x|^2)^{1/2}$.

Let $S^{(M_p)} = S^{(M_p)}(\mathbb{R}^n)$ and $S^{\{M_p\}} = S^{\{M_p\}}(\mathbb{R}^n)$ be the projective (as $m \to \infty$) and the inductive (as $m \to 0$) limits of the space $S_2^{(M_p),m}$ respectively.

The dual spaces of $S^{(M_p)}$ and $S^{\{M_p\}}$ are denoted by $S'^{(M_p)}$ and $S'^{\{M_p\}}$ respectively. These are the spaces of tempered ultradistributions of Beurling and Roumie type, respectively.

In the case when the sequence (M_p) is defined with $M_p = p!^s, \quad s > 1$, the spaces of tempered ultradistributions $S'^{(M_p)}$ and $S'^{\{M_p\}}$ will be denote with $S'^{(s)}$ and $S'^{\{s\}}$, respectively. These spaces are studied in Grudzinski [3] and Pilipović [8].

A non-trivial example, in case $n = 1$, of an element of the space S'^* is

$$< f, \varphi >= \int_R f\varphi dx, \quad \varphi \in S^*,$$

where f is a locally integrable function of the ultrapolynomial growth of the class $*$, i.e.

$$|f(x)| \leq P(x), \quad x \in \mathbb{R}$$

where P is an ultrapolynomial of the class $*$ ($*$ denotes (M_p) or $\{M_p\}$). Note that if (2.4) is fulfilled the function f is of the ultrapolynomial growth of the class (M_α) (respectively,$\{M_p\}$) if and only if for some $m > 0$ and some $C > 0$ (respectively, for every $m > 0$ there exists $C > 0$) such that

$$|f(x)| \leq CexpM(m|x|), \quad x \in \mathbb{R}.$$

Let f be an analytic function in the upper half-plane $\Pi^+ = \{z : Imz > 0\}$ and let $p > 0$. Then $f \in H^p(\Pi^+) = H^p$ if

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < +\infty.$$

We need the following results.

Theorem 2.1 ([6]). *Let $f \in H^p(\Pi^+), p \geq 1$. Then there exists $f^* \in L^p(\mathbb{R})$ such that for almost every $t \in \mathbb{R}$ the nontangential limit*

$$\lim_{z \to t} f(z) = f^*(t).$$

Theorem 2.2 ([6]). *If $f \in H^p(\Pi^+), 1 \leq p$, then*

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} f^*(t)dt, z = x + iy.$$

Also, if $h \in L^p, (1 \leq p)$ and

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} h(t)dt, z = x + iy$$

is an analytic function in Π^+, then $f \in H^p(\Pi^+)$, and for its boundary function it is true that $f^*(t) = h(t)$ almost everywhere in $\mathbb{R}$.

Theorem 2.3 ([2]). *If $f \in H^p, 1 \leq p$, then*

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt, Imz = y > 0$$

and the integral is equal to zero for each $Imz = y < 0$.

Also, the opposite is true. If $h \in L^p, (1 \le p \le \infty)$ and if

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{h(t)}{t-z}dt = 0, Imz = y < 0$$

then for each $y > 0$ the integral represents the function $f \in H^p(\Pi^+)$ with the, boundary function $f^*(x) = h(x)$ almost everywhere in $\mathbb{R}$.

Theorem 2.4 ([13]). *Let f be an analytic function at $V \setminus \Omega$. If $f(x \pm i0) = \lim_{y\to 0^+} f(x \pm iy)$ exist in $D'^*(\Omega)$, if $f(x \pm i0)$ is bounded in Ω and if $f(x+i0) = f(x-i0)$, then f is analytic at V.*

3. Main Results

In Rajna's paper [11], a characterization of the functions from the $H^p(\Pi^+)$ -space, with asymptotic behavior and distributional boundary values, in the space of distribution $(S^1)'$, is given through Theorem 1.1. The distributions $(S^1)'$ are defined in the following way: S^1 is the space of all functions ϕ which are infinite differential on $\mathbb{R}$ such that $|\phi^{(n)}(x)| \le C_n e^{-a|x|}, \quad n \in \mathbb{N}$, where $C_n > 0$ and $a > 0$ depend upon ϕ. The space S^1 is the image of S_1 by a Fourier transformation. Their duals distributional spaces are denoted by $(S_1)'$ and $(S^1)'$, respectively.

Let (M_p) be a sequence satisfying conditions (M.1), (M.2) and (M.3). Let $m_p = M_p/M_{p-1}, p \in N$. A polynomial

$$P_L(z) = \prod_{p=1}^{\infty}(1 + \frac{L}{m_p}z), Rez > 0$$

where $L > 0$ is some constant is an ultrapolynomial of (M_p) class.

The next inequality is true, and it is given in [4]. There exist $K_1 > 0, C_1 > 0$ such that

$$e^{M(L|z|)} \le |P_L(z)| \le C_1 e^{M(K_1|z|)}, Rez > 0. \tag{3.1}$$

Lemma 3.1. *Let $s > 1$ and f be an analytic function in the upper half plain $\Pi^+ = \{z : Imz > 0\}$. Then, for every $\delta > 0$ there exist $C_\delta > 0$ and $K_\delta > 0$ such that:*

$$|f(z)| \le C_\delta e^{K_\delta |z|^{1/s}}, Imz \ge \delta$$

if and only if for every $\delta > 0$ there exist $C_\delta > 0$ and the ultrapolynomial P_L of $(p!^s)$ - class, such that

$$|f(z)| \le C_\delta |P_L(iz)|, Imz \ge \delta.$$

Proof. Let for every $\delta > 0$ there exist $C_\delta > 0$ and the ultrapolynomial $P_L(z)$ of $(p!^s)$ - class such that

$$|f(z)| \le C_\delta |P_L(iz)|, Imz \ge \delta.$$

The inequality (3.1) implies that there exist $C_1 > 0$ and $L_1 > 0$ such that

$$|P_L(-iz)| \leq C_1 e^{M(L|z|)} \approx C_1 e^{KL^{1/s}|z|^{1/s}}, \ Imz \geq \delta > 0$$

(We used the fact that $M(|z|) \approx C|z|^{1/s}$, for some $C > 0$).

Now we will show that the opposite holds. Let for every $\delta > 0$ there exist $C > 0$ and $K_\delta > 0$ such that

$$|f(z)| \leq C_\delta e^{K_\delta |z|^{1/s}}, Imz \geq \delta$$

From (3.1), it follows

$$e^{L|z|^{1/s}} \leq |P_L(z)|, Rez > 0.$$

□

The next theorem is the main result of the paper. It answers positively the posed question, as we noted in Introduction.

Theorem 3.2. *Let $s > 1$ and let f be an analytic function in the upper half-plane Π^+ and let for each $\delta > 0$ there exist $C_\delta > 0$ and the ultrapolynomial $P_L(z)$ of $(p!^s)$-class, such that*

$$|f(z)| \leq C_\delta |P_L(iz)|, \quad Imz \geq \delta. \tag{3.2}$$

Then, $f \in H^p(\Pi^+)$, $(1 \leq p \leq \infty)$ if and only if $f(z)$ converges to $f(x) \in L^p(\mathbb{R})$ in the sense of the ultradistributions $S^{'(s)}$.

Proof. Let $f(z) \in H^p(\Pi^+)$. We will show that $f(x+iy) \to f(x)$ when $y \to 0$ in the sense of ultradistributions $S^{'(s)}$ ($f(x)$ is a bounded function for $f(z)$). Note that the following is true: for every $y > 0, f_y(x) = f(x+iy)$ is ultradistribution in $S^{'(s)}$, because $f_y(x)$ is locally integrable and it is ultrapolinomial bounded, i.e. there exists an ultrapolinomial P so that $|f_y(x)| \leq P(x)$ holds for every $x \in \mathbb{R}$. This is true because of the condition (3.2). Now, we will show that $f(x+iy) \to f(x)$ when $y \to 0$ in the sense of ultradistribution $S^{'(s)}$. Let $\phi \in S^{(s)}$. We have:

$$\begin{aligned} |< f_y, \phi > - < f, \phi > | &= | < f_y - f, \phi > | \\ &\leq \int_{-\infty}^{\infty} |f(x+iy) - f(x)||\phi(x)|dx \\ \leq \left(\int_{-\infty}^{\infty} |f(x+iy) - f(x)|^p dx \right)^{\frac{1}{p}} &\left(\int_{-\infty}^{\infty} |\phi(x)|^{p'} \right)^{\frac{1}{p'}} \to 0 \end{aligned}$$

when $y \to 0$. Where $\frac{1}{p} + \frac{1}{p'} = 1$.

Now, we will prove the opposite. Let $f(x+iy)$ converge to $f(x) \in L^p(-\infty, \infty)$ in the sense of the ultradistribution $S^{'(s)}$ as $y \to 0$. We will prove that $f \in H^p(\Pi^+)$.

Let $N_p = (p!)^{s-\rho}$, where $s - \rho > 1$, $\rho > 0$. Let $\widetilde{P}_L(z)$ be an ultrapolynomial of $(p!^s)$-class which corresponds to the sequence (N_p).

Let $\epsilon > 0$. We will define a function $g_\epsilon(z) = \frac{f(z)}{\psi_\epsilon(z)}$, where $\psi_\epsilon(z) = \widetilde{P}_L(i\epsilon z)$, $Rez > 0$.

Step one:

Let $\epsilon > 0$ be fixed. We will show that for every fixed $y > 0$,

$$G_\epsilon(u) = \frac{1}{\sqrt{2\pi}} e^{yu} \int_{-\infty}^{\infty} g_\epsilon(x+iy) e^{-ixu} dx, u \in \mathbb{R}$$

is a smooth function independent of $y > 0$.

Let $y_1 > y_2 > 0$ be fixed numbers and let $\delta > 0$ such that $y_2 > \delta$.

We will approximate $|g_\epsilon(z)|$ in $\{z : Imz \geq \delta\}$.

We will use the inequalities in (3.1). Takeing into consideration (3.2) we have that

$$|g_\epsilon(z)| = \frac{|f(z)|}{|\psi_\epsilon(z)|} \leq \frac{C_\delta |P_L(-iz)|}{\widetilde{P}_L(-i\epsilon z)|}$$

$$\leq \frac{C_\delta C e^{M(L_1|z|)}}{e^{2N(L\epsilon|z|)}}, Imz \geq \delta > 0$$

Since $M(t) \approx Kt^{\frac{1}{s}}$ and $N(t) \approx K_1 t^{\frac{1}{s-\rho}}$, for some $K, K_1 > 0$, we obtain

$$\frac{e^{M(L_1|z|)}}{e^{2N(L\epsilon|z|)}} \approx e^{KL_1^{\frac{1}{s}}|z|^{\frac{1}{s}} - 2K_1 L^{\frac{1}{(s-\rho)}} \epsilon^{\frac{1}{(s-\rho)}} |z|^{\frac{1}{(s-\rho)}}} \to 0, |z| \to \infty$$

So there exists $K_{\epsilon,\delta} > 0$ such that, for every $z \in \mathbb{C}\, Imz \geq \delta > 0$ there holds:

$$|g_\epsilon(z)| \leq K_{\epsilon,\delta} e^{-d|z|^{\frac{1}{(s-\rho)}}}$$

where $d = K_1 L^{\frac{1}{(s-\rho)}} \epsilon^{\frac{1}{(s-\rho)}} > 0$. This implies that g_ϵ is a smooth function.

We will use the integral $\int_\Gamma g_\epsilon(z) e^{-izu} dz$, where the contour Γ is the boundary of $\Omega = \{z : -a < Rez < a, y_1 < Imz < y_2\}$.

For the fixed $u \in \mathbb{R}$ the function $z \mapsto g_{\epsilon(z)} e^{-izu}$, $z \in \Pi^+$ is an analytic function in Ω, so by Cauchy's Theorem we obtain $\int_\Gamma g_\epsilon(z) e^{-izu} dz = 0$.

Hence,

$$e^{y_2 u} \int_{-a}^{a} g_\epsilon(x+iy_2) e^{-ixu} dx + e^{y_1 u} \int_{a}^{-a} g_\epsilon(x+iy) e^{-ixu} dx$$

$$+ e^{-iau} \int_{y_2}^{y_1} g_\epsilon(a+iy) e^{yu} dy + e^{iau} \int_{y_1}^{y_2} g_\epsilon(-a+iy) e^{yu} du = 0.$$

Because of $|g_\epsilon(\pm a + iy)| \leq K_{\epsilon,\delta} e^{-d|\pm a+iy|^{1/(s-\rho)}} \leq K_{\epsilon,\delta} e^{-d|a|^{1/(s-\rho)}}$ we obtain

$$\lim_{a\to\infty} \int_{y_1}^{y_2} |g_\epsilon(a+iy)| e^{yu} dy = \lim_{a\to\infty} \int_{y_2}^{y_1} |g_\epsilon(-a+iy)| e^{yu} dy = 0.$$

So $e^{y_2 u}\int_{-\infty}^{\infty} g_\epsilon(x+iy_2)e^{-ixu}dx = e^{y_1 u}\int_{-\infty}^{\infty} g_\epsilon(x+iy_1)e^{-ixu}dx$ i.e.

$$G_\epsilon(u) = \frac{1}{\sqrt{2\pi}}e^{yu}\int_{-\infty}^{\infty} g_\epsilon(x+iy)e^{-ixu}dx \tag{3.3}$$

is independent of $y>0$. So, we have proved step one.

Step two:

We shall prove that, for $G_\epsilon(u) = \frac{1}{\sqrt{2\pi}}e^{yu}\int_{-\infty}^{\infty} g_\epsilon(x+iy)e^{-ixu}dx$, $u\in\mathbb{R}$, it is true that $G_\epsilon(u)=0$ for $u<0$ and $G_\epsilon(u)$ has an exponential growth, when $u>0$.

Since, $\int_{-\infty}^{\infty}|g_\epsilon(x+iy)|dx < +\infty$, there exists $K_\epsilon>0$ such that for every $u\in\mathbb{R}$ and $y\geq\delta$ it is true that $|G_\epsilon(u)|\leq K_\epsilon e^{uy}$.

So, if $u<0$, we obtain $G_\epsilon(u)=0$ and if $u>0$ we obtain that $|G_\epsilon(u)|\leq A_{\delta,\epsilon}e^{\delta u}$ for some constant $A_{\delta,\epsilon}$.

Step three:

Let $\epsilon>0$ be fixed. We shall prove that $e^{-yu}G_\epsilon(u)\to G_\epsilon(u)$ in the sense of $S^{'(s)}$ when $y\to 0$, i.e.

(3.4)
$$< e^{-yu}G_\epsilon(u),\phi(u) > \to < G_\epsilon(u),\phi(u) >, \text{ when } y\to 0 \text{ for every } \phi\in S^{(s)}.$$

Let $\phi_1,\phi_2\in S^{(s)}$, that they are equal at $(-\infty,p)$ for some $p>0$. Because of $suupG_\epsilon\subset[0,\infty)$ it is true $< G_\epsilon(u),\phi_1 > = < G_\epsilon(u),\phi_2 >$. So, if $\gamma\in S^{(s)}$ such that $\gamma(u)=0$ for $u<-2$ and $\gamma(u)=1$ for $u>-1$ we obtain that

$$< e^{-yu}G_\epsilon(u),\phi > = < G_\epsilon(u), e^{-yu}\phi > = < G_\epsilon(u),\gamma(u)e^{-yu}\phi >$$

To prove (3.4), it suffices to show that $\gamma e^{-yu}\phi\to\phi$ when $y\to 0$ in $S^{(s)}$.

Let $h>0$ be fixed. We will show that $\|e^{-yu}\theta(u)-\theta(u)\|_h\to 0$ when $y\to 0$, where $\theta(u)=\gamma(u)\phi(u)$, $u\in\mathbb{R}$, and $\|\theta\|_h=\sigma_{h,\infty}(\phi)$. We have (for every $u\in R$),

$$|\frac{h^{\alpha+\beta}(1+u^2)^{\alpha/2}}{\alpha!^s\beta!^s}(e^{-yu}\theta(u)-\theta(u))^{(\beta)}|$$

$$=\frac{h^{\alpha+\beta}(1+u^2)^{\alpha/2}}{\alpha!^s\beta!^s}|\sum_{j=0}^{\beta}\binom{\beta}{j}(e^{-yu}-1)^{(\beta-j)}\theta^{(j)}(u)|$$

$$=\frac{h^{\alpha+\beta}(1+u^2)^{\alpha/2}}{\alpha!^s\beta!^s}|(e^{-yu}-1)\theta^{(\beta)}(u)+\sum_{j=0}^{\beta-1}\binom{\beta}{j}(-y)^{(\beta-j)}e^{-yu}\theta^{(j)}(u)|$$

$$\leq (e^{-yu}-1)\sup_{\alpha,\beta}\frac{h^{\alpha+\beta}(1+u^2)^{\alpha/2}|\theta^{(\beta)}(u)|}{\alpha!^s\beta!^s}$$

$$+e^{-yu}|y|\frac{(2h)^{\alpha+\beta}(1+u^2)^{\alpha/2}}{\alpha!^s\beta!^s}\frac{1}{2^{\alpha+\beta}}|\sum_{j=0}^{\beta-1}\binom{\beta}{j}(-1)^{\beta-j}y^{\beta-j-1}e^{-yu}\theta^{(j)}(u)|$$

$$= \|\theta\|_h(e^{-yu}-1)$$

$$+e^{-yu}|y|\frac{1}{2^{\alpha+\beta}}|\sum_{j=0}^{\beta-1}\binom{\beta}{j}\frac{(-1)^{\beta-j}y^{\beta-j-1}}{(\beta-j)!^s}\frac{(2h)^{\alpha+\beta}(1+u^2)^{\alpha/2}\theta^{(j)}(u)}{\alpha!^s j!^s}|$$

$$\le \|\theta\|_h(e^{-yu}-1)+e^{-yu}|y|\frac{\|\theta\|_{2h}}{2^{\alpha+\beta}}\sum_{j=0}^{\beta-1}\binom{\beta}{j}\frac{|y|^{\beta-j-1}}{(\beta-j)!^s}$$

$$\le \|\theta\|_h(e^{-yu}-1)+e^{-yu}|y|\frac{\|\theta\|_{2h}}{2^{\alpha+\beta}}(1+|y|)^{\beta-1}.$$

The last expression, for a sufficient small $|y|$ will be less than

$$\|\theta\|_h(e^{-yu}-1)+e^{-yu}|y|\frac{\|\theta\|_{2h}}{2^{\alpha+1}},$$

and this yields

$$\|e^{-yu}\theta(u)-\theta(u)\|_h \le \|\theta\|_h(e^{-yu}-1)+e^{-yu}|y|\|\theta\|_{2h}, u\in R.$$

Thus, it is proved that $\gamma e^{-yu}\phi \to \gamma\phi$ in $S^{(s)}$ when $y\to 0$. So, for every $\epsilon>0$ it is true that $e^{-yu}G_\epsilon \to G_\epsilon$ in $S^{'(s)}$ when $y\to 0$. This completes the prof of step 3.

Notice that $g_\epsilon(x+iy)$ is a Fourier transform of $e^{-yu}G_\epsilon(u)$. It is known that the Fourier transformation $F: S^{'(s)} \to S^{'(s)}$, is continuous in week topology. Hence, the ultradistributional Fourier transform of $g_\epsilon(x+iy)$ converges in $S^{'(s)}$ when $y\to 0$ to the ultradistributional Fourier transform of G_ϵ which we denote by g_ϵ.

Step four:

We will prove that $g_\epsilon(x)=\frac{f(x)}{\psi_\epsilon(x)}, x\in\mathbb{R}$, as an ultradistribution in $S^{'(s)}$.

Notice that, $(1/\psi_\epsilon)(x+iy)\to(1/\psi_\epsilon)(x)$ in the sense of ultradistribution $S^{'(s)}$ when $y\to 0$, and because $f(x+iy)\to f(x)$ in the sense of ultradistribution $S^{'(s)}$ when $y\to 0$, we get $g_\epsilon(x+iy)=(f/\psi_\epsilon)(x+iy)\to(f/\psi_\epsilon)(x)$ in the sense of ultradistribution $S^{'(s)}$ when $y\to 0$ (as a product of regular ultraultradistribution). It holds that $g_\epsilon(x+iy)\to g_\epsilon(x)$ in $S^{'(s)}$ when $y\to 0$. Thus we obtain that $g_\epsilon(x)=\frac{f(x)}{\psi_\epsilon(x)}, x\in\mathbb{R}$, as an ultradistribution in $S^{'(s)}$.

Step five:

We shall show that $g_\epsilon\in H^p$,$p\in[2,\infty)$, for every $\epsilon>0$.

We need the following result:

If $f \in L^2(\mathbb{R})$, then $F(\tilde{f}) = \tilde{F(f)}$, where on the left-hand side is an $F-$transformation in the sense of ultradistribution $S'^{(s)}$, and on the right-hand side is a regularization of F-transformation of f defined by $\tilde{f} = F(f) = \lim_{n\to\infty} F(\phi_n)$, where (ϕ_n) is a sequence in $S^{(s)}$ which converges to f in the space $L^2(\mathbb{R})$.

Because, $f \in L^p$, in view of HÖlder's inequality, we get

$$\int_{-\infty}^{\infty} |g_\epsilon(x)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 \frac{1}{|\psi_\epsilon(x)|^2} dx$$

$$\leq (\int_{-\infty}^{\infty} (|f(x)|^2)^{p/2} dx)^{2/p} (\int_{-\infty}^{\infty} \frac{dx}{|\psi_\epsilon(x)|^{2q}})^{1/q}.$$

where $1/p + 1/q = 1$. So $g_\epsilon(x) \in L^2(-\infty, \infty)$. Because the function $1/\psi_\epsilon$ is bounded, we get that $g_\epsilon(x) \in L^p(-\infty, \infty)$. Also, because g_ϵ is a Fourier transform of G_ϵ, and $g_\epsilon \in L^2(-\infty, \infty)$, we obtain $G_\epsilon \in L^2(0, \infty)$,$(G_\epsilon(u) = 0$ for $u < 0)$.

Now, because g_ϵ is a Fourier transform of $e^{-yu} G_\epsilon$ we get

$$g_\epsilon(z) = g_\epsilon(x + iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G_\epsilon(u) e^{-yu} e^{ixu} du =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} G_\epsilon(u) e^{i(xu+iyu)} du = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} G_\epsilon(u) e^{iuz} du, z \in \mathbb{C}, y > 0$$

Now, from the Theorem (Paley-Wiener)[2] we obtain $g_\epsilon \in H^2$ and from Poisson's Theorem [2] for integral representation we obtain that $g_\epsilon \in H^p$. Thus step five is proved.

Step six:

Now we will prove that $f(z) \in H^p$ for $p \in [2, \infty)$. It is true that $g_\epsilon(x) \to f(x)$ in $L^p(-\infty, \infty)$ when $\epsilon \to 0$. It follows that $g_\epsilon(z) \to f_1(z)$ when $\epsilon \to 0$, where $f_1 \in H^p$ and $f(x)$ is its bounded function. The above is true, because of the next arguments.

Let $f_1(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-t)^2} f(t) dt \in H^p(\Pi^+)$.

Now we obtain

$$|g_\epsilon(z) - f_1(z)| = \frac{1}{\pi} | \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-t)^2} (g_\epsilon(t) - f(t)) dt|$$

$$\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-t)^2} |g_\epsilon(t) - f(t)| dt$$

$$\leq \frac{1}{\pi} (\int_{-\infty}^{\infty} |g_\epsilon(t) - f(t)|^p dt)^{1/p} (\int_{-\infty}^{\infty} (\frac{y}{y^2 + (x-t)^2})^2 dt)^{1/q} \to 0 \, , \epsilon \to 0$$

So $\lim_{\epsilon\to 0} g_\epsilon(z) = f_1(z)$, $Imz > 0$.

Now, we obtain $(f - f_1)(x + iy) \to 0$ in the sense of ultradistribution $S^{'(s)}$ when $y \to 0$. Because $S^{'(s)} \subseteq D^{'(s)}(\Omega)\,, \Omega = (-R, R)$ we can use Theorem 2.4.

So, from $(f - f_1)(x + iy) \to 0$ in the sense of $D^{'(s)}$, when $y \to 0$, we obtain $f - f_1 \equiv 0$ in the neighborhood of Ω, i.e. there exist $r > 0$ such that $(f - f_1)(x + iy) = 0$ for $|x| < R$ and $0 < y < r$. With analytic continuation, we get that $f(z) = f_1(z)$ for every $z \in \Pi^+$ i.e. $f \in H^p$.

Step seven:

We will show that $g_\epsilon \in H^p(\Pi^+)\,, p \in [1, 2)$ for every $\epsilon > 0$. Under the condition of Theorem 2.3, $f \in L^p(\mathbb{R})$. So, since $1/\psi_\epsilon$ is bounded, it is true that $g_\epsilon \in L^p(-\infty, \infty)$ and

$$G_\epsilon \in L^q(0, \infty)\,, (q = p/(p - 1).$$

Again, we get the representation (3.3) with G_ϵ as the Fourier transform of some function in $L^p(-\infty, \infty)$.

The following is true:

$$\begin{aligned} \frac{1}{i(t-z)} &= \int_0^\infty e^{-itu} e^{izu} du\,, (Imz > 0) \\ &= -\int_{-\infty}^0 e^{-itu} e^{izu} du\,, (Imz < 0) \end{aligned} \tag{3.5}$$

Now we use Fubini's Theorem the fact that g_ϵ is the Fourier transform of G_ϵ and the equality (3.5) and obtain

$$\int_{-\infty}^{\infty} \frac{g_\epsilon(t)}{i(t - z)} dt = \int_{-\infty}^{\infty} (g_\epsilon(t) \int_0^\infty e^{-itu} e^{izu} du) dt =$$

$$\int_0^\infty (e^i zu \int_{-\infty}^{\infty} g_\epsilon(t) e^{-itu} dt) du = \sqrt{2\pi} \int_0^\infty G_\epsilon(u) e^{izu} du = 2\pi g_\epsilon(z), \ (Imz > 0).$$

The same argument gives that $g_\epsilon(z) = 0$ for $Imz < 0$ because that $G_\epsilon(u) = 0$ for $u < 0$.

Thus, Theorem 2.3 implies $g_\epsilon \in H^p$.

Step eight:

The proof that $f \in H^p(\mathbb{R})$ for $p \in [1, 2)$ is the same as in step six.

□

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