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# SPACES WITH $\sigma$-LOCALLY COUNTABLE LINDELÖF $sn$-NETWORKS

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**Abstract.** In this paper, we prove that a space $X$ has a $\sigma$-locally countable Lindelöf $sn$-network if and only if $X$ is a compact-covering compact $msss$-image of a locally separable metric space, if and only if $X$ is a sequentially-quotient $\pi$ and $msss$-image of a locally separable metric space, where "compact-covering" (or "sequentially-quotient") can not be replaced by "sequence-covering". As an application, we give a new characterizations of spaces with $\sigma$-locally countable Lindelöf weak bases.

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## 1. Introduction

In [15], S. Lin introduced the concept of $msss$-maps to characterize spaces with certain $\sigma$-locally countable networks by $msss$-images of metric spaces. After that, Z. Li, Q. Li, and X. Zhou gave some characterizations for certain $msss$-images of metric spaces ([14]). Recently, N. V. Dung gave some characterizations for certain $msss$-images of locally separable metric spaces ([3]).

In this paper, we prove that a space $X$ has a $\sigma$-locally countable Lindelöf $sn$-network if and only if $X$ is a compact-covering compact $msss$-image of a locally separable metric space, if and only if $X$ is a sequentially-quotient $\pi$ and $msss$-image of a locally separable metric space, where "compact-covering" (or "sequentially-quotient") can not be replaced by "sequence-covering". As an application, we give a new characterizations of spaces with $\sigma$-locally countable Lindelöf weak bases.

Throughout this paper, all spaces are assumed to be $T_1$ and regular, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of $X$ and $x \in X$, we denote $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\mathrm{st}(x, \mathcal{P}) = \bigcup(\mathcal{P})_x$ and $\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$. For a sequence $\{x_n\}$ converging to $x$ and $P \subset X$, we say that $\{x_n\}$ is *eventually* in $P$ if $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in $P$ if some subsequence of $\{x_n\}$ is eventually in $P$.

## 2. Definitions

**Definition 2.1.** Let $X$ be a space, $P \subset X$ and let $\mathcal{P}$ be a cover of $X$.

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1. $P$ is a *sequential neighborhood* of $x$ in $X$ [5], if each sequence $S$ converging to $x$ is eventually in $P$.

2. $P$ is a *sequentially open* subset of $X$ [5], if $P$ is a sequential neighborhood of $x$ in $X$ for every $x \in P$.

3. $\mathcal{P}$ is an *so-cover* for $X$ [20], if each element of $\mathcal{P}$ is sequentially open in $X$.

4. $\mathcal{P}$ is a *cfp-cover* for $X$ [27], if whenever $K$ is compact subset of $X$, there exists a finite family $\{K_i : i \leq n\}$ of closed subsets of $K$ and $\{P_i : i \leq n\} \subset \mathcal{P}$ such that $K = \bigcup\{K_i : i \leq n\}$ and each $K_i \subset P_i$.

5. $\mathcal{P}$ is an *cs*-cover* for $X$ [26], if every convergent sequence is frequently in some $P \in \mathcal{P}$.

**Definition 2.2.** Let $\mathcal{P}$ be a family of subsets of a space $X$.

1. For each $x \in X$, $\mathcal{P}$ is a *network* at $x$ in $X$ [17], if $x \in \bigcap \mathcal{P}$, and if $x \in U$ with $U$ open in $X$, then there exists $P \in \mathcal{P}$ such that $x \in P \in U$.

2. $\mathcal{P}$ is a *cs-network* for $X$ [26], if each sequence $S$ converging to a point $x \in U$ with $U$ open in $X$, $S$ is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

3. $\mathcal{P}$ is a *cs*-network* for $X$ [26], if for each sequence $S$ converging to a point $x \in U$ with $U$ open in $X$, $S$ is frequently in $P \subset U$ for some $P \in \mathcal{P}$.

4. $\mathcal{P}$ is *Lindelöf*, if each element of $\mathcal{P}$ is a Lindelöf subset of $X$.

5. $\mathcal{P}$ is *point-countable* [4], if each point $x \in X$ belongs to only countably many members of $\mathcal{P}$.

6. $\mathcal{P}$ is *locally countable* [4], if for each $x \in X$, there exists a neighborhood $V$ of $x$ such that $V$ meets only countably many members of $\mathcal{P}$.

7. $\mathcal{P}$ is *locally finite* [4], if for each $x \in X$, there exists a neighborhood $V$ of $x$ such that $V$ meets only finite many members of $\mathcal{P}$.

8. $\mathcal{P}$ is *star-countable* [24], if each $P \in \mathcal{P}$ meets only countably many members of $\mathcal{P}$.

**Definition 2.3.** Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space $X$ satisfying that, for every $x \in X$, $\mathcal{P}_x$ is a network at $x$ in $X$, and if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

1. $\mathcal{P}$ is a *weak base* for $X$ [1], if $G \subset X$ such that for every $x \in G$, there exists $P \in \mathcal{P}_x$ satisfying $P \subset G$, then $G$ is open in $X$. Here, $\mathcal{P}_x$ is a *weak base* at $x$ in $X$.

2. $\mathcal{P}$ is an *sn-network* for $X$ [16], if each member of $\mathcal{P}_x$ is a sequential neighborhood of $x$ for all $x \in X$. Here, $\mathcal{P}_x$ is an *sn-network* at $x$ in $X$.

**Definition 2.4.** Let $X$ be a space.

1. $X$ is an *sn-first countable space* [6], if there is a countable $sn$-network at $x$ in $X$ for all $x \in X$.

2. $X$ is a *cosmic space* [22], if $X$ has a countable network.

3. $X$ is an $\aleph_0$-*space* [22], if $X$ has a countable $cs$-network.

4. $X$ is a *sequential space* [5], if each sequentially open subset of $X$ is open.

5. $X$ is a *Fréchet space* [4], if for each $x \in \overline{A}$, there exists a sequence in $A$ converging to $x$.

**Definition 2.5.** Let $f : X \longrightarrow Y$ be a map.

1. $f$ is *sequence-covering* [23], if for each convergent sequence $S$ of $Y$, there exists a convergent sequence $L$ of $X$ such that $f(L) = S$. Note that a sequence-covering map is a *strong sequence-covering* map in the sense of [12].

2. $f$ is *compact-covering* [22], if for each compact subset $K$ of $Y$, there exists a compact subset $L$ of $X$ such that $f(L) = K$.

3. $f$ is *pseudo-sequence-covering* [11], if for each convergent sequence $S$ of $Y$, there exists a compact subset $K$ of $X$ such that $f(K) = S$.

4. $f$ is a *subsequence-covering* [18], if for every convergent sequence $S$ of $Y$, there is a compact subset $K$ of $X$ such that $f(K)$ is a subsequence of $S$.

5. $f$ is *sequentially-quotient* [2], if for each convergent sequence $S$ of $Y$, there exists a convergent sequence $L$ of $X$ such that $f(L)$ is a subsequence of $S$.

6. $f$ is a *quotient map* [4], if whenever $U \subset Y$, $U$ open in $Y$ if and only if $f^{-1}(U)$ open in $X$.

7. $f$ is an *msss-map* [15], if $X$ is a subspace of the product space $\prod_{i\in\mathbb{N}} X_i$ of a family $\{X_i : i \in \mathbb{N}\}$ of metric spaces and for each $y \in Y$, there is a sequence $\{V_i : i \in \mathbb{N}\}$ of open neighborhoods of $y$ such that each $p_i f^{-1}(V_i)$ is separable in $X_i$.

8. $f$ is *compact* [4], if each $f^{-1}(y)$ is compact in $X$.

9. $f$ is a $\pi$-*map* [11], if for each $y \in Y$ and for each neighborhood $U$ of $y$ in $Y$, $d\big(f^{-1}(y), X - f^{-1}(U)\big) > 0$, where $X$ is a metric space with a metric $d$.

**Definition 2.6** ([17])**.** Let $\{\mathcal{P}_i\}$ be a cover sequence of a space $X$. $\{\mathcal{P}_i\}$ is called a *point-star network*, if $\{\mathrm{st}(x, \mathcal{P}_i) : i \in \mathbb{N}\}$ is a network of $x$ for each $x \in X$.

For some undefined or related concepts, we refer the reader to [4], [11] and [17].

## 3. Main results

**Lemma 3.1.** *Let $f : M \longrightarrow X$ be a sequentially-quotient msss-map, and $M$ be a locally separable metric space. Then, $X$ has a $\sigma$-locally countable Lindelöf cs-network.*

*Proof.* By Lemma 1.2 [15], there exists a base $\mathcal{B}$ of $M$ such that $f(\mathcal{B})$ is a $\sigma$-locally countable network for $X$. Since $M$ is locally separable, for each $a \in M$, there exists a separable open neighborhood $U_a$. Denote

$$\mathcal{C} = \{B \in \mathcal{B} : B \subset U_a \text{ for some } a \in M\}.$$

Then, $\mathcal{C} \subset \mathcal{B}$ and $\mathcal{C}$ is a separable base for $M$. If we put $\mathcal{P} = f(\mathcal{C})$, then $\mathcal{P} \subset f(\mathcal{B})$, and $\mathcal{P}$ is a $\sigma$-locally countable Lindelöf network. Since $f$ is sequentially-quotient and $\mathcal{C}$ is a base for $M$, $\mathcal{P}$ is a $cs^*$-network. Therefore, $\mathcal{P}$ is a $\sigma$-locally countable Lindelöf $cs^*$-network.

Let $\mathcal{P} = \bigcup\{\mathcal{P}_i : i \in \mathbb{N}\}$, we can assume that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$. Since each element of $\mathcal{P}_i$ is Lindelöf, each $\mathcal{P}_i$ is star-countable. It follows from Lemma 2.1 [24] that for each $i \in \mathbb{N}$, $\mathcal{P}_i = \bigcup\{\mathcal{Q}_{i,\alpha} : \alpha \in \Lambda_i\}$, where $\mathcal{Q}_{i,\alpha}$ is a countable subfamily of $\mathcal{P}_i$ for all $\alpha \in \Lambda_i$ and $(\bigcup \mathcal{Q}_{i,\alpha}) \cap (\bigcup \mathcal{Q}_{i,\beta}) = \emptyset$ for all $\alpha \neq \beta$. For each $i \in \mathbb{N}$ and $\alpha \in \Lambda_i$, we put

$$\mathcal{R}_{i,\alpha} = \{\bigcup \mathcal{F} : \mathcal{F} \text{ is a finite subfamily of } \mathcal{Q}_{i,\alpha}\}.$$

Since each $\mathcal{R}_{i,\alpha}$ is countable, we can write $\mathcal{R}_{i,\alpha} = \{R_{i,\alpha,j} : j \in \mathbb{N}\}$. Now, for each $i, j \in \mathbb{N}$, put $\mathcal{F}_{i,j} = \{R_{i,\alpha,j} : \alpha \in \Lambda_i\}$, and denote $\mathcal{G} = \bigcup\{\mathcal{F}_{i,j} : i, j \in \mathbb{N}\}$. Then, each $R_{i,\alpha,j}$ is Lindelöf and each family $\mathcal{F}_{i,j}$ is locally countable. Now, we shall show that $\mathcal{G}$ is a $cs$-network. In fact, let $\{x_n\}$ be a sequence converging to $x \in U$ with $U$ is open in $X$. Since $\mathcal{P}$ is a point-countable $cs^*$-network, it follows from Lemma 3 [25] that there exists a finite family $\mathcal{A} \subset (\mathcal{P})_x$ such that $\{x_n\}$ is eventually in $\bigcup \mathcal{A} \subset U$. Furthermore, since $\mathcal{A}$ is finite and $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ for all $i \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $\mathcal{A} \subset \mathcal{P}_i$. So, there exists unique $\alpha \in \Lambda_i$ such that $\mathcal{A} \subset \mathcal{Q}_{i,\alpha}$, and $\bigcup \mathcal{A} \in \mathcal{R}_{i,\alpha}$. Thus, $\bigcup \mathcal{A} = R_{i,\alpha,j}$ for some $j \in \mathbb{N}$. Hence, $\bigcup \mathcal{A} \in \mathcal{G}$, and $\mathcal{G}$ is a $cs$-network. Therefore, $\mathcal{G}$ is a $\sigma$-locally countable Lindelöf $cs$-network. □

**Theorem 3.2.** *The following are equivalent for a space $X$.*

1. *$X$ is a space with a $\sigma$-locally countable sn-network and has an so-cover consisting of $\aleph_0$-subspaces;*
2. *$X$ has a $\sigma$-locally countable Lindelöf sn-network;*
3. *$X$ is a compact-covering compact and msss-image of a locally separable metric space;*
4. *$X$ is a pseudo-sequence-covering compact and msss-image of a locally separable metric space;*
5. *$X$ is a subsequence-covering compact and msss-image of a locally separable metric space;*

6. *$X$ is a sequentially-quotient $\pi$ and msss-image of a locally separable metric space.*

*Proof.* (1) $\Longrightarrow$ (2). Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a $\sigma$-locally countable *sn*-network and $\mathcal{O}$ be an *so*-cover consisting of $\aleph_0$-subspaces for $X$. For each $x \in X$, pick $O_x \in \mathcal{O}$ such that $x \in O_x$ and put

$$\mathcal{G}_x = \{P \in \mathcal{P}_x : P \subset O_x\}, \;\; \mathcal{G} = \bigcup\{\mathcal{G}_x : x \in X\}.$$

Then, $\mathcal{G}$ is a $\sigma$-locally countable Lindelöf *sn*-network for $X$.

(2) $\Longrightarrow$ (3). Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\} = \{\mathcal{P}_n : n \in \mathbb{N}\}$ be a $\sigma$-locally countable Lindelöf *sn*-network for $X$, where each $\mathcal{P}_n$ is locally countable and each $\mathcal{P}_x$ is an *sn*-network at $x$. Since $X$ is a regular space, we can assume that each element of $\mathcal{P}$ is closed. Since each element of $\mathcal{P}_i$ is Lindelöf, each $\mathcal{P}_i$ is star-countable. It follows from Lemma 2.1 [24] that for each $i \in \mathbb{N}$, $\mathcal{P}_i = \bigcup\{\mathcal{Q}_{i,\alpha} : \alpha \in \Phi_i\}$, where $\mathcal{Q}_{i,\alpha}$ is a countable subfamily of $\mathcal{P}_i$ for all $\alpha \in \Phi_i$ and $(\bigcup \mathcal{Q}_{i,\alpha}) \cap (\bigcup \mathcal{Q}_{i,\beta}) = \emptyset$ for all $\alpha \neq \beta$. Since each $\mathcal{Q}_{i,\alpha}$ is countable, we can write $\mathcal{Q}_{i,\alpha} = \{P_{i,\alpha,j} : j \in \mathbb{N}\}$. Now, for each $i, j \in \mathbb{N}$, put $\mathcal{F}_{i,j} = \{P_{i,\alpha,j} : \alpha \in \Phi_i\}$, and

$$A_{i,j} = \{x \in X : \mathcal{P}_x \cap \mathcal{F}_{i,j} = \emptyset\}, \;\; \mathcal{H}_{i,j} = \mathcal{F}_{i,j} \cup \{A_{i,j}\}.$$

Then, $\mathcal{P} = \bigcup\{\mathcal{F}_{i,j} : i, j \in \mathbb{N}\}$, and

(a) Each $\mathcal{H}_{i,j}$ is locally countable. It is obvious.

(b) Each $\mathcal{H}_{i,j}$ is a *cfp*-cover. Let $K$ be a non-empty compact subset of $X$. We shall show that there exists a finite subset of $\mathcal{H}_{i,j}$ which forms a *cfp*-cover of $K$. In fact, since $X$ has a $\sigma$-locally countable *sn*-network, $K$ is metrizable. Noting that each $\bigcup \mathcal{Q}_{i,\alpha}$ is sequentially open and $(\bigcup \mathcal{Q}_{i,\alpha}) \cap (\mathcal{Q}_{i,\beta}) = \emptyset$ for all $\alpha \neq \beta$. Then, $K$ meets only finitely many members of $\{\bigcup \mathcal{Q}_{i,\alpha} : \alpha \in \Phi_i\}$. If not, for each $\alpha \in \Phi_i$, take $x_\alpha \in (\bigcup \mathcal{Q}_{i,\alpha}) \cap K$. Thus, there exists a sequence $\{x_{\alpha,n} : n \in \mathbb{N}\} \subset \{x_\alpha : \alpha \in \Phi_i\}$ such that $\{x_{\alpha,n} : n \in \mathbb{N}\}$ converges to $x \in K$. Hence, there exists $\alpha_0 \in \Phi_i$ such that $\{x_{\alpha,n} : n \in \mathbb{N}\}$ is eventually in $\bigcup \mathcal{Q}_{i,\alpha_0}$. This is a contradiction to $x_{\alpha,n} \notin \bigcup \mathcal{Q}_{i,\alpha_0}$ for all $\alpha \neq \alpha_0$. Therefore, $K$ meets only finitely many members of $\mathcal{H}_{i,j}$. Let

$$\Gamma_{i,j} = \{\alpha \in \Phi_i : P_{i,\alpha,j} \in \mathcal{H}_{i,j}, P_{i,\alpha,j} \cap K \neq \emptyset\}.$$

For each $\alpha \in \Gamma_{i,j}$, put $K_{i,\alpha,j} = P_{i,\alpha,j} \cap K$, then $K_{i,j} = \overline{K - \bigcup_{\alpha \in \Gamma_{i,j}} K_{i,\alpha,j}}$. It is obvious that all $K_{i,\alpha,j}$ and $K_{i,j}$ are closed subset of $K$, and $K = K_{i,j} \cup (\bigcup_{\alpha \in \Gamma_{i,j}} K_{i,\alpha,j})$. Now, we only need to show $K_{i,j} \subset A_{i,j}$. Let $x \in K_{i,j}$, then there exists a sequence $\{x_n\}$ of $K - \bigcup_{\alpha \in \Gamma_{i,j}} K_{i,\alpha,j}$ converging to $x$. If $P \in \mathcal{P}_x \cap \mathcal{H}_{i,j}$, then $P$ is a sequential neighborhood of $x$ and $P = P_{i,\alpha,j}$ for some $\alpha \in \Gamma_{i,j}$. Thus, $x_n \in P$ whenever $n \geq m$ for some $m \in \mathbb{N}$. Hence, $x_n \in K_{i,\alpha,j}$ for some $\alpha \in \Gamma_{i,j}$, a contradiction. So, $\mathcal{P}_x \cap \mathcal{H}_{i,j} = \emptyset$, and $x \in A_{i,j}$. This implies that $K_{i,j} \subset A_{i,j}$ and $\{A_{i,j}\} \bigcup \{P_{i,\alpha,j} : \alpha \in \Gamma_{i,j}\}$ is a *cfp*-cover of $K$.

(c) $\{\mathcal{H}_{i,j} : i, j \in \mathbb{N}\}$ is a point-star network for $X$. Let $x \in U$ with $U$ is open in $X$. Then, $x \in P \subset U$ for some $P \in \mathcal{P}_x$. Thus, there exists $i \in \mathbb{N}$ such that $P \in \mathcal{P}_i$. Hence, there exists a unique $\alpha \in \Phi_i$ such that $P \in \mathcal{Q}_{i,\alpha}$. This implies that $P = P_{i,\alpha,j} \in \mathcal{H}_{i,j}$ for some $j \in \mathbb{N}$. Since $P \in \mathcal{P}_x \cap \mathcal{H}_{i,j}$, $x \notin A_{i,j}$. Noting that $P \cap P_{i,\alpha,j} = \emptyset$ for all $j \neq i$. Then, $\mathrm{st}(x, \mathcal{H}_{i,j}) = P \subset U$.

Next, we write $\{\mathcal{H}_{m,n} : m, n \in \mathbb{N}\} = \{\mathcal{G}_i : i \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, put $\mathcal{G}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ and endow $\Lambda_n$ with the discrete topology. Then,

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ forms a network at some point } x_\alpha \in X\Big\}$$

is a metric space and the point $x_\alpha$ is unique in $X$ for every $\alpha \in M$. Define $f : M \longrightarrow X$ by $f(\alpha) = x_\alpha$. It follows from Lemma 13 [21] that $f$ is a compact-covering and compact map. On the other hand, we have

Claim 1. $M$ is locally separable.

Let $a = (\alpha_i) \in M$. Then, $\{P_{\alpha_i}\}$ is a network at some point $x_a \in X$, and $x_a \in P$ for some $P \in \mathcal{P}_{x_a}$. Thus, there exists $m \in \mathbb{N}$ such that $P \in \mathcal{P}_m$. Hence, there exists a unique $\alpha \in \Phi_m$ such that $P \in \mathcal{Q}_{m,\alpha}$. Therefore, $P = P_{m,\alpha,n} \in \mathcal{H}_{m,n}$ for some $n \in \mathbb{N}$. Since $P \in \mathcal{P}_{x_a} \cap \mathcal{H}_{m,n}$, $x_a \notin A_{m,n}$. Noting that $P \cap P_{m,\alpha,n} = \emptyset$ for every $n \in \mathbb{N}$ such that $n \neq m$. This implies that $\mathrm{st}(x, \mathcal{H}_{m,n}) = P$. Then, $\mathcal{H}_{m,n} = \mathcal{G}_{i_0}$ for some $i_0 \in \mathbb{N}$ and $P = P_{\alpha_{i_0}}$. Thus, $P_{\alpha_{i_0}}$ is Lindelöf. Put

$$U_a = M \cap \Big\{(\beta_i) \in \prod_{i\in\mathbb{N}} \Lambda_i : \beta_i = \alpha_i, i \le i_0\Big\}.$$

Then, $U_a$ is an open neighborhood of $a$ in $M$. Now, for each $i \le i_0$, put $\Delta_i = \{\alpha_i\}$, and for each $i > i_0$, we put $\Delta_i = \{\alpha \in \Lambda_i : P_\alpha \cap P_{\alpha_{i_0}} \neq \emptyset\}$. Then, $U_a \subset \prod_{i\in\mathbb{N}} \Delta_i$. Furthermore, since each $\mathcal{P}_i$ is locally countable and $P_{\alpha_{i_0}}$ is Lindelöf, $\Delta_i$ is countable for every $i > i_0$. Thus, $U_a$ is separable, and $M$ is locally separable.

Claim 2. $f$ is an $msss$-map.

Let $x \in X$. For each $n \in \mathbb{N}$, since $\mathcal{G}_n$ is locally countable, there is an open neighborhood $V$ such that $V_n$ intersects at most countable members of $\mathcal{G}_n$. Put

$$\Theta_n = \{\alpha \in \Lambda_n : P_\alpha \cap V_n \neq \emptyset\}$$

Then, $\Theta_n$ is countable and $p_n f^{-1}(V_n) \subset \Theta_n$. Hence, $p_n f^{-1}(V_n)$ is a separable subset of $\Lambda_n$, so $f$ is an $msss$-map.

(3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6). It is obvious.

(6) $\Longrightarrow$ (1). Let $f : M \longrightarrow X$ be a sequentially-quotient $\pi$ and $msss$-map, where $M$ be a locally separable metric space. By Corollary 2.9 [7], $X$ has a point-star network $\{\mathcal{U}_n\}$, where each $\mathcal{U}_n$ is a $cs^*$-cover. For each $n \in \mathbb{N}$, put $\mathcal{G}_n = \bigwedge_{i\le n} \mathcal{U}_i$. Now, for each $x \in X$, let $\mathcal{G}_x = \{\mathrm{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$. Since each $\mathcal{U}_n$ is a $cs^*$-cover, it implies that $\bigcup\{\mathcal{G}_x : x \in X\}$ is an $sn$-network for $X$. Hence, $X$ is an $sn$-first countable space. On the other hand, since $f$ is a sequentially-quotient $msss$-map, it follows from Lemma 3.1 that $X$ has a $\sigma$-locally countable Lindelöf $cs$-network $\mathcal{P}$. We can assume that each $\mathcal{P}$ is closed under finite intersections. Then, each element of $\mathcal{P}$ is a cosmic subspace. By Theorem 3.4 [20], $X$ has an $so$-cover consisting of $\aleph_0$-subspaces. Now, we only need to prove that $X$ has a $\sigma$-locally countable $sn$-network. In fact, since $X$

is $sn$-first countable, $X$ has an $sn$-network $\mathcal{Q} = \bigcup\{\mathcal{Q}_x : x \in X\}$ with each $\mathcal{Q}_x = \{Q_n(x) : n \in \mathbb{N}\}$ is a countable weak base at $x$. For each $x \in X$, put

$$\mathcal{P}_x = \{P \in \mathcal{P} : Q_n(x) \subset P \text{ for some } n \in \mathbb{N}\}.$$

By using proof of Lemma 7 [19], we obtain that $\mathcal{P}_x$ is an $sn$-network at $x$. Then, $\mathcal{G} = \bigcup\{\mathcal{P}_x : x \in X\}$ is an $sn$-network for $X$. Since $\mathcal{G} \subset \mathcal{P}$, it implies that $\mathcal{G}$ is locally countable. Thus, $X$ has a $\sigma$-locally countable $sn$-network. □

By Theorem 3.2, the following corollary holds.

**Corollary 3.3.** *The following are equivalent for a space $X$.*

1. *$X$ is a local $\aleph_0$-subspace with a $\sigma$-locally countable weak base;*
2. *$X$ has a $\sigma$-locally countable Lindelöf weak base;*
3. *$X$ is a compact-covering quotient compact and msss-image of a locally separable metric space;*
4. *$X$ is a pseudo-sequence-covering quotient compact and msss-image of a locally separable metric space;*
5. *$X$ is a subsequence-covering quotient compact and msss-image of a locally separable metric space;*
6. *$X$ is a quotient $\pi$ and msss-image of a locally separable metric space.*

**Example 3.4.** Let $C_n$ be a convergent sequence containing its limit point $p_n$ for each $n \in \mathbb{N}$, where $C_m \cap C_n = \emptyset$ if $m \neq n$. Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ be the set of all rational numbers of the real line $\mathbb{R}$. Put $M = (\bigoplus\{C_n : n \in \mathbb{N}\}) \oplus \mathbb{R}$ and let $X$ be the quotient space obtained from $M$ by identifying each $p_n$ in $C_n$ with $q_n$ in $\mathbb{R}$. Then, by the proof of Example 3.1 [10], $X$ has a countable weak base and $X$ is not a sequence-covering quotient $\pi$-image of a metric space. Hence,

1. A space with a $\sigma$-locally countable Lindelöf $sn$-network $\not\Rightarrow$ a sequence-covering $\pi$ and $msss$-image of a locally separable metric space.
2. A space with a $\sigma$-locally countable Lindelöf weak base $\not\Rightarrow$ a sequence-covering quotient $\pi$ and $msss$-image of a locally separable metric space.

**Example 3.5.** Using Example 3.1 [9], it is easy to see that $X$ is Hausdorff, non-regular and $X$ has a countable base, but it is not a sequentially-quotient $\pi$-image of a metric space. This shows that regular properties of $X$ can not be omitted in Theorem 3.2 and Corollary 3.3.

**Example 3.6.** $S_\omega$ is a Fréchet and $\aleph_0$-space, but it is not first countable. Thus, $S_\omega$ has a $\sigma$-locally countable Lindelöf $cs$-network. It follows from Theorem 2.8 [3] that $X$ is a sequence-covering $msss$-image of a locally separable metric space. Furthermore, since $S_\omega$ is not first countable, it has not point-countable $sn$-network. Hence,

1. A space with a $\sigma$-locally countable Lindelöf $cs$-network $\not\Rightarrow$ a sequentially-quotient $\pi$ and $msss$-image of a locally separable metric space.
2. A sequence-covering quotient $msss$-image of a locally separable metric space $\not\Rightarrow$ $X$ has a $\sigma$-locally countable Lindelöf $sn$-network.

**Example 3.7.** Using Example 2.7 [13], it is easy to see that $X$ is a compact-covering quotient and compact image of a locally compact metric space, but it has no point-countable $cs$-network. Thus, a compact-covering quotient and compact image of a locally separable metric space $\not\Rightarrow$ $X$ has a $\sigma$-locally countable Lindelöf $sn$-network.

**Example 3.8.** There exists a space $X$ has a locally countable $sn$-network, which is not an $\aleph$-space (see Example 2.19 [8]). Then, a space with a $\sigma$-locally countable Lindelöf $sn$-network $\not\Rightarrow$ $X$ has a $\sigma$-locally finite Lindelöf $sn$-network.

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