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INSEPARABLE SEQUENCES AND FORCING

Boris Šobot[1]

Abstract. An inseparable sequence is an almost disjoint family $\langle A_\xi : \xi < \omega_1\rangle$ of subsets of ω such that for no $D \subseteq \omega$ there are uncountably many A_ξ such that $A_\xi \cap D$ is finite and uncountably many A_ξ such that $A_\xi \setminus D$ is finite. We investigate under which conditions is an inseparable sequence destroyed by forcing. We translate these conditions into forcing language and use them to obtain several necessary conditions for destroying inseparability. In particular, we prove that in order to get such a notion of forcing with c.c.c. we must assume at least $\neg$MA, and in order to get such a notion of forcing that is proper we must assume at least $\neg$PFA.

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1. Introduction

A family $\mathcal{A} = \langle A_\xi : \xi < \omega_1\rangle$ of infinite subsets of ω is *an almost disjoint family* if $|A_\xi \cap A_\zeta| < \aleph_0$ for all $\xi \neq \zeta$. *An inseparable sequence* is an almost disjoint family such that there is no $D \subseteq \omega$ such that

$$\exists X \in [\omega_1]^{\aleph_1}\ \forall \xi \in X\ |A_\xi \cap D| < \aleph_0$$
$$\exists Y \in [\omega_1]^{\aleph_1}\ \forall \xi \in Y\ |A_\xi \setminus D| < \aleph_0.$$

Since there are only $\aleph_0$ finite subsets of ω, we can find uncountable subsets of X and Y that enumerate subsets with same intersections and differences with D, and replace the above conditions by

$$\exists S \in [\omega]^{<\aleph_0}\ \exists X \in [\omega_1]^{\aleph_1}\ \forall \xi \in X\ A_\xi \cap D = S \tag{1}$$
$$\exists T \in [\omega]^{<\aleph_0}\ \exists Y \in [\omega_1]^{\aleph_1}\ \forall \xi \in Y\ A_\xi \setminus D = T. \tag{2}$$

Lusin in [7] defined a stronger concept, a *Lusin sequence*, and proved that it can be constructed in ZFC. Abraham and Shelah in [1] proved that it is independent from ZFC whether every inseparable sequence has a Lusin subsequence.

2. When forcing destroys inseparability

Our goal is to investigate the effect of forcing on inseparable sequences. One way to kill an inseparable sequence would be, of course, to collapse $\aleph_1$.

[1]Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, e-mail: sobot@dmi.uns.ac.rs

However, we are interested only in forcing notions that preserve $\aleph_1$, so we will concentrate our attention on destroying separability by means of adding a set D that satisfies (1) and (2).

A forcing preserving $\aleph_1$ adds a set D destroying the inseparability of $\mathcal{A}$ iff

$$\begin{aligned} 1 \quad \Vdash \quad & \exists D \subseteq \check{\omega}\, (\exists S \in ([\omega]^{<\aleph_0})\check{}\ \exists X \in [\check{\omega}_1]^{\check{\aleph}_1}\ \forall \xi \in X\ \check{A}_\xi \cap D = S \\ & \wedge \exists T \in ([\omega]^{<\aleph_0})\check{}\ \exists Y \in [\check{\omega}_1]^{\check{\aleph}_1}\ \forall \xi \in Y\ \check{A}_\xi \setminus D = T). \end{aligned}$$

Let $N_n(\kappa)$ denote the set of nice names for subsets of κ (see [3], Lemma VII 5.12). Using the Maximum principle ([2], Lemma 14.19) we get: a forcing preserving $\aleph_1$ adds a set D destroying the inseparability of $\mathcal{A}$ iff

$$\begin{aligned} & \exists \sigma, \pi, \mu \in N_n(\omega)\ \exists \theta, \tau \in N_n(\omega_1)\ 1 \Vdash (|\pi| < \aleph_0 \wedge |\mu| < \aleph_0 \\ & \wedge |\theta| = \aleph_1 \wedge |\tau| = \aleph_1 \wedge \forall \xi \in \theta\ (\check{A}_\xi \cap \sigma = \pi) \wedge\ \forall \xi \in \tau\ (\check{A}_\xi \setminus \sigma = \mu)). \end{aligned}$$

To make things as simple as possible, we will assume that we force with a complete Boolean algebra $\mathbb{B}$. Thus we can take our nice names to be of the form

$$\begin{aligned} \sigma &= \{\langle \check{n}, \sigma(n) \rangle : n \in \omega\} \\ \pi &= \{\langle \check{n}, \pi(n) \rangle : n \in \omega\} \\ \mu &= \{\langle \check{n}, \mu(n) \rangle : n \in \omega\} \\ \theta &= \{\langle \check{\alpha}, \theta(\alpha) \rangle : \alpha \in \omega_1\} \\ \tau &= \{\langle \check{\alpha}, \tau(\alpha) \rangle : \alpha \in \omega_1\}. \end{aligned}$$

In this case we have $\|\check{\xi} \in \rho\| = \rho(\xi)$, for each name ρ and $\xi \in \mathrm{dom}(\rho)$.

Now we state a modification of Lemma 1 from [6]. The quantifier $\exists^\kappa \alpha$ means "there are κ-many ordinals α". Our forcing-including formulas will be adjusted for complete Boolean algebras: $\exists^+ r \leq q$ means "there is $r \leq q$ such that $r > 0$, and similarly for $\forall^+ r \leq q$. $q \parallel r$ means that the elements $q, r \in B$ are compatible in $B \setminus \{0\}$, i.e. $q \wedge r > 0$.

Lemma 1. If $\mathbb{B}$ is a complete Boolean algebra, $p \in B^+$, $\rho \in N_n(\kappa)$ and $1 \Vdash \mathrm{reg}(\check{\kappa})$, then:

(a) $p \Vdash |\rho| = \check{\kappa}$ iff $\forall^+ q \leq p\ \exists^\kappa \alpha\ q \parallel \rho(\alpha)$;

(b) $p \Vdash |\rho| < \check{\kappa}$ iff $\forall^+ r \leq p\ \exists^+ q \leq r\ \exists \beta < \kappa\ \forall \alpha \geq \beta\ q \leq (\rho(\alpha))'$.

Proof. Using $1 \Vdash \mathrm{reg}(\check{\kappa})$ we get

$$\begin{aligned} & p \Vdash |\rho| = \check{\kappa} \\ \Leftrightarrow\quad & p \Vdash \forall \beta < \check{\kappa}\ \exists \alpha > \beta\ \alpha \in \rho \\ \Leftrightarrow\quad & \forall \beta < \kappa\ \forall^+ q \leq p\ \exists^+ r \leq q\ \exists \alpha \geq \beta\ r \Vdash \check{\alpha} \in \rho \\ \Leftrightarrow\quad & \forall^+ q \leq p\ \forall \beta < \kappa\ \exists \alpha \geq \beta\ \exists^+ r \leq q\ r \leq \rho(\alpha) \\ \Leftrightarrow\quad & \forall^+ q \leq p\ \forall \beta < \kappa\ \exists \alpha \geq \beta\ q \parallel \rho(\alpha) \end{aligned}$$

which proves (a), and (b) is proved similarly. □

Lemma 2. If $\mathbb{B}$ is a complete Boolean algebra, φ is a formula and $\rho \in N_n(\kappa)$, then the following conditions are equivalent:

(i) $1 \Vdash \check{\xi} \in \rho \Rightarrow \varphi(\check{\xi})$.

(ii) $\rho(\xi) \Vdash \varphi(\check{\xi})$.

Proof. We have $1 \Vdash \check{\xi} \in \rho \Rightarrow \varphi(\check{\xi})$ iff $\|\check{\xi} \in \rho\| \leq \|\varphi(\check{\xi})\|$ iff $\rho(\xi) \Vdash \varphi(\check{\xi})$. $\square$

Using the preceding two lemmas, we get

Lemma 3. Let V be the ground model, $\mathcal{A}$ an inseparable sequence in V and $\mathbb{B}$ a complete Boolean algebra. The inseparability of $\mathcal{A}$ is destroyed in every forcing extension by $\mathbb{B}$ iff there are $\sigma, \pi, \mu \in N_n(\omega)$ and $\theta, \tau \in N_n(\omega_1)$ such that

$$\forall p \in B^+ \ \exists^{\aleph_1} \xi \ p \parallel \theta(\xi) \tag{3}$$

$$\forall p \in B^+ \ \exists^{\aleph_1} \xi \ p \parallel \tau(\xi) \tag{4}$$

$$\forall p \in B^+ \ \exists^+ q \leq p \ \exists m \in \omega \ \forall n \geq m \ q \leq (\pi(n))' \tag{5}$$

$$\forall p \in B^+ \ \exists^+ q \leq p \ \exists m \in \omega \ \forall n \geq m \ q \leq (\mu(n))' \tag{6}$$

$$\forall \xi \in \omega_1 \ \theta(\xi) \Vdash \check{A}_\xi \cap \sigma = \pi \tag{7}$$

$$\forall \xi \in \omega_1 \ \tau(\xi) \Vdash \check{A}_\xi \setminus \sigma = \mu. \tag{8}$$

3. Some necessary conditions

In this section we prove several conditions necessary for a complete Boolean algebra to destroy the inseparability of a sequence $\mathcal{A}$.

We remind the reader of a few facts about distributivity. A maximal antichain in a Boolean algebra $\mathbb{B}$ is also called *a partition* (of unity). A partition U is a *refinement* of a partition W if for every $w \in W$ there is $u \in U$ such that $u \leq w$. $\mathbb{B}$ is (κ, λ)-*distributive* if every family $\{W_\alpha : \alpha < \kappa\}$ of partitions of cardinality λ has a common refinement.

Lemma 4. If forcing with a complete Boolean algebra $\mathbb{B}$ destroys the inseparability of a sequence $\mathcal{A}$, then $\mathbb{B}$ is not $(\omega, 2)$-distributive.

Proof. It is well-known that $\mathbb{B}$ is $(\kappa, 2)$-distributive iff forcing with $\mathbb{B}$ does not add any subsets of κ ([2], Theorem 15.38). So if $\mathbb{B}$ were $(\omega, 2)$-distributive, the set $D \in V[G]$ that separates $\mathcal{A}$ would also have to belong to V. Of course, the finite sets S and T from (1) and (2) are also in V. But "$A_\xi \cap D = S$" and "$A_\xi \setminus D = T$" are absolute, so D would separate $\mathcal{A}$ in V as well. $\square$

We say that forcing with $\mathbb{P}$ adds an *unsupported* subset of κ if in $V_{\mathbb{P}}[G]$ there is a set $U \in [\kappa]^\kappa$ such that for no $W \in ([\kappa]^\kappa)^V$ $W \subseteq U$ holds. With r.o.($\mathbb{P}$) we denote the completion of $\mathbb{P}$.

Lemma 5. If forcing with $\mathbb{P}$ destroys the inseparability of a sequence $\mathcal{A}$, then it also adds an unsupported subset of ω_1, so r.o.($\mathbb{P}$) can not contain a countable dense subset.

Proof. Let $V[G]$ be a generic extension via $\mathbb{P}$ and let $S, T, D, X, Y \in V[G]$ be as in (1) and (2). We will show that at least one of the sets X and Y is unsupported. To prove that, suppose the opposite: let $Z, W \in ([\omega_1]^{\aleph_1})^V$ be such that $Z \subseteq X$ and $W \subseteq Y$. We define $C = \bigcup_{\xi \in W}(A_\xi \setminus T)$. Now, since $A_\xi \setminus T \subseteq D$, we have $C \subseteq D$.

Of course, $C \in V$ and $A_\xi \setminus C \subseteq T$ for $\xi \in W$. But for every $\xi \in Z$ we also have $\xi \in X$, so $A_\xi \cap C \subseteq A_\xi \cap D = S$. Thus $A_\xi \cap C$ is finite in $V[G]$, and hence in V as well. It follows that C separates $\mathcal{A}$, a contradiction.

The last statement follows directly from [5], Proposition 5. □

Theorem 6. (a) Let $V \models \mathrm{MA}_{\aleph_1}$. If the inseparability of a sequence $\mathcal{A}$ is destroyed in every forcing extension by $\mathbb{P}$, then $\mathbb{P}$ is not c.c.c.

(b) Let $V \models \mathrm{PFA}$. If the inseparability of a sequence $\mathcal{A}$ is destroyed in every forcing extension by $\mathbb{P}$, then $\mathbb{P}$ is not proper.

Proof. (a) Since $\mathbb{P}$ is c.c.c. iff its completion is, we may assume without loss of generality that we force with a complete Boolean algebra $\mathbb{B}$. So suppose the opposite, that $\mathbb{B}$ is c.c.c. in V. Let $\pi, \mu, \sigma, \theta, \tau$ be as in Lemma 3. We define, for $\alpha \in \omega_1$ and $n \in \omega$,

$$\begin{aligned}
D_\alpha &= \{p \in B^+ : \exists \xi \geq \alpha\ p \leq \theta(\xi)\} \\
E_\alpha &= \{q \in B^+ : \exists \zeta \geq \alpha\ q \leq \tau(\zeta)\} \\
F &= \{p \in B^+ : \exists m \in \omega\ \forall n \geq m\ p \leq (\pi(n))'\} \\
H &= \{q \in B^+ : \exists m \in \omega\ \forall n \geq m\ q \leq (\mu(n))'\} \\
S_n &= \{r \in B^+ : r \leq \sigma(n) \vee r \leq \sigma'(n)\}.
\end{aligned}$$

The conditions (3), (4), (5) and (6) clearly imply that D_α, E_α (for all $\alpha < \omega_1$), F and H are dense in $B \setminus \{0\}$; moreover they are open dense. Therefore $D'_\alpha = D_\alpha \cap F$ and $E'_\alpha = E_\alpha \cap H$ are dense for $\alpha < \omega_1$. It is obvious that S_n is dense for all $n \in \omega$. Hence it follows from $\mathrm{MA}_{\aleph_1}$ that there is a filter $G \in V$ intersecting all D'_α, all E'_α and all S_n.

For each $n \in \omega$ there is $r \in S_n \cap G$. This means that either $\sigma(n) \in G$ or $\sigma'(n) \in G$. Let us define $D = \{n \in \omega : \sigma(n) \in G\}$, and prove that D separates $\mathcal{A}$.

For each $\alpha < \omega_1$ let $p_\alpha \in D'_\alpha \cap G, q_\alpha \in E'_\alpha \cap G$, and let ξ_α, ζ_α be such that $p_\alpha \leq \theta(\xi_\alpha)$ and $q_\alpha \leq \tau(\zeta_\alpha)$. The sets $X = \{\xi_\alpha : \alpha < \omega_1\}$ and $Y = \{\zeta_\alpha : \alpha < \omega_1\}$ are uncountable, since each ξ can appear as ξ_α only for $\alpha \leq \xi$.

Let $\xi = \xi_\alpha \in X$; we want to prove $|A_\xi \cap D| < \aleph_0$. Suppose the opposite. From (7) we have $p_\alpha \Vdash \check{A}_\xi \cap \sigma \subseteq \pi$, which can also be written as $\forall n \in A_\xi\ p_\alpha \wedge \sigma(n) \leq \pi(n)$. Since $p_\alpha \in F$ there is $m \in \omega$ such that $p_\alpha \leq (\pi(n))'$ for all $n \geq m$. Let $n \in A_\xi \cap D$ be greater than m; we have $p_\alpha \wedge \sigma(n) \in G$, so $p_\alpha \wedge \sigma(n) \neq 0$. But this element is below both $\pi(n)$ and $(\pi(n))'$, a contradiction.

Analogously we prove that $|A_\xi \setminus D| < \aleph_0$ for $\xi \in Y$.

Practically the same proof works for (b). □

4. Concluding remarks

It would be interesting to find an example of a notion of forcing that destroys inseparability. The results in the previous section may serve to direct such a construction. Since one needs to preserve $\aleph_1$, one should try either to construct a c.c.c.notion of forcing (and by Theorem 6(a) has to assume at least $\neg \mathrm{MA}_{\aleph_1}$) or a proper notion of forcing (assuming, by Theorem 6(b), at least $\neg$PFA). Todorčević's method, described for example in [4], could prove useful for this purpose.

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