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DECOMPOSABLE HAMILTON SPACES

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Abstract. The authors deal with $2(n+m)$ dimensional Hamilton space of the first order and consider linear connections. The general linear connection has $4^3 = 64$ types of connection coefficients. Different special kinds of linear connection, as almost d-connections, d-connection, strongly distinguished and almost strongly distinguished connections are defined. The transformation law of connection coefficients are determined. Different covariant derivatives which transform as tensors are obtained. For mentioned different kind of covariant derivatives the torsion and curvature tensors are calculated.

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1. Definitions, natural and adapted bases

The generalization of Finsler spaces began in the second half of the last century. The first big steps in the direction of spaces of higher orders are given in [18], [19], which leads to Lagrange spaces of higher order. It is natural that its dual spaces, the Hamilton spaces, are also investigated. Lagrange and Hamilton spaces of different orders have been investigated by many authors. Some of them are mentioned in references [1]-[24].

Let H be a $2(n+m)$ dimensional manifold, where some point $u \in H$ in some local chart has the coordinates

$$u = (x^a, x^\alpha, p_a, p_\alpha) \quad a, b, c, \ldots = \overline{1, n}, \quad \alpha, \beta, \gamma, \ldots = \overline{1, m}. \tag{1.1}$$

If $(x^{a'}, x^{\alpha'}, p_{a'}, p_{\alpha'})$ are the coordinates of the same point u in some other coordinate system, then the allowable coordinate transformations are given by 1-1 C^∞ functions

$$\begin{aligned} x^{a'} &= x^{a'}(x^a) \Leftrightarrow x^a = x^a(x^{a'}) \\ x^{\alpha'} &= x^{\alpha'}(x^\alpha) \Leftrightarrow x^\alpha = x^\alpha(x^{\alpha'}) \\ p_{a'} &= B^a_{a'} p_a \Leftrightarrow p_a = B^{a'}_a p_{a'} \\ p_{\alpha'} &= B^\alpha_{\alpha'} p_\alpha \Leftrightarrow p_\alpha = B^{\alpha'}_\alpha p_{\alpha'}, \end{aligned} \tag{1.2}$$

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where

$$B_a^{a'} = \partial_a x^{a'}, \quad B_{a'}^a = \partial_{a'} x^a, \quad B_\alpha^{\alpha'} = \partial_\alpha x^{\alpha'}, \quad B_{\alpha'}^\alpha = \partial_{\alpha'} x^\alpha. \tag{1.3}$$

Definition 1.1. *The $2(n+m)$ dimensional C^∞ manifold, whose points have coordinates as in (1.1) and the allowable coordinate transformations are prescribed by (1.2) and (1.3) is called the decomposable Hamilton space.*

Remark 1.1. *The space H is called decomposable Hamilton space, with the components H_{2n} and H_{2m}, where $(x^a, p_a) \in H_{2n}$ and $(x^\alpha, p_\alpha) \in H_{2m}$.*

As the transformations (1.2) are regular, so there exist inverse transformations and we have

$$B_a^{a'} B_{b'}^a = \delta_{b'}^{a'}, \quad B_{a'}^a B_b^{a'} = \delta_b^a, \quad B_\alpha^{\alpha'} B_{\beta'}^\alpha = \delta_{\beta'}^{\alpha'}, \quad B_{\alpha'}^\alpha B_\beta^{\alpha'} = \delta_\beta^\alpha. \tag{1.4}$$

Theorem 1.1. *The transformations of type (1.2) form a pseudo-group.*

One example of space H is such a space in which $p_a = \frac{\partial}{\partial x^a}$, $p_\alpha = \frac{\partial}{\partial x^\alpha}$. These elements satisfy the transformation laws given by (1.2). In this special case $p_a x^b = \delta_a^b$, $p_\alpha x^\beta = \delta_\alpha^\beta$. In the further examination we shall suppose that p_a and p_α are arbitrary elements, which transform as it is prescribed by (1.2).

The natural basis $\bar{B}$ of $T(H)$ is

$$\bar{B} = \left\{ \partial_a = \frac{\partial}{\partial x^a}, \partial_\alpha = \frac{\partial}{\partial x^\alpha}, \partial^{\bar{a}} = \frac{\partial}{\partial p_a}, \partial^{\bar{\alpha}} = \frac{\partial}{\partial p_\alpha} \right\}. \tag{1.5}$$

We shall use overline indices when it is necessary to affirm that they are related to p_a or p_α. Where there is no confusion we omit the overlines, for instance in (1.1).

The elements of $\bar{B}$ transform in the following way:

$$\begin{aligned}
\partial_a &= B_a^{a'} \partial_{a'} + B_{a'}{}^{b}{}_{b'} B_a^{b'} p_b \partial^{\bar{a}'} \\
\partial_\alpha &= B_\alpha^{\alpha'} \partial_{\alpha'} + B_{\alpha'}{}^{\beta}{}_{\beta'} B_\alpha^{\beta'} p_\beta \partial^{\bar{\alpha}'} \\
\partial^{\bar{a}} &= B_{a'}^a \bar{\partial}^{a'}, \quad \partial^{\bar{\alpha}} = B_{\alpha'}^\alpha \partial^{\bar{\alpha}'}
\end{aligned} \tag{1.6}$$

The first two elements of $\bar{B}$ do not transform as tensors, the last two have this property.

The adapted basis B of $T(H)$ is

$$B = \{\delta_a, \delta_\alpha, \delta^{\bar{a}}, \delta^{\bar{\alpha}}\}, \tag{1.7}$$

where

$$\begin{aligned}
\delta_a &= \partial_a - N_{ab} \partial^{\bar{b}}, \qquad \delta^{\bar{a}} = \partial^{\bar{a}}, N_{ab} = N_{ab}(x^a, p_a), \\
\delta_\alpha &= \partial_\alpha - N_{\alpha\beta} \partial^{\bar{\beta}}, \qquad \delta^{\bar{\alpha}} = \partial^{\bar{\alpha}}, N_{\alpha\beta} = N_{\alpha\beta}(x^\alpha, p_\alpha).
\end{aligned} \tag{1.8}$$

Theorem 1.2. *The necessary and sufficient conditions, that the elements of B transform as tensors, i.e.*

$$\delta_a = B_a^{a'} \delta_{a'}, \quad \delta_\alpha = B_\alpha^{\alpha'} \delta_\alpha \tag{1.9}$$

are the following relations

$$N_{a'c'} = N_{ac} B_{a'}^a B_{c'}^c - B_{a'}{}^b{}_{c'} p_b \tag{1.10}$$

$$N_{\alpha'\beta'} = N_{\alpha\beta} B_{\alpha'}^\alpha B_{\beta'}^\beta - B_{\alpha'}{}^\beta{}_{\beta'} p_\beta.$$

Proof. From $\delta_{a'} = B_{a'}^a \delta_a$ using (1.6) and (1.9) we get

$$\partial_{a'} - N_{a'c'} \partial^{c'} = B_{a'}^a (\partial_a - N_{ab} \partial^b) =$$

$$B_{a'}^a (B_a^{c'} \partial_{c'} + B_{a'}{}^b{}_{c'} B_a^{c'} p_b \partial^{a'} - N_{ac} B_{c'}^c \partial^{c'}).$$

From (1.4) and the above equation it follows the first equation of (1.10). The second equation can be proved in a similar way. □

The natural basis $\bar{B}^*$ of $T^*(H)$ is

$$\bar{B}^* = \{dx^a, dx^\alpha, dp_a, dx_\alpha\}.$$

From (1.2) it follows

$$dx^a = B_{a'}^a dx^{a'}, \quad dx^\alpha = B_{\alpha'}^\alpha dx^{\alpha'} \tag{1.11}$$

$$dp_a = B_a{}^{a'}_b p_{a'} dx^b + B_a^{a'} dp_{a'}, \quad dp_\alpha = B_{\alpha\beta}^{\alpha'} p_{\alpha'} dx^\beta + B_\alpha^{\alpha'} dp_{\alpha'}.$$

The adapted basis B^* of $T^*(H)$ is given by

$$B^* = \{\delta x^a, \delta x^\alpha, \delta p_a, \delta p_\alpha\}, \tag{1.12}$$

where

$$\delta x^a = dx^a, \quad \delta x^\alpha = dx^\alpha \tag{1.13}$$

$$\delta p_{\bar{a}} = dp_{\bar{a}} + M_{ab} dx^b,$$

$$\delta p_{\bar{\alpha}} = dp_{\bar{\alpha}} + M_{\alpha\beta} dx^\beta,$$

where $M_{ab} = M_{ab}(x^a, p_a)$, $M_{\alpha\beta} = M_{\alpha\beta}(x^\alpha, p_\alpha)$.

Theorem 1.3. *The necessary and sufficient conditions that the elements of B^* transform as tensors, i.e.*

$$\delta x^a = B_{a'}^a \delta x^{a'}, \quad \delta x^\alpha = B_{\alpha'}^\alpha \delta x^{\alpha'} \tag{1.14}$$

$$\delta p_{\bar{a}} = B_a^{a'} \delta p_{\bar{a}'}, \quad \delta p_{\bar{\alpha}} = B_\alpha^{\alpha'} \delta p_{\bar{\alpha}'}$$

are the following relations:

$$M_{a'b'} = B_{a\ b}^{\ c'} B_{a'}^{a} B_{b'}^{b} p_{c'} + M_{ab} B_{a'}^{a} B_{b'}^{b} \tag{1.15}$$

$$M_{\alpha'\beta'} = B_{\alpha\ \beta}^{\ \gamma'} B_{\alpha'}^{\alpha} B_{\beta'}^{\beta} p_{\gamma'} + M_{\alpha\beta} B_{\alpha'}^{\alpha} B_{\beta'}^{\beta}. \tag{1.16}$$

Proof. The proof is similar to the proof of Theorem 1.2. □

Theorem 1.4. *The duality of natural bases is coordinate invariant, i.e. from the duality of $\bar{B}^{*'}$ and $\bar{B}'$ follows the duality of $\bar{B}^*$ and $\bar{B}$.*

Proof. By assumption

$$\begin{bmatrix} dx^{b'} \\ dx^{\beta'} \\ dp_{\bar{b}'} \\ dp_{\bar{\beta}'} \end{bmatrix} [\partial_{a'} \partial_{\alpha'} \partial^{\bar{a}'} \partial^{\bar{\alpha}'}] = \begin{bmatrix} \delta_{a'}^{b'} & 0 & 0 & 0 \\ 0 & \delta_{\alpha'}^{\beta'} & 0 & 0 \\ 0 & 0 & \delta_{\bar{b}'}^{\bar{a}'} & 0 \\ 0 & 0 & 0 & \delta_{\bar{\beta}'}^{\bar{\alpha}'} \end{bmatrix} \tag{1.17}$$

Using the matrix form of (1.6) and (1.11) we have

$$\begin{bmatrix} dx^{b} \\ dt^{\beta} \\ dp_{\bar{b}} \\ dp_{\bar{\beta}} \end{bmatrix} = A \begin{bmatrix} dx^{b'} \\ dt^{\beta'} \\ dp^{\bar{b}'} \\ dp_{\bar{\beta}'} \end{bmatrix} \tag{1.18}$$

$$[\partial_a \partial_\alpha \partial^{\bar{a}} \partial^{\bar{\alpha}}] = [\partial_{a'} \partial_{\alpha'} \partial^{\bar{a}'} \partial^{\bar{\alpha}'}] B, \tag{1.19}$$

where

$$A = \begin{bmatrix} B_{b'}^{b} & 0 & 0 & 0 \\ 0 & B_{\beta'}^{\beta} & 0 & 0 \\ B_{b\ c}^{\ c'} p_{c'} B_{b'}^{c} & 0 & B_{b}^{b'} & 0 \\ 0 & B_{\beta\ \gamma}^{\ \gamma'} p_{\gamma'} B_{\beta'}^{\gamma} & 0 & B_{\beta}^{\beta'} \end{bmatrix},$$

$$B = \begin{bmatrix} B_{a}^{a'} & 0 & 0 & 0 \\ 0 & B_{\alpha}^{\alpha'} & 0 & 0 \\ B_{a'\ b'}^{\ b} B_{a}^{b'} p_b & 0 & B_{a'}^{a} & 0 \\ 0 & B_{\alpha'\ \gamma'}^{\ \beta} B_{\alpha}^{\gamma'} p_\beta & 0 & B_{\alpha'}^{\alpha} \end{bmatrix}.$$

Using (1.17), (1.18) and (1.19) we get

$$\begin{bmatrix} dx^b \\ dt^\beta \\ dp_{\bar b} \\ dp_{\bar\beta} \end{bmatrix} [\partial_a \partial_\alpha \partial^{\bar a} \partial^{\bar\alpha}] = \begin{bmatrix} \delta^b_a & 0 & 0 & 0 \\ 0 & \delta^\beta_\alpha & 0 & 0 \\ \alpha & 0 & \delta^{\bar a}_{\bar b} & 0 \\ 0 & \beta & 0 & \delta^{\bar\alpha}_{\bar\beta} \end{bmatrix},$$

where

$$\alpha = B_b{}^{c'}{}_c p_{c'} \delta^c_a + B^{b'}_b B_{b'}{}^d{}_{c'} B^{c'}_a p_d$$

$$\beta = B_\beta{}^{\gamma'}{}_\gamma p_{\gamma'} \delta^\gamma_\alpha + B^{\beta'}_\beta B_{\beta'}{}^\gamma{}_{\gamma'} B^{\gamma'}_\alpha p_\gamma.$$

We have to prove $\alpha = 0$, $\beta = 0$.

From $B^\gamma_{\gamma'} B^{\gamma'}_\beta = \delta^\gamma_\beta$, taking the partial derivative with respect to x^α we get

$$B_{\gamma'}{}^\gamma{}_{\beta'} B^{\beta'}_\alpha B^{\gamma'}_\beta + B^\gamma_{\gamma'} B_\alpha{}^{\gamma'}{}_\beta = 0.$$

The multiplication of the above equation with p_γ results $\beta = 0$.

The relation $\alpha = 0$ can be proved in the similar way. □

Theorem 1.5. *The adapted basis B^* of $T^*(H)$ is dual to the adapted basis B of $T(H)$ if $\bar B^*$ is dual to $\bar B$ and*

$$M_{ac} = N_{ca}, \quad M_{\alpha\gamma} = N_{\gamma\alpha}. \tag{1.20}$$

Proof. The equations (1.8) and (1.13) can be written in the matrix form as follows:

$$\begin{bmatrix} \delta x^a \\ \delta x^\alpha \\ \delta p_{\bar a} \\ \delta p_{\bar\alpha} \end{bmatrix} = \begin{bmatrix} \delta^a_b & 0 & 0 & 0 \\ 0 & \delta^\alpha_\beta & 0 & 0 \\ M_{ab} & 0 & \delta^{\bar b}_{\bar a} & 0 \\ 0 & M_{\alpha\beta} & 0 & \delta^{\bar\beta}_{\bar\alpha} \end{bmatrix} \begin{bmatrix} dx^b \\ dx^\beta \\ dp_{\bar b} \\ dp_{\bar\beta} \end{bmatrix},$$

$$[\delta_c \delta_\gamma \delta^{\bar c} \delta^{\bar\alpha}] = [\partial_d \partial_\delta \delta^{\bar d} \delta^{\bar\delta}] \begin{bmatrix} \delta^d_c & 0 & 0 & 0 \\ 0 & \delta^\delta_\gamma & 0 & 0 \\ -N_{cd} & 0 & \delta^{\bar c}_{\bar d} & 0 \\ 0 & -N_{\gamma\delta} & 0 & \delta^{\bar\gamma}_{\bar\delta} \end{bmatrix}.$$

If we multiply the above two equations and use the duality of $\bar B^*$ and $\bar B$ we get

$$\begin{bmatrix} \delta x^a \\ \delta x^\alpha \\ \delta p_{\bar a} \\ \delta p_{\bar\alpha} \end{bmatrix} [\delta_c \delta_\gamma \delta^{\bar c} \delta^{\bar\gamma}] = \begin{bmatrix} \delta^a_c & 0 & 0 & 0 \\ 0 & \delta^\alpha_\gamma & 0 & 0 \\ \alpha & 0 & \delta^{\bar c}_{\bar a} & 0 \\ 0 & \beta & 0 & \delta^{\bar\gamma}_{\bar\alpha} \end{bmatrix},$$

where

$$\alpha = M_{ac} - N_{ca} \qquad \beta = M_{\alpha\gamma} - N_{\gamma\alpha}.$$

From the above it follows (1.20). □

2. Different kinds of connections on $T(H)$

To define different kinds of connections on $T(H)$ we need different subspaces of $T(H)$ and $T^*(H)$. Some point $u \in H = H_{2(n+m)}$ is given by its coordinate $(x^a, x^\alpha, p_a, p_\alpha)$, $a = \overline{1,n}$, $\alpha = \overline{1,m}$. We can consider two families of subspaces: H_{2n} and H_{2m} of $H_{2(n+m)}$ whose point are given by $H_{2n} : (x^a, C^\alpha, p_a, C_\alpha)$ and $H_{2n} : (C^a, x^\alpha, C_a, p_\alpha)$ respectively, where C^a, C^α, C_a, C_α are constants. From (1.2) it follows that H_{2n} and H_{2m} are Hamilton spaces and

$$T(H) = T(H_{2(n+m)}) = T(H_{2n}) \oplus T(H_{2m}),$$

$$T^*(H) = T^*(H_{2(n+m)}) = T^*(H_{2n}) \oplus T^*(H_{2m}),$$

$$T(H_{2n}) \text{ is generated by } \{\delta_a, \delta^{\bar{a}}\}$$

$$T(H_{2m}) \text{ is generated by } \{\delta_\alpha, \delta^{\bar{\alpha}}\}$$

$$T^*(H_{2n}) \text{ is generated by } \{\delta x^a, \delta p_{\bar{a}}\}$$

$$T^*(H_{2m}) \text{ is generated by } \{\delta x^\alpha, \delta p_{\bar{\alpha}}\}.$$

Let us denote by T_1, T_2, T_3, T_4 the subspaces of $T(H)$ generated by $\{\delta_a\}$, $\{\delta_\alpha\}$, $\{\delta^{\bar{a}}\}$, $\{\delta^{\bar{\alpha}}\}$ and by T_1^*, T_2^*, T_3^*, T_4^* the subspaces of $T^*(H)$ generated by $\{\delta x^a\}$, $\{\delta x^\alpha\}$, $\{\delta p_{\bar{a}}\}$, $\{\delta p_{\bar{\alpha}}\}$ respectively.

Some tensor T on $T_1 \otimes T_1^* \otimes T_2 \otimes T_2^* \otimes T_3 \otimes T_3^* \otimes T_4 \otimes T_4^*$ is given by

$$T = T^{a \cdot \alpha \cdot \; \cdot \; \bar{d} \cdot \bar{\delta}}_{\cdot b \; \cdot \beta \bar{c} \; \cdot \bar{\gamma} \cdot} \delta_a \otimes \delta x^b \otimes \delta_\alpha \otimes \delta x^\beta \otimes \delta^{\bar{c}} \otimes \delta p_{\bar{d}} \otimes \delta^{\bar{\gamma}} \otimes \delta p_{\bar{\delta}}.$$

In the chart (U', φ') T has the same form, only all indices obtain the sign $'$. Using the transformation law of elements of adapted bases ((1.9), (1.14)) B and B^* in $U \cap U'$ we obtain:

$$T^{a \cdot d \; \cdot \; \cdot \; \bar{d} \; \cdot \; \bar{\delta}}_{\cdot b \; \cdot \beta \bar{c} \; \cdot \bar{\gamma} \; \cdot} = T^{a' \cdot \; d' \cdot \; \cdot \; \bar{d}' \cdot \; \bar{\delta}'}_{\cdot \; b' \; \cdot \beta' \bar{c}' \; \cdot \bar{\gamma}' \cdot} B^{ab'd\beta'\bar{c}'\bar{d}\bar{\gamma}'\bar{\delta}}_{a'bd'\beta\bar{c}\bar{d}'\bar{\gamma}\bar{\delta}'}. \tag{2.1}$$

Let us suppose that on $T^*(H) \otimes T^*(H)$ one metric tensor g is given.

Theorem 2.1. *The necessary and sufficient conditions that the subspaces T_1^*, T_2^*, T_3^*, T_4^* of $T^*(H)$ are mutually orthogonal with respect to g is that the metric tensor has the form:*

$$g = g_{ab} \delta x^a \otimes \delta x^b + g_{\alpha\beta} \delta x^\alpha \otimes \delta x^\beta + g^{\bar{a}\bar{b}} \delta p_{\bar{a}} \otimes \delta p_{\bar{b}} + g^{\bar{\alpha}\bar{\beta}} \delta p_{\bar{\alpha}} \otimes \delta p_{\bar{\beta}}. \tag{2.2}$$

It is supposed that all components of g are symmetric and rank $g = 2(n+m)$.

Theorem 2.2. *The following equations are coordinate invariant*

$$g_{ab}\delta x^a = \delta_b \Leftrightarrow g^{ab}\delta_b = \delta x^a \tag{2.3}$$

$$g_{\alpha\beta}\delta x^\alpha = \delta_\beta \Leftrightarrow g^{\alpha\beta}\delta_\beta = \delta x^\alpha$$

$$g^{\bar{a}\bar{b}}\delta p_{\bar{a}} = \delta^{\bar{b}} \Leftrightarrow g_{\bar{a}\bar{b}}\delta^{\bar{b}} = \delta p_{\bar{a}}$$

$$g^{\bar{\alpha}\bar{\beta}}\delta p_{\bar{\alpha}} = \delta^{\bar{\beta}} \Leftrightarrow g_{\bar{\alpha}\bar{\beta}}\delta^{\bar{\beta}} = \delta p_{\bar{\alpha}}$$

where

$$g_{ab}g^{bc} = \delta^c_a, \quad g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha, \quad g_{\bar{a}\bar{b}}g^{\bar{b}\bar{c}} = \delta^{\bar{c}}_{\bar{a}}, \quad g_{\bar{\alpha}\bar{\beta}}g^{\bar{\beta}\bar{\gamma}} = \delta^{\bar{\gamma}}_{\bar{\alpha}}.$$

Proof. From the first equation of (2.3) it follows

$$g_{a'b'}\delta x^{a'} = B^{a}_{a'}{}^{b}_{b'} g_{ab} B^{a'}_{c}\delta x^c = \delta^a_c g_{ab} B^b_{b'}\delta x^c =$$

$$B^b_{b'} g_{ab}\delta x^a = B^b_{b'}\delta_b = \delta_{b'}.$$

The other relations of (2.3) can be obtained in the similar way. □

The necessary conditions for (2.3) are:

$$g_{ab} = g_{ab}(x^a, p_a) \quad , \quad g^{\bar{a}\bar{b}} = g^{\bar{a}\bar{b}}(x^a, p_a)$$

$$g_{\alpha\beta} = g_{\alpha\beta}(x^\alpha, p_\alpha) \quad , \quad g^{\bar{\alpha}\bar{\beta}} = g^{\bar{\alpha}\bar{\beta}}(x^\alpha, p_\alpha),$$

i.e. the metric tensor is also decomposable, its matrix has four diagonal blocks, the first two of them are $n \times n$, the other two $m \times m$ matrices. The first two are functiones of (x^a, p_a), the other two of (x^α, p_α).

We shall use the notations (if $T_1 : \{\delta_a\}, T_2 : \{\delta_\alpha\}, T_3 : \{\delta^{\bar{a}}\}, T_4 : \{\delta^{\bar{\alpha}}\}$)

$$T_h(H) = T_1 \oplus T_2 \quad T_v(H) = T_3 \oplus T_4. \tag{2.4}$$

We have

$$T(H_{2n}) = T_1 \oplus T_3, \quad T(H_{2m}) = T_2 \oplus T_4. \tag{2.5}$$

It is obvious that

$$T(H) = T_h(H) \oplus T_v(H) = T(H_{2n}) \oplus T(H_{2m}). \tag{2.6}$$

Relations (2.4)-(2.6) are valid if everywhere T is substituted by T^*, i.e. the decomposition of the dual tangent space is possible in the same way.

Definition 2.1. *The linear connection* $\nabla(T(H)\times T(H)\to T(H)$, $\nabla:(X,Y)\mapsto\nabla_X Y$ *for every* $X,Y\in T(H)$ *is defined by*

$$\text{(2.7)}\qquad \begin{aligned}
&(a)\quad \text{For } x\in\{a,\alpha\},\quad y\in\{b,\beta\}\\
&\qquad \nabla_{\delta_x}\delta_y=\Gamma_{y\ x}^{\ c}\delta_c+\Gamma_{y\ x}^{\ \gamma}\delta_\gamma+\Gamma_{y\bar{c}x}\delta^{\bar{c}}+\Gamma_{y\bar{\gamma}x}\delta^{\bar{\gamma}}\\
&(b)\quad \text{for } x\in\{a,\alpha\},\quad \bar{y}\in\{\bar{b},\bar{\beta}\}\\
&\qquad \nabla_{\delta_x}\delta^{\bar{y}}=\Gamma^{\bar{y}c}_{\ \ x}\delta_c+\Gamma^{\bar{y}\gamma}_{\ \ x}\delta_\gamma+\Gamma^{\bar{y}}_{\ \bar{c}x}\delta^{\bar{c}}+\Gamma^{\bar{y}}_{\ \bar{\gamma}x}\delta^{\bar{\gamma}},\\
&(c)\quad \text{for } \bar{x}\in\{\bar{a},\bar{\alpha}\},\quad y\in\{b,\beta\}\\
&\qquad \nabla_{\delta^{\bar{x}}}\delta_y=\Gamma_{y}^{\ c\bar{x}}\delta_c+\Gamma_{y}^{\ \gamma\bar{x}}\delta_\gamma+\Gamma_{y\bar{c}}^{\ \ \bar{x}}\delta^{\bar{c}}+\Gamma_{y\bar{\gamma}}^{\ \ \bar{x}}\delta^{\bar{\gamma}},\\
&(d)\quad \text{for } \bar{x}\in\{\bar{a},\bar{\alpha}\},\quad \bar{y}\in\{\bar{b},\bar{\beta}\},\\
&\qquad \nabla_{\delta^{\bar{x}}}\delta^{\bar{y}}=\Gamma^{\bar{y}c\bar{x}}\delta_c+\Gamma^{\bar{y}\gamma\bar{x}}\delta_\gamma+\Gamma^{\bar{y}\ \bar{x}}_{\ \bar{c}}\delta^{\bar{c}}+\Gamma^{\bar{y}\ \bar{x}}_{\ \bar{\gamma}}\delta^{\bar{\gamma}}.
\end{aligned}$$

From the above it is obvious that the linear connection has $16\cdot 4=64$ types of connection coefficients.

Definition 2.2. *The almost d-connection of the first type* (*a.d.c.f.t.*) *is a linear connection for which* Y *and* $\nabla_X Y$ *both belong to* $T(H_{2n})$ *or* $T(H_{2m})$ *for every* $X\in T(H)$. *For* (*a.d.c.f.t.*) *we have*

$$\text{(2.8)}\qquad \begin{aligned}
\nabla_{\delta_a}\delta_b&=\Gamma_{b\ a}^{\ c}\delta_c+\Gamma_{b\bar{c}a}\delta^{\bar{c}}\\
\nabla_{\delta_a}\delta^{\bar{b}}&=\Gamma^{\bar{b}c}_{\ \ a}\delta_c+\Gamma^{\bar{b}}_{\ \bar{c}a}\delta^{\bar{c}}\\
\nabla_{\delta^{\bar{a}}}\delta_b&=\Gamma_{b}^{\ c\bar{a}}\delta_c+\Gamma_{b\bar{c}}^{\ \ \bar{a}}\delta^{\bar{c}}\\
\nabla_{\delta^{\bar{a}}}\delta^{\bar{b}}&=\Gamma^{\bar{b}c\bar{a}}\delta_c+\Gamma^{\bar{b}\ \bar{a}}_{\ \bar{c}}\delta^{\bar{c}}.
\end{aligned}$$

In the above equations X, Y and $\nabla_X Y$ belong to $T(H_{2n})$.

If in (2.8) we substitute $a\to\alpha$, $\bar{a}\to\bar{\alpha}$ we obtain the other 8 types of connection coefficients for (*a.d.c.f.t.*), where $X\in T(H_{2m})$ and Y, $\nabla_X Y$ belong to $T(H_{2n})$.

If in (2.8) we make the following changes:

$$b\to\beta,\quad \bar{b}\to\bar{\beta},\quad c\to\gamma,\quad \bar{c}\to\bar{\gamma},$$

we obtain 8 types of (*a.d.c.f.t.*), where $X\in T(H_{2n})$, Y and $\nabla_X Y$ belong to $T(H_{2m})$.

If in thus obtained equations we substitute

$$a\to\alpha,\quad \bar{a}\to\bar{\alpha}$$

we obtain the last 8 types of (*a.d.c.f.t.*), where X, Y, $\nabla_X Y$ belong to $T(H_{2m})$.

From the above it follows that (*a.d.c.f.t.*) has $4\cdot 8=32$ types of connection coefficients.

Definition 2.3. *The almost d-connection of the second type* $(a.d.c.s.t.)$ *is a linear connection in which* Y *and* $\nabla_X Y$ *both belong to* $T_h(H)$ *or* $T_v(H)$ *for every* $X \in T(H)$.

For $(a.d.c.s.t.)$ we have

$$\begin{aligned}\nabla_{\delta_a}\delta_b &= \Gamma_{b\ a}^{\ c}\delta_c + \Gamma_{b\ a}^{\ \gamma}\delta_\gamma \\ \nabla_{\delta^{\bar a}}\delta_b &= \Gamma_b^{\ c\bar a}\delta_c + \Gamma_b^{\ \gamma\bar a}\delta_\gamma \\ \nabla_{\delta_a}\delta_\beta &= \Gamma_{\beta\ a}^{\ c}\delta_c + \Gamma_{\beta\ a}^{\ \gamma}\delta_\gamma \\ \nabla_{\delta^{\bar a}}\delta_\beta &= \Gamma_\beta^{\ c\bar a}\delta_c + \Gamma_\beta^{\ \gamma\bar a}\delta_\gamma.\end{aligned} \tag{2.9}$$

In (2.9), Y and $\nabla_X Y$ belong to $T_h(H)$.

If in (2.9) we change everywhere $a \to \alpha$, $\bar a \to \bar\alpha$ we obtain other 8 types of connection coefficients of $(a.d.c.s.t.)$, where Y and $\nabla_X Y$ belong to $T_h(H)$. If in (2.9) we make the substitution

$$b \to \bar b, \quad \beta \to \bar\beta, \quad c \to \bar c, \quad \gamma \to \bar\gamma$$

exchanging the upper and lower indexes we obtain such connection coefficients of $(a.d.c.s.t.)$ in which Y and $\nabla_X Y$ belong to $T_v(H)$. In thus obtained the formulae we make the substitution $a \to \alpha$, $\bar a \to \bar\alpha$.

In this way we obtain $4 \cdot 8 = 32$ connection coefficients of $(a.d.c.s.t.)$.

Definition 2.4. *The d-connection is a linear connection in which* Y *and* $\nabla_X Y$ *belong to the same subspace* T_1 *or* T_2, *or* T_3, *or* T_4 *of* $T(H)$.

The d-connection is defined by the following equations:

For $x \in \{a, \alpha\}$

$$\begin{aligned}\nabla_{\delta_x}\delta_b &= \Gamma_{b\ x}^{\ c}\delta_c & \nabla_{\delta_x}\delta_\beta &= \Gamma_{\beta\ x}^{\ \gamma}\delta_\gamma \\ \nabla_{\delta_x}\delta^{\bar b} &= \Gamma^{\bar b}_{\bar c x}\delta^{\bar c} & \nabla_{\delta_x}\delta^{\bar\beta} &= \Gamma^{\bar\beta}_{\bar\gamma x}\delta^{\bar\gamma}\end{aligned} \tag{2.10}$$

and for $\bar x \in \{\bar a, \bar\alpha\}$

$$\begin{aligned}\nabla_{\delta^{\bar x}}\delta_b &= \Gamma_b^{\ c\bar x}\delta_c & \nabla_{\delta^{\bar x}}\delta_\beta &= \Gamma_\beta^{\ \gamma\bar x}\delta_\gamma \\ \nabla_{\delta^{\bar x}}\delta^{\bar b} &= \Gamma^{\bar b\ \bar x}_{\ \bar c}\delta^{\bar c} & \nabla_{\delta^{\bar x}}\delta^{\bar\beta} &= \Gamma^{\bar\beta\ \bar x}_{\ \bar\gamma}\delta^{\bar\gamma}.\end{aligned} \tag{2.11}$$

From (2.10) and (2.11) it is obvious that for the d-connection there are $8 \cdot 2 = 16$ different types of connection coefficients.

Definition 2.5. *The strongly distinguished connection* $(s.d.c.)$ *is a linear connection in which* X, Y *and* $\nabla_X Y$ *belong to the same subspace* T_1 *or* T_2 *or* T_3 *or* T_4 *of* $T(H)$.

For s.d.c. we have

$$\begin{aligned}\nabla_{\delta_a}\delta_b &= \Gamma_{b\ a}^{\ c}\delta_c & \nabla_{\delta_\alpha}\delta_\beta &= \Gamma_{\beta\ \alpha}^{\ \gamma}\delta_\gamma \\ \nabla_{\delta^{\bar a}}\delta^{\bar b} &= \Gamma^{\bar b\ \bar a}_{\ \bar c}\delta^{\bar c} & \nabla_{\delta^{\bar\alpha}}\delta^{\bar\beta} &= \Gamma^{\bar\beta\ \bar\alpha}_{\ \bar\gamma}\delta^{\bar\gamma}.\end{aligned} \tag{2.12}$$

The other connection coefficients are equal to zero.

Definition 2.6. *The almost strongly distinguished connection* (*a.s.d.c.*) *is a linear connection in which* X, Y *and* $\nabla_X Y$ *belong to one of* $T(H_{2n})$ *or* $T(H_{2m})$.

For X, Y, $\nabla_X Y \in T(H_{2n})$ we have

$$\begin{aligned}
\nabla_{\delta_a}\delta_b &= \Gamma_{b\ a}^{\ c}\delta_c + \Gamma_{b\bar{c}a}\delta^{\bar{c}}\\
\nabla_{\delta^{\bar{a}}}\delta_b &= \Gamma_b^{\ c\bar{a}}\delta_c + \Gamma_{b\bar{c}}^{\ \ \bar{a}}\delta^{\bar{c}}\\
\nabla_{\delta_a}\delta^{\bar{b}} &= \Gamma^{\bar{b}c}{}_a\delta_c + \Gamma^{\bar{b}}{}_{\bar{c}a}\delta^{\bar{c}}\\
\nabla_{\delta^{\bar{a}}}\delta^{\bar{b}} &= \Gamma^{\bar{b}c\bar{a}}\delta_c + \Gamma^{\bar{b}}{}_{\bar{c}}{}^{\bar{a}}\delta^{\bar{c}}.
\end{aligned} \tag{2.13}$$

For X, Y, $\nabla_X Y \in T(H_{2m})$ the ($a.s.d.c.$) connection is obtained from (2.13) when the substitution $(a, b, c, \bar{a}, \bar{b}, \bar{c}) \to (\alpha, \beta, \gamma, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is done.

3. The transformation law of connection coefficients

Theorem 3.1. *Among 64 types of connection coefficients determined by* (2.7) *only four of them:* $\Gamma_{b\ a}^{\ c}$ $\Gamma_{\beta\ \alpha}^{\ \gamma}$, $\Gamma^{\bar{b}}{}_{\bar{c}a}$, $\Gamma^{\bar{\beta}}{}_{\bar{\gamma}\alpha}$ *are transforming as connection coefficients (see* (3.1)-(3.4)*, the other are transforming as tensors.*

Proof. If in (2.7) we put $x = a$, $y = b$ we get

$$\begin{aligned}
\nabla_{\delta_{a'}}\delta_{b'} \;=\;& \nabla_{B^a_{a'}\delta_a} B^b_{b'}\delta_b = B^a_{a'}(\delta_a B^b_{b'})\delta_b + B^a_{a'}B^b_{b'}\nabla_{\delta_a}\delta_b =\\
& B^a_{a'}B_{b'}{}^{c}{}_{c'}B^{c'}_a\delta_c + B^a_{a'}B^b_{b'}(\Gamma_{b\ a}^{\ c}\delta_c + \Gamma_{b\ a}^{\ \alpha}\delta_\alpha + \Gamma_{b\bar{c}a}\delta^{\bar{c}} + \Gamma_{b\bar{\gamma}a}\delta^{\bar{\gamma}}) =\\
& \Gamma_{b'\ a'}^{\ c'}B^c_{c'}\delta_c + \Gamma_{b'\ a'}^{\ \alpha'}B^\alpha_{\alpha'}\delta_\alpha + \Gamma_{b'\bar{c}'a'}B^{c'}_c\delta^{\bar{c}} + \Gamma_{b'\bar{\gamma}'a'}B^{\gamma'}_\gamma\delta^{\bar{\gamma}}.
\end{aligned}$$

From the above we get

$$\begin{aligned}
\delta_c : \Gamma_{b'\ a'}^{\ c'}B^c_{c'} &= \Gamma_{b\ a}^{\ c}B^a_{a'}B^b_{b'} + B_{b'}{}^{c}{}_{a'},\\
\delta_\alpha : \Gamma_{b'\ a'}^{\ \alpha'}B^\alpha_{\alpha'} &= \Gamma_{b\ a}^{\ \alpha}B^a_{a'}B^b_{b'}\\
\delta^{\bar{c}} : \Gamma_{b'\bar{c}'a'}B^{c'}_c &= \Gamma_{b\bar{c}a}B^a_{a'}B^b_{b'}\\
\delta^{\bar{\gamma}} : \Gamma_{b'\bar{\gamma}'a'}B^{\gamma'}_\gamma &= \Gamma_{b\bar{\gamma}a}B^a_{a'}B^b_{b'}.
\end{aligned} \tag{3.1}$$

If in (2.7)(a) we put $x = \alpha$, $y = \beta$ and using the relation

$$\delta_\alpha B^\beta_{\beta'} = B_{\beta'}{}^{\beta}{}_{\gamma'}B^{\gamma'}_\alpha$$

by the same procedure as before we get

$$\Gamma_{\beta'\ \alpha'}^{\ \gamma'}B^\gamma_{\gamma'} = \Gamma_{\beta\ \alpha}^{\ \gamma}B^\alpha_{\alpha'}B^\beta_{\beta'} + B_{\alpha'}{}^{\gamma}{}_{\beta'} \tag{3.2}$$

and the other connection coefficients: $\Gamma_{\beta'}{}^{c'}{}_{\alpha'}$, $\Gamma_{\beta'\bar{c}'\alpha'}$, $\Gamma_{\beta'\bar{\gamma}'\alpha'}$ are transforming as tensor.

If in (2.7)(a) we put $x = a$, $y = \beta$ or $x = \alpha$, $y = b$ we obtain connection coefficients which are transforming as tensors, because

$$\delta_a B^{\beta'}_{\beta} = 0, \quad \delta_\alpha B^{b'}_{b} = 0.$$

From (2.7)(b) and $x = a$, $\bar{y} = \bar{b}$, using the former procedure we get:

$$\Gamma^{\bar{b}'}{}_{\bar{c}'a'} B^{c'}_{c} = \Gamma^{\bar{b}}{}_{\bar{c}a} B^{a}_{a'} B^{b'}_{b} + B^{a}_{a'} B_{c}{}^{b'}{}_{a} \tag{3.3}$$

and for $x = \alpha$, $\bar{y} = \bar{\beta}$ we get

$$\bar{\Gamma}^{\bar{\beta}'}{}_{\bar{\gamma}'\alpha'} B^{\gamma'}_{\gamma} = \Gamma^{\bar{\beta}}{}_{\bar{\gamma}\alpha} B^{\alpha}_{\alpha'} B^{\beta'}_{\beta} + B^{\alpha}_{\alpha'} B_{\gamma}{}^{\beta'}{}_{\alpha}. \tag{3.4}$$

The other connection coefficients appearing in (2.7)(b) are transforming as tensors.

As $B^{b}_{b'}$, $B^{b'}_{b}$, $B^{\beta}_{\beta'}$, $B^{\beta'}_{\beta}$ are not functiones of p_a and p_α, we have

$$\begin{array}{llll} \delta^{\bar{a}} B^{b}_{b'} = 0 & \delta^{\bar{a}} B^{\beta}_{\beta'} = 0 & \delta^{\bar{\alpha}} B^{b}_{b'} = 0 & \delta^{\bar{\alpha}} B^{\beta}_{\beta'} = 0 \\ \delta^{\bar{a}} B^{b'}_{b} = 0 & \delta^{\bar{a}} B^{\beta'}_{\beta} = 0 & \delta^{\bar{\alpha}} B^{b'}_{b} = 0 & \delta^{\bar{\alpha}} B^{\beta'}_{\beta} = 0. \end{array} \tag{3.5}$$

From the above equations it follows that all connection coefficients appearing in (2.7)(c) and (2.7)(d) are transforming as tensors. □

4. Covariant differentials

Using the connections we want to obtain some tensors for which we know the transformation law.

If X and Y are vector fields in $T(H)$, then in the adapted basis B they have the decomposition:

$$X = X^a \delta_a + X^\alpha \delta_\alpha + X_{\bar{a}} \delta^{\bar{a}} + X_{\bar{\alpha}} \delta^{\bar{\alpha}}$$

$$Y = Y^b \delta_b + Y^\beta \delta_\beta + Y_{\bar{b}} \delta^{\bar{b}} + Y_{\bar{\beta}} \delta^{\bar{\beta}}.$$

Theorem 4.1. *For the generalized linear connection ∇ we have:*

$$\begin{aligned} \nabla_X Y \ = \ & (Y^b{}_{|a} X^a + Y^b{}_{|\alpha} X^\alpha + Y^b|^{\bar{a}} X_{\bar{a}} + Y^b|^{\bar{\alpha}} X_{\bar{\alpha}}) \delta_b + \\ & (Y^\beta{}_{|a} X^a + Y^\beta{}_{|\alpha} X^\alpha + Y^\beta|^{\bar{a}} X_{\bar{a}} + Y^\beta|^{\bar{\alpha}} X_{\bar{\alpha}}) \delta_\beta + \\ & (Y_{\bar{b}|a} X^a + Y_{\bar{b}|\alpha} X^\alpha + Y_{\bar{b}}|^{\bar{a}} X_{\bar{a}} + Y_{\bar{b}}|^{\bar{\alpha}} X_{\bar{\alpha}}) \delta^{\bar{b}} + \\ & (Y_{\bar{\beta}|a} X^a + Y_{\bar{\beta}|\alpha} X^\alpha + Y_{\bar{\beta}}|^{\bar{a}} X_{\bar{a}} + Y_{\bar{\beta}}|^{\bar{\alpha}} X_{\bar{\alpha}}) \delta^{\bar{\beta}}, \end{aligned} \tag{4.1}$$

where

$$Y^b{}_{|a} = \delta_a Y^b + \Gamma_c{}^b{}_a Y^c + \Gamma_\gamma{}^b{}_a Y^\gamma + \Gamma^{\bar{c}b}{}_a Y_{\bar{c}} + \Gamma^{\bar{\gamma}b}{}_a Y_{\bar{\gamma}}. \tag{4.2}$$

If in (4.2) we make the changes $a \to \alpha$ we obtain $Y^b{}_{|\alpha}$, $b \to \beta$ we obtain $Y^\beta{}_{|a}$, $a \to \alpha \wedge b \to \beta$ we obtain $Y^\beta{}_{|\alpha}$.

$$(4.3) \qquad Y^b|^{\bar{a}} = \delta^{\bar{a}} Y^b + \Gamma_c{}^{b\bar{a}} Y^c + \Gamma_\gamma{}^{b\bar{a}} Y^\gamma + \Gamma^{\bar{c}b\bar{a}} Y_{\bar{c}} + \Gamma^{\bar{\gamma}b\bar{a}} Y_{\bar{\gamma}}.$$

If in (4.3) we make the changes $\bar{a} \to \bar{\alpha}$, $b \to \beta$, $\bar{a} \to \bar{\alpha} \wedge b \to \beta$ we obtain $Y^b|^{\bar{\alpha}}$, $Y^\beta|^{\bar{a}}$, $Y^\beta|^{\bar{\alpha}}$ respectively. Further:

$$(4.4) \qquad Y_{\bar{b}|a} = \delta_a Y_{\bar{b}} + \Gamma_{c\bar{b}a} Y^c + \Gamma_{\gamma\bar{b}a} Y^\gamma + \Gamma^{\bar{c}}{}_{\bar{b}a} Y_{\bar{c}} + \Gamma^{\bar{\gamma}}{}_{\bar{b}a} Y_{\bar{\gamma}}.$$

If in (4.4) we make the changes $a \to \alpha$, $\bar{b} \to \bar{\beta}$, $a \to \alpha \wedge \bar{b} \to \bar{\beta}$ we obtain $Y_{\bar{b}|\alpha}$, $Y_{\bar{\beta}|a}$, $Y_{\bar{\beta}|\alpha}$ respectively.

We have:

$$(4.5) \qquad Y_{\bar{b}}|^{\bar{a}} = \delta^{\bar{a}} Y_{\bar{b}} + \Gamma_{c\bar{b}}{}^{\bar{a}} Y^c + \Gamma_{\gamma\bar{b}}{}^{\bar{a}} Y^\gamma + \Gamma^{\bar{c}}{}_{\bar{b}}{}^{\bar{a}} Y_{\bar{c}} + \Gamma^{\bar{\gamma}}{}_{\bar{b}}{}^{\bar{a}} Y_{\bar{\gamma}}.$$

If in (4.5) we make the changes $\bar{a} \to \bar{\alpha}$, $\bar{b} \to \bar{\beta}$, $\bar{a} \to \bar{\alpha} \wedge \bar{b} \to \bar{\beta}$ we obtain $Y_{\bar{b}}|^{\bar{\alpha}}$, $Y_{\bar{\beta}}|^{\bar{a}}$ and $Y_{\bar{\beta}}|^{\bar{\alpha}}$ respectively.

Theorem 4.2. *All covariant derivatives determined by* (4.2)-(4.5) *and those obtained by changing the indices, i.e. all 16 appearing in* (4.1) *transform as tensors.*

Proof. The proof follows from (3.1)-(3.5). For instance, we prove

$$(4.6) \qquad Y^{b'}{}_{|a'} = B^{b'}_b B^a_{a'} Y^b{}_{|a}.$$

From (4.2) it follows that we have to prove that $\delta_a Y^b + \Gamma_c{}^b{}_a Y^c$ is a tensor, because the next three terms in $Y^b{}_{|a}$ are tensors.

From (3.1) and $\delta_{a'} Y^{b'} = B^a_{a'} B^{b'}_b (\delta_a Y^b) + B_a{}^{b'}{}_b B^a_{a'} Y^b$ the statement follows. □

In the further consideration we shall restrict ourselves to the almost strongly d. connection defined by (2.13).

Theorem 4.3. *For $X \in T(H_{2n})$, $Y \in T(H)$, i.e.*

$$X = X^a \delta_a + X_{\bar{a}} \delta^{\bar{a}}, \quad Y = Y^b \delta_b + Y^\beta \delta_\beta + Y_{\bar{b}} \delta^{\bar{b}} + Y_{\bar{\beta}} \delta^{\bar{\beta}}$$

and for a.s.d. connection we have

$$(4.7) \qquad \begin{aligned} \nabla_X Y \;=\; & Y^b{}_{|a} X^a \delta_b + Y^b|^{\bar{a}} X_{\bar{a}} \delta_b + Y_{\bar{b}|a} X^a \delta^{\bar{b}} + Y_{\bar{b}}|^{\bar{a}} X_{\bar{a}} \delta^{\bar{b}} + \\ & Y^\beta{}_{|a} X^a \delta_\beta + Y^\beta|^{\bar{a}} X_{\bar{a}} \delta_\beta + Y_{\bar{\beta}|a} X^a \delta^{\bar{\beta}} + Y_{\bar{\beta}}|^{\bar{a}} X_{\bar{a}} \delta^{\bar{\beta}}, \end{aligned}$$

where

$$Y^b{}_{|a} = \delta_a Y^b + \Gamma_c{}^b{}_a Y^c + \Gamma^{\bar{c}b}{}_a Y_{\bar{c}} \tag{4.8}$$

$$Y^b|^{\bar{a}} = \delta^{\bar{a}} Y^b + \Gamma_c{}^{b\bar{a}} Y^c + \Gamma^{\bar{c}b\bar{a}} Y_{\bar{c}}$$

$$Y_{\bar{b}|a} = \delta_a Y_{\bar{b}} + \Gamma_{c\bar{b}a} Y^c + \Gamma^{\bar{c}}{}_{\bar{b}a} Y_{\bar{c}}$$

$$Y_{\bar{b}}|^{\bar{a}} = \delta^{\bar{a}} Y_{\bar{b}} + \Gamma_{c\bar{b}}{}^{\bar{a}} Y^c + \Gamma^{\bar{c}}{}_{\bar{b}}{}^{\bar{a}} Y_{\bar{c}}$$

$$Y^\beta{}_{|a} = \delta_a Y^\beta, \quad Y^\beta|^{\bar{a}} = \delta^{\bar{a}} Y^\beta, \quad Y_{\bar{\beta}|a} = \delta_a Y_{\bar{\beta}}, \quad Y_{\bar{\beta}}|^{\bar{a}} = \delta^{\bar{a}} Y_{\bar{\beta}}.$$

If

$$Y^\beta = Y^\beta(x^\alpha, p_\alpha) \quad \text{and} \quad Y_{\bar{\beta}} = Y_{\bar{\beta}}(x^\alpha, p_\alpha),$$

then

$$Y^\beta_{|a} = 0,\ Y^\beta|^{\bar{a}} = 0,\ Y_{\bar{\beta}|a} = 0,\ Y_{\bar{\beta}}|^{\bar{a}} = 0.$$

For $X \in T(H_{2m})$, i.e. $X = X^\alpha \delta_\alpha + X_{\bar{\alpha}} \delta^{\bar{\alpha}}$ and $Y \in T(H)$ (4.7) and (4.8) change in such a way that $(a, \bar{a}, b, \bar{b}, c, \bar{c}, \beta, \bar{\beta}, a, \bar{a})$ from (4.7) and (4.8) became $(\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}, b, \bar{b}, \alpha, \bar{\alpha})$ respectively.

Theorem 4.4. *For the s.d. connection we have*

$$\nabla_X Y = Y^b{}_{|a} X^a \delta_b + Y^\beta{}_{|\alpha} X^\alpha \delta_\beta + Y_{\bar{b}}|^{\bar{a}} X_{\bar{a}} \delta^{\bar{b}} + Y_{\bar{\beta}}|^{\bar{\alpha}} X_{\bar{\alpha}} \delta^{\bar{\beta}}, \tag{4.9}$$

where the covariant derivative of Y are given by

$$Y^b{}_{|a} = \delta_a Y^b + \Gamma_c{}^b{}_a Y^c \tag{4.10}$$

$$Y^\beta{}_{|\alpha} = \delta_\alpha Y^\beta + \Gamma_\gamma{}^\beta{}_\alpha Y^\gamma \tag{4.11}$$

$$Y_{\bar{b}}|^{\bar{a}} = \delta^{\bar{a}} Y_{\bar{b}} + \Gamma^{\bar{c}}{}_{\bar{b}}{}^{\bar{a}} Y_{\bar{c}} \tag{4.12}$$

$$Y_{\bar{\beta}}|^{\bar{\alpha}} = \delta^{\bar{\alpha}} Y_{\bar{\beta}} + \Gamma^{\bar{\gamma}}{}_{\bar{\beta}}{}^{\bar{\alpha}} Y_{\bar{\gamma}}. \tag{4.13}$$

If ω is a 1-form field in $T^*(H)$, then

$$\omega = \omega_a \delta x^a + \omega_\alpha \delta x^\alpha + \omega^{\bar{a}} \delta p_{\bar{a}} + \omega^{\bar{\alpha}} \delta p_{\bar{\alpha}}. \tag{4.14}$$

For the s.d. connection we define

$$\nabla_{\delta_a} \delta x^b = \bar{\Gamma}^b{}_{ca} \delta x^c, \quad \nabla_{\delta_\alpha} \delta x^\beta = \bar{\Gamma}^\beta{}_{\gamma\alpha} \delta p^\gamma \tag{4.15}$$

$$\nabla_{\delta^{\bar{a}}} \delta p_{\bar{b}} = \bar{\Gamma}_{\bar{b}}{}^{\bar{c}\bar{a}} \delta p_{\bar{c}}, \quad \nabla_{\delta^{\bar{\beta}}} \delta p_{\bar{\alpha}} = \bar{\Gamma}_{\bar{\alpha}}{}^{\bar{\gamma}\bar{\beta}} \delta p_{\bar{\gamma}}.$$

Theorem 4.5. *For the s.d. connection the following relations are valid*

$$\bar{\Gamma}^{b}{}_{ca} = -\Gamma_{c}{}^{b}{}_{a},\ \bar{\Gamma}^{\beta}{}_{\gamma\alpha} = -\Gamma_{\gamma}{}^{\beta}{}_{\alpha},\ \bar{\Gamma}_{\bar{b}}{}^{\bar{c}\bar{a}} = -\Gamma^{\bar{c}}{}_{\bar{b}}{}^{\bar{a}},\ \bar{\Gamma}_{\bar{\alpha}}{}^{\bar{\gamma}\bar{\beta}} = -\Gamma^{\bar{\gamma}}{}_{\bar{\alpha}}{}^{\bar{\beta}}. \tag{4.16}$$

Proof. As B and B^* are dual adapted bases of $T(H)$ and $T^*(H)$ respectively, we have

$$\langle \delta x^b, \delta_a \rangle = \delta^b_a, \quad \langle \delta x^\beta, \delta_\alpha \rangle = \delta^\beta_\alpha,$$

$$\langle \delta_{\bar{b}}, \delta^{\bar{a}} \rangle = \delta^{\bar{a}}_{\bar{b}}, \quad \langle \delta p_{\bar{\beta}}, \delta^{\bar{\alpha}} \rangle = \delta^{\bar{\alpha}}_{\bar{\beta}}.$$

From the above equations it follows

$$\nabla_{\delta_c} \langle \delta x^b, \delta_a \rangle = 0 \Rightarrow$$

$$\langle \nabla_{\delta_c} \delta x^b, \delta_a \rangle + \langle \delta x^b, \nabla_{\delta_c} \delta_a \rangle = 0$$

$$\langle \bar{\Gamma}^{b}{}_{dc} \delta x^d, \delta_a \rangle + \langle \delta x^b, \Gamma_{a}{}^{d}{}_{c} \delta_d \rangle = 0$$

$$\bar{\Gamma}^{b}{}_{ac} + \Gamma_{a}{}^{b}{}_{c} = 0 \Rightarrow \bar{\Gamma}^{b}{}_{ac} = -\Gamma_{a}{}^{b}{}_{c}.$$

The other relations in (4.16) can be proved in the same way. □

5. The torsion and curvature tensors

We shall consider the almost strongly distinguished connection (given by Definition 2.6). As

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{5.1}$$

we first calculate the Lie bracket of basis vectors of $T(H_{2n})$. We have

$$[\delta_a \delta_b] = \Omega_{a\bar{c}b} \partial^{\bar{c}} = -(\partial_a N_{bc} - N_{ad} \partial^{\bar{d}} N_{bc}) \partial^{\bar{c}} + (a|b) \tag{5.2}$$

$$[\delta_a, \delta^{\bar{b}}] = \Omega_{a\bar{c}}{}^{\bar{b}} \partial^{\bar{c}} = \partial^{\bar{b}} N_{ac} \partial^{\bar{c}}$$

$$[\delta^{\bar{a}}, \delta_b] = \Omega^{\bar{a}}{}_{\bar{c}b} \partial^{\bar{c}} = -\partial^{\bar{a}} N_{bc} \partial^{\bar{c}}$$

$$[\delta^{\bar{a}}, \delta^{\bar{b}}] = 0,$$

see (1.8) and $(a|b)$ is the previous expression in which a and b change the places.

Theorem 5.1. *For $X, Y \in T(H_{2n})$ and almost strongly distinguished connection we have*

$$\begin{aligned} T(X, Y) \ = \ & (T_b{}^c{}_a \delta_c + T_{b\bar{c}a} \delta^{\bar{c}}) X^a Y^b + \\ & (T^{\bar{b}c}{}_a \delta_c + T^{\bar{b}}{}_{\bar{c}a} \delta^{\bar{c}}) X^a Y_{\bar{b}} + \\ & (T_b{}^{c\bar{a}} \delta_c + T_{b\bar{c}}{}^{\bar{a}} \delta^{\bar{c}}) X_{\bar{a}} Y^b + \\ & (T^{\bar{b}c\bar{a}} \delta_c + T^{\bar{b}}{}_{\bar{c}}{}^{\bar{a}} \delta^{\bar{c}}) X_{\bar{a}} Y_{\bar{b}}, \end{aligned} \tag{5.3}$$

where

$$(5.4)\qquad \begin{aligned} T_{b\ a}^{\ c} &= \Gamma_{a\ b}^{\ c} - \Gamma_{b\ a}^{\ c} & T_{b\bar{c}a} &= \Gamma_{b\bar{c}a} - \Gamma_{a\bar{c}b} - \Omega_{a\bar{c}b} \\ T^{\bar{b}c}_{\ \ a} &= \Gamma^{\bar{b}c}_{\ \ a} - \Gamma_{a}^{\ c\bar{b}} & T^{\bar{b}}_{\ \bar{c}a} &= \Gamma^{\bar{b}}_{\ \bar{c}a} - \Gamma_{a\bar{c}}^{\ \ \bar{b}} - \Omega_{a\bar{c}}^{\ \ \bar{b}} \\ T_{b}^{\ c\bar{a}} &= \Gamma_{b}^{\ c\bar{a}} - \Gamma^{\bar{a}c}_{\ \ b} & T_{b\bar{c}}^{\ \ \bar{a}} &= \Gamma_{b\bar{c}}^{\ \ \bar{a}} - \Gamma^{\bar{a}}_{\ \bar{c}b} - \Omega^{\bar{a}}_{\ \bar{c}b} \\ T^{\bar{b}c\bar{a}} &= \Gamma^{\bar{b}c\bar{a}} - \Gamma^{\bar{a}c\bar{b}} & T^{\bar{b}\ \bar{a}}_{\ \bar{c}} &= \Gamma^{\bar{b}\ \bar{a}}_{\ \bar{c}} - \Gamma^{\bar{a}\ \bar{b}}_{\ \bar{c}}. \end{aligned}$$

For $X, Y \in T(H_{2m})$ (5.1)-(5.4) are valid, only the changes $(a, b, c, d) \to (\alpha, \beta, \gamma, \delta)$ have to be made.

It is obvious that for the almost distinguished connection in $T(H_{2n+2m})$ we obtain the torsion tensor of $T(H_{2n})$ and $T(H_{2m})$ separately.

Theorem 5.2. *For the s.d. connection and $X, Y \in T(H_{2n})$ the torsion tensor has the form*

$$(5.5)\qquad \begin{aligned} T(X,Y) \;&=\; (T_{b\ a}^{\ c}\delta_c - \Omega_{a\bar{c}b}\delta^{\bar{c}})X^aY^b + \\ &\quad T^{\bar{b}\ \bar{a}}_{\ \bar{c}}\delta^{\bar{c}}X_{\bar{a}}Y_{\bar{b}} - \Omega_{a\bar{c}}^{\ \ \bar{b}}\delta^{\bar{c}}X^aY_{\bar{b}} - \Omega^{\bar{a}}_{\ \bar{c}b}\delta^{\bar{c}}X_{\bar{a}}Y^b, \end{aligned}$$

where the above components of T and Ω are given by (5.2) *and* (5.4)*.*

For the s.d. connection and $X, Y \in T(H_{2m})$ (5.2) and (5.5) are valid, only the changes $(a, b, c, d) \to (\alpha, \beta, \gamma, \delta)$ have to be made.

The curvature tensor is as usually defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Theorem 5.3. *For $X, Y, Z \in T(H_{2n})$ i.e.*

$$(5.6)\qquad X = X^a\delta_a + X_{\bar{a}}\delta^{\bar{a}},\ Y = Y^b\delta_b + Y_{\bar{b}}\delta^{\bar{b}},\ Z = Z^c\delta_c + Z_{\bar{c}}\delta^{\bar{c}}$$

and a.s.d. connection, the curvature tensor is given by

$$(5.7)\qquad \begin{aligned} &R(X,Y)Z = \\ &(R_{c\ ba}^{\ d}\delta_d + R_{c\bar{d}ba}\delta^{\bar{d}})X^aY^bZ^c + (R_{c\ b}^{\ d\ \bar{a}}\delta_d + R_{c\bar{d}b}^{\ \ \ \bar{a}}\delta^{\bar{d}})X_{\bar{a}}Y^bZ^c + \\ &(R_{c\ \ a}^{\ d\bar{b}}\delta_d + R_{c\bar{d}\ a}^{\ \ \bar{b}}\delta^{\bar{d}})X^aY_{\bar{b}}Z^c + (R^{\bar{c}d}_{\ \ ba}\delta_d + R^{\bar{c}}_{\ \bar{d}ba}\delta^{\bar{d}})X^aY^bZ_{\bar{c}} + \\ &(R_{c}^{\ d\bar{b}\bar{a}}\delta_d + R_{c\bar{d}}^{\ \ \bar{b}\bar{a}}\delta^{\bar{d}})X_{\bar{a}}Y_{\bar{b}}Z^c + (R^{\bar{c}d\ \bar{a}}_{\ \ b}\delta_d + R^{\bar{c}\ \ \bar{a}}_{\ \bar{d}b}\delta^{\bar{d}})X_{\bar{a}}Y^bZ_{\bar{c}} + \\ &(R^{\bar{c}d\bar{b}}_{\ \ \ a}\delta_d + R^{\bar{c}\ \bar{b}}_{\ \bar{d}\ a}\delta^{\bar{d}})X^aY_{\bar{b}}Z_{\bar{c}} + (R^{\bar{c}d\bar{b}\bar{a}}\delta_d + R^{\bar{c}\ \bar{b}\bar{a}}_{\ \bar{d}}\delta^{\bar{d}})X_{\bar{a}}Y_{\bar{b}}Z_{\bar{c}}. \end{aligned}$$

The components of the curvature tensors have the form:

$$R_{c\ ba}^{\ d} = K_{c\ ba}^{\ d} + \Omega_{a\bar{c}b}\Gamma_{c}^{\ d\bar{c}},$$

$$K_{c\ ba}^{\ d} = (\delta_a \Gamma_{c\ b}^{\ d} + \Gamma_{c\ b}^{\ f}\Gamma_{f\ a}^{\ d} + \Gamma_{c\bar{f}b}\Gamma^{\bar{f}d}_{\ \ a}) - (a|b),$$

$$R_{c\bar{d}ba} = K_{c\bar{d}ba} + \Omega_{a\bar{c}b}\Gamma_{c\bar{d}}^{\ \bar{c}},$$

$$K_{c\bar{d}ba} = (\delta_a \Gamma_{c\bar{d}b} + \Gamma_{c\ b}^{\ f}\Gamma_{f\bar{d}a} + \Gamma_{c\bar{f}b}\Gamma^{\bar{f}}_{\ \bar{d}a}) - (a|b),$$

$$R_{c\ b}^{\ d\ \bar{a}} = K_{c\ b}^{\ d\ \bar{a}} + \Omega^{\bar{a}}_{\ \bar{c}b}\Gamma_{c}^{\ d\bar{c}},$$

$$K_{c\ b}^{\ d\ \bar{a}} = \delta^{\bar{a}}\Gamma_{c\ b}^{\ d} + \Gamma_{c\ b}^{\ f}\Gamma_{f}^{\ d\bar{a}} + \Gamma_{c\bar{f}b}\Gamma^{\bar{f}d\bar{a}} - (\bar{a}|b),$$

$$R_{c\bar{d}b}^{\ \ \ \bar{a}} = K_{c\bar{d}b}^{\ \ \ \bar{a}} + \Omega^{\bar{a}}_{\ \bar{c}b}\Gamma_{c\bar{d}}^{\ \bar{c}},$$

$$K_{c\bar{d}b}^{\ \ \ \bar{a}} = \delta^{\bar{a}}\Gamma_{c\bar{d}b} + \Gamma_{c\ b}^{\ f}\Gamma_{f\bar{d}}^{\ \ \bar{a}} + \Gamma_{c\bar{f}b}\Gamma^{\bar{f}\ \bar{a}}_{\ \bar{d}} - (\bar{a}|b), \dots,$$

or shorter

$$R_{xyuv} = K_{xyuv} + \Omega_{v\bar{c}u}\Gamma_{xy}^{\ \ \bar{c}} \tag{5.8}$$

$$x \in \{{}_c, {}^{\bar{c}}\},\ y \in \{{}^d, {}_{\bar{d}}\},\ u \in \{{}_b, {}^{\bar{b}}\},\ v \in \{{}_a, {}^{\bar{a}}\}$$

$$K_{xyuv} = (\delta_v \Gamma_{xyu} + \Gamma_{x\ u}^{\ f}\Gamma_{fyv} + \Gamma_{x\bar{f}u}\Gamma^{\bar{f}}_{\ yv}) - (u|v)$$

and only $\Omega_{a\bar{c}b}$, $\Omega_{a\bar{c}}^{\ \ \bar{b}}$, $\Omega^{\bar{a}}_{\ \bar{c}b}$ given by (5.2) are different from zero.

Remark 5.1. *For $X, Y, Z \in T(H_{2m})$ and the a.s.d. connection Theorem 5.3 is valid when the Latin indices change to the corresponding Greek indices.*

Theorem 5.4. *For $X, Y, Z \in T(H_{2n})$ (see (5.6)) and s.d. connection the curvature tensor is given by*

$$R(X,Y)Z = R_{c\ ba}^{\ d}X^aY^bZ^c\delta_d + R^{\bar{c}\ \bar{b}\bar{a}}_{\ \bar{d}}X_{\bar{a}}Y_{\bar{b}}Z_{\bar{c}}\delta^{\bar{d}},$$

where

$$R_{c\ ba}^{\ d} = (\delta_a \Gamma_{c\ b}^{\ d} + \Gamma_{c\ b}^{\ f}\Gamma_{f\ a}^{\ d}) - (a|b)$$

$$R^{\bar{c}\ \bar{b}\bar{a}}_{\ \bar{d}} = (\delta^{\bar{a}}\Gamma^{\bar{c}\ \bar{b}}_{\ \bar{d}} + \Gamma^{\bar{c}\ \bar{b}}_{\ \bar{f}}\Gamma^{\bar{f}\ \bar{a}}_{\ \bar{d}}) - (\bar{a}|\bar{b}).$$

Remark 5.2. For $X, Y, Z \in T(H_{2m})$ and s.d. connection Theorem 5.4 is valid when the Latin indices change to the corresponding Greek indices.

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