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SOLUTIONS OF PERTURBED NONLINEAR NABLA FRACTIONAL DIFFERENCE EQUATIONS

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Abstract. In the present work, we discuss the differentiability properties of solutions of nabla fractional difference equations of order α ($0 < \alpha < 1$) with respect to the initial conditions. Further, we develop a nonlinear variation of parameters formula to obtain the solution of a perturbed nonlinear nabla fractional difference equation.

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1. Introduction

Fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering, such as, viscoelasticity, diffusion, neurology, control theory, and statistics [6]. The analogous theory for discrete fractional calculus was initiated and properties of the theory of fractional sums and differences were established. Recently, a series of papers continuing this research has appeared.

The study of the theory of fractional differential equations was initiated and existence and uniqueness of solutions for different types of fractional differential equations have been established recently [6]. Very little progress has been made to develop the theory of analogous fractional difference equations.

The variation of parameters formula is an important tool in the study of qualitative properties of perturbed problems. The main advantage of this formula is that we obtain the solution of the perturbed problem in terms of the solution of the unperturbed problem.

In 1967, V.M. Alekseev [1] established the relation between the solutions of unperturbed problem

$$u'(t) = f(t, u(t)), \quad u(t_0) = u_0 \tag{1}$$

and the perturbed problem

$$v'(t) = f(t, v(t)) + g(t, v(t)), \quad v(t_0) = u_0. \tag{2}$$

Later in 1989 and in 1990, Lakshmikantham and others [2] proposed a different version of the variation of parameters formula. The nonlinear variation of parameters formula for nonlinear difference is given by Lakshmikantham and Trigiante [3]. The present article is organized as follows.

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In Section 2, we discuss the continuous dependence of solutions of fractional difference equations on the initial conditions and parameters. In Section 3, we establish the variation of parameters formula for fractional difference equations.

Throughout this article, we use the following notations: $\mathbb{N}$ is the set of natural numbers including zero and $\mathbb{Z}$ is the set of integers. $\mathbb{N}_a^+ = \{a, a+1, a+2, ...\}$ for $a \in \mathbb{Z}$. Let $u(n)$ be a real-valued function defined on $\mathbb{N}_0^+$. Then for all $n_1, n_2 \in \mathbb{N}_0^+$ and $n_1 > n_2$, $\sum_{j=n_1}^{n_2} u(j) = 0$ and $\prod_{j=n_1}^{n_2} u(j) = 0$, i.e. empty sums and products are taken to be 0 and 1 respectively. If n and $n+1$ are in $\mathbb{N}_0^+$, the backward difference operator ∇ is defined as $\nabla u(n+1) = u(n+1) - u(n)$.

Now, we introduce some basic definitions and results concerning nabla discrete fractional calculus. The extended binomial coefficient $\binom{a}{n}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ is defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & \text{n} > 0 \\ 1 & \text{n} = 0 \\ 0 & \text{n} < 0. \end{cases} \tag{3}$$

Definition 1.1. For any complex numbers α and β,

$$(\alpha)_\beta = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} & \text{when } \alpha,\ \alpha+\beta \text{ are neither zero nor negative integers} \\ 1 & \text{when } \alpha = \beta = 0 \\ 0 & \text{when } \alpha = 0,\ \beta \text{ is neither zero nor negative integer} \\ undefined & \text{otherwise.} \end{cases}$$

Remark 1.2. For any complex numbers α and β, when α, β and $\alpha+\beta$ are neither zero nor negative integers,

$$(\alpha+\beta)_n = \sum_{k=0}^{n} \binom{n}{k} (\alpha)_{n-k} (\beta)_k \tag{4}$$

for any positive integer n.

B.G. Pachpatte [5] established the following remarkable inequalites in discrete calculus. Let $u(n)$, $v(n)$, $a(n)$, $b(n)$, $c(n)$ and $p(n)$ be real-valued nonnegative functions defined on $\mathbb{N}_0^+$.

Theorem 1.3. *For $n \in \mathbb{N}_0^+$, if*

$$u(n) \le a(n) + p(n) \sum_{j=0}^{n-1} [b(j)u(j) + c(j)] \tag{5}$$

then

$$u(n) \le a(n) + p(n) \sum_{j=0}^{n-1} [a(j)b(j) + c(j)] \prod_{k=j+1}^{n-1} \Big[1 + b(k)p(k)\Big].$$

Theorem 1.4. *Let $g(n, j, u)$ be defined on $\mathbb{N}_0^+ \times \mathbb{R} \times \mathbb{R}$ and nondecreasing with respect to u. Suppose that for $n \in \mathbb{N}_0^+$,*

$$u(n) \le p(n) + \sum_{j=0}^{n-1} g(n, j, u(j)).$$

Then, $u(0) \leq p(0)$ implies $u(n) \leq v(n)$, where $v(n)$ is the solution of the difference equation

$$v(n) = p(n) + \sum_{j=0}^{n-1} g(n, j, v(j)), \quad v(0) = p(0).$$

Let $u(n) : \mathbb{N}_0^+ \to \mathbb{R}$ and $m - 1 < \alpha \leq m$ where $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}_1^+$.

Definition 1.5. The fractional sum operator of order α is defined as

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u(j). \tag{6}$$

Definition 1.6. The Riemann-Liouville type fractional difference operator of order α is defined as

$$\nabla^{\alpha} u(n) = \nabla^m [\nabla^{-(m-\alpha)} u(n)] = \nabla^m \Big[\sum_{j=1}^{n} \binom{n-j+m-\alpha-1}{n-j} u(j) \Big]. \tag{7}$$

Now, we simplify the above definition for our convenience as follows.

Corollary 1.7. *The equivalent form of (7) is*

$$\nabla^{\alpha} u(n) = \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j), \quad \alpha \notin \mathbb{N}_1^+. \tag{8}$$

Proof. Consider

$$\begin{aligned}
\nabla^{\alpha} u(n) &= \nabla^m \Big[\sum_{j=1}^{n} \binom{n-j+m-\alpha-1}{n-j} u(j) \Big] \\
&= \nabla^{m-1} \nabla \Big[\sum_{j=1}^{n} \binom{n-j+m-\alpha-1}{n-j} u(j) \Big] \\
&= \nabla^{m-1} \Big[\sum_{j=1}^{n} \binom{n-j+m-\alpha-1}{n-j} u(j) \\
&\quad - \sum_{j=1}^{n-1} \binom{n-j+m-\alpha-2}{n-j-1} u(j) \Big] \\
&= \nabla^{m-1} \Big[\sum_{j=1}^{n} \binom{n-j+m-\alpha-2}{n-j} u(j) \Big].
\end{aligned}$$

By applying the similar procedure $m-1$ times, we get (8). □

The unified definition for fractional sums and differences is as follows.

Definition 1.8. Let $u(n) : \mathbb{N}_0^+ \to \mathbb{R}$ and $m - 1 < \alpha \leq m$ where $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}_1^+$. Then

1. the α^{th}-order fractional sum of $u(n)$ is given by

$$(9)\qquad \nabla^{-\alpha}u(n)=\sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}u(j).$$

2. the α^{th}-order fractional difference of $u(n)$ is given by

$$(10)\qquad \nabla^{\alpha}u(n)=\begin{cases}\sum_{j=1}^{n}\binom{n-j-\alpha-1}{n-j}u(j), & \alpha\notin\mathbb{N}_1^+,\\ \nabla^m u(n), & \alpha=m.\end{cases}$$

Theorem 1.9. *Let $u(n)$ and $v(n):\mathbb{N}_0^+\to\mathbb{R}$; $\alpha,\beta>0$ and c, d are scalars. Then*

1. $\nabla^{-\alpha}\nabla^{-\beta}u(n)=\nabla^{-(\alpha+\beta)}u(n)=\nabla^{-\beta}\nabla^{-\alpha}u(n).$
2. $\nabla^{\alpha}[cu(n)+dv(n)]=c\nabla^{\alpha}u(n)+d\nabla^{\alpha}v(n).$
3. $\nabla\nabla^{-\alpha}u(n)=\nabla^{-(\alpha-1)}u(n).$
4. $\nabla^{-\alpha}\nabla u(n)=\nabla^{(1-\alpha)}u(n)-\binom{n+\alpha-2}{n-1}u(0).$

Proof. (1) Consider

$$\begin{aligned}
\nabla^{-\alpha}\nabla^{-\beta}u(n) &= \nabla^{-\alpha}\Big[\nabla^{-\beta}u(n)\Big]\\
&= \sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}\nabla^{-\beta}u(j)\\
&= \sum_{j=1}^{n}\sum_{k=1}^{j}\binom{n-j+\alpha-1}{n-j}\binom{j-k+\beta-1}{j-k}u(k)\\
&= \sum_{k=1}^{n}\sum_{j=0}^{n-k}\frac{\Gamma(n-j-k+\alpha)}{\Gamma(n-j-k+1)\Gamma(\alpha)}\frac{\Gamma(j+\beta)}{\Gamma(j+1)\Gamma(\beta)}u(k)\\
&= \sum_{k=1}^{n}\frac{u(k)}{\Gamma(n-k+1)}\sum_{j=0}^{n-k}\binom{n-k}{j}(\alpha)_{n-k-j}(\beta)_j\\
&= \sum_{k=1}^{n}\frac{u(k)}{\Gamma(n-k+1)}(\alpha+\beta)_{n-k}\ \text{(using (4))}\\
&= \sum_{k=1}^{n}\frac{\Gamma(n-k+\alpha+\beta)}{\Gamma(n-k+1)\Gamma(\alpha+\beta)}u(k)\\
&= \sum_{k=1}^{n}\binom{n-k+\alpha+\beta-1}{n-k}u(k)=\nabla^{-(\alpha+\beta)}u(n).
\end{aligned}$$

(3) Consider

$$\begin{aligned}
\nabla\nabla^{-\alpha}u(n) &= \nabla\Big[\sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}u(j)\Big] \\
&= \Big[\sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}u(j)-\sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}u(j)\Big] \\
&= \Big[\sum_{j=1}^{n}\binom{n-j+\alpha-2}{n-j}u(j)\Big] \\
&= \nabla^{(1-\alpha)}u(n).
\end{aligned}$$

(4) Consider

$$\begin{aligned}
\nabla^{-\alpha}\nabla u(n) &= \sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}\nabla u(j) \\
&= \sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}u(j)-\sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}u(j-1) \\
&= \sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}u(j)-\sum_{j=0}^{n-1}\binom{n-j+\alpha-2}{n-j-1}u(j) \\
&= \sum_{j=1}^{n}\binom{n-j+\alpha-2}{n-j}u(j)-\binom{n+\alpha-2}{n-1}u(0) \\
&= \nabla^{(1-\alpha)}u(n)-\binom{n+\alpha-2}{n-1}u(0).
\end{aligned}$$

$\square$

Theorem 1.10. *(Leibniz Rule) Let $u(n), v(n) : \mathbb{N}_0^+ \to \mathbb{R}$; $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Then*

$$\nabla^{\alpha}u(n)v(n)=\sum_{k=0}^{n-1}\binom{\alpha}{k}\Big[\nabla^{\alpha-k}u(n-k)\Big]\nabla^{k}v(n). \tag{11}$$

Proof. Consider

$$\nabla^{\alpha}u(n)v(n)=\sum_{j=1}^{n}\binom{n-j-\alpha-1}{n-j}u(j)v(j). \tag{12}$$

By induction it can be shown that

$$\sum_{k=0}^{j}\binom{j}{k}(-1)^{k}\nabla^{k}v(n)=v(n-j). \tag{13}$$

Thus

$$\nabla^{\alpha} u(n)v(n) = \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k \nabla^k v(n).$$

Since

$$\frac{\Gamma(\alpha+1)}{\Gamma(k-\alpha)} \frac{\Gamma(-\alpha)}{\Gamma(\alpha-k+1)} = (-1)^k, \tag{14}$$

for any nonnegative integer k,

$$\begin{aligned}
&\nabla^{\alpha} u(n)v(n) \\
&= \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k \nabla^k v(n) \\
&= \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) \sum_{k=0}^{n-j} \binom{n-j}{k} \frac{\Gamma(\alpha+1)}{\Gamma(k-\alpha)} \frac{\Gamma(-\alpha)}{\Gamma(\alpha-k+1)} \nabla^k v(n) \\
&= \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \binom{n-j}{k} \binom{n-j-\alpha-1}{n-j} u(j) \frac{\Gamma(\alpha+1)}{\Gamma(k-\alpha)} \frac{\Gamma(-\alpha)}{\Gamma(\alpha-k+1)} \nabla^k v(n) \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \sum_{j=1}^{n-k} \binom{n-j-\alpha-1}{n-k-j} u(j) \nabla^k v(n) \\
&= \sum_{k=0}^{n-1} \binom{\alpha}{k} \left[\nabla^{\alpha-k} u(n-k)\right] \nabla^k v(n).
\end{aligned}$$

□

Definition 1.11. Let $f(n,r) : \mathbb{N}_0^+ \times \mathbb{R} \to \mathbb{R}$. Then a nonlinear difference equation of order α, $0 < \alpha < 1$, together with an initial condition, is of the form

$$\nabla^{\alpha} u(n+1) = f(n, u(n)), \quad n \in \mathbb{N}_1^+, \quad u(1) = u_0. \tag{15}$$

The problem of existence and uniqueness of solutions of difference equations becomes easy as the solutions are expressed as recurrence relations involving the values of the unknown function of the previous arguments. Applying $\nabla^{-\alpha}$ on both sides of (15), we have

$$\nabla^{-\alpha} \nabla^{\alpha} u(n+1) = \nabla^{-\alpha} f(n, u(n)). \tag{16}$$

Using 3 and 4 of Theorem 1.9, we get

$$\begin{aligned}
&\nabla^{-\alpha} \nabla^{\alpha} u(n+1) \\
&= \nabla^{-\alpha} \nabla \left[\nabla^{-(1-\alpha)} u(n+1)\right] \\
&= \nabla^{(1-\alpha)} \nabla^{-(1-\alpha)} u(n+1) - \binom{n+\alpha-1}{n} \left[\nabla^{-(1-\alpha)} u(n+1)\right]_{n=0} \\
&= u(n+1) - \binom{n+\alpha-1}{n} u(1).
\end{aligned}$$

Thus, in view of (16),

$$u(n+1) = \binom{n+\alpha-1}{n} u_0 + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} f(j, u(j))$$

$$\text{(17)} \qquad or \quad u(n) = \binom{n+\alpha-2}{n-1} u_0 + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} f(j, u(j))$$

The recursive iteration to this sum equation shows the existence of the unique solution to the initial value problem (15).

2. Dependence on initial conditions and parameters

The initial value problem (15) describes a model of a physical problem in which often some parameters such as lengths, masses, temperature, etc. are involved. The values of these parameters can be measured only up to a certain degree of accuracy. Thus, in (15) the initial value u_0, as well as the function $f(n, u(n))$, may be subject to some errors either by necessity or for convenience. Hence, it is important to know how the solution changes when u_0 and $f(n, u(n))$ are slightly altered. We shall discuss this question quantitatively in the following:

Theorem 2.1. *Let the following conditions be satisfied.*

1. *$f(n, u(n))$ is defined on $\mathbb{N}_0^+ \times \mathbb{R}$ and for all $(n, u(n)), (n, v(n)) \in \mathbb{N}_0^+ \times \mathbb{R}$,*

$$\text{(18)} \qquad |f(n, u(n)) - f(n, v(n))| \le \lambda(n)|u(n) - v(n)|$$

   where $\lambda(n)$ is a nonnegative function defined on $\mathbb{N}_0^+$.

2. *$g(n, u(n))$ is defined on $\mathbb{N}_0^+ \times \mathbb{R}$ and for all $(n, u(n)) \in \mathbb{N}_0^+ \times \mathbb{R}$,*

$$\text{(19)} \qquad |g(n, u(n))| \le \mu(n)$$

   where $\mu(n)$ is a nonnegative function defined on $\mathbb{N}_0^+$.

Then, for the solutions $u(n)$ and $v(n)$ of the initial value problems (15) and

$$\text{(20)} \qquad \nabla^\alpha v(n+1) = f(n, v(n)) + g(n, v(n)), \quad n \in \mathbb{N}_1^+, \quad v(1) = v_0$$

the following inequality holds

$$\text{(21)} \quad |u(n) - v(n)| \le \binom{n+\alpha-2}{n-1}|u_0 - v_0| + |u_0 - v_0| \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} \Big[\binom{j+\alpha-1}{j}\lambda(j) + \mu(j)\Big] \prod_{k=j+1}^{n-1} \Big[1 + \binom{n-k+\alpha-2}{n-k-1}\lambda(k)\Big].$$

Proof. Using (17), the initial value problems (16) and (20) are equivalent to

$$u(n) = \binom{n+\alpha-2}{n-1}u_0 + \sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}f(j,u(j)),$$
$$v(n) = \binom{n+\alpha-2}{n-1}v_0 + \sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}[f(j,v(j)) + g(j,v(j)].$$

Then

$$\begin{aligned} u(n) - v(n) = \binom{n+\alpha-2}{n-1}[u_0 - v_0] - \sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}g(j,v(j)) \\ + \sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}[f(j,u(j)) - f(j,v(j))]. \end{aligned}$$

Thus, from (18) and (19) it, follows that

$$\begin{aligned} |u(n) - v(n)| \le \binom{n+\alpha-2}{n-1}|u_0 - v_0| \\ + \sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}[\lambda(j)|u(j) - v(j)| + \mu(j)] \end{aligned}$$

Now, the application of Theorem 1.3 yields (21). □

Hereafter, to emphasize the dependence of the initial point $(1, u_0)$ we shall denote the solutions of the initial value problem (15) as $u(n, 1, u_0)$. In our next result we shall show that $u(n, 1, u_0)$ is differentiable with respect to u_0.

Theorem 2.2. *Let for all $(n, u(n)) \in \mathbb{N}_0^+ \times \mathbb{R}$, the function $f(n, u(n))$ be defined and the partial derivative $\frac{\partial f}{\partial u}$ exist. Further, let the solution $u(n) = u(n, 1, u_0)$ of the initial value problem (15) exists on $\mathbb{N}_1^+$ and*

$$H(n,1,u_0) = \frac{\partial f(n, u(n,1,u_0))}{\partial u}, \tag{22}$$

then,

$$\Phi(n,1,u_0) = \frac{\partial u(n,1,u_0)}{\partial u_0} \tag{23}$$

exists, and is the solution of the initial value problem

$$\nabla^\alpha \Phi(n+1,1,u_0) = H(n,1,u_0)\Phi(n,1,u_0), \quad n \in \mathbb{N}_1^+, \quad \Phi(1,1,u_0) = I. \tag{24}$$

Proof. Since $u(n, 1, u_0)$ is the solution of (15), we have

$$\nabla^\alpha u(n+1,1,u_0) = f(n, u(n,1,u_0)), \quad u(1,1,u_0) = u_0. \tag{25}$$

Consider

$$\begin{aligned}
&\frac{\partial}{\partial u_0}\Big[\nabla^\alpha \Phi(n+1,1,u_0)\Big] \\
&= \lim_{h\to 0}\frac{1}{h}\Big[\nabla^\alpha \Phi(n+1,1,u_0+h) - \nabla^\alpha \Phi(n+1,1,u_0)\Big] \\
&= \nabla^\alpha\Big[\lim_{h\to 0}\frac{\Phi(n+1,1,u_0+h)-\Phi(n+1,1,u_0)}{h}\Big] \\
&= \nabla^\alpha\Big[\frac{\partial}{\partial u_0}u(n+1,1,u_0)\Big] \\
&= \nabla^\alpha \Phi(n+1,1,u_0) \qquad (26)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial u_0}\Big[f(n,u(n,1,u_0))\Big] &= \frac{\partial}{\partial u}\Big[f(n,u(n,1,u_0))\Big]\frac{\partial}{\partial u_0}\Big[u(n,1,u_0)\Big] \\
&= H(n,1,u_0)\Phi(n,1,u_0). \qquad (27)
\end{aligned}$$

Thus, in view of (25), we get (24). □

Theorem 2.3. *Assume*

$$|f(n,u(n)) - f(n,v(n))| \le g(n,|u(n)-v(n)|) \tag{28}$$

for all $(n,u(n)),(n,v(n)) \in \mathbb{N}_0^+ \times \mathbb{R}$*, where* $g(n,r)$ *is defined on* $\mathbb{N}_0^+ \times \mathbb{R}$ *and nondecreasing in* r *for any fixed* $n \in \mathbb{N}_0^+$*. Further, let* $u(n,1,u_1)$ *and* $u(n,1,u_2)$ *be solutions of (15) exist on* $\mathbb{N}_1^+$*. Then, for all* $n \in \mathbb{N}_0^+$*,*

$$|u(n,1,u_1) - u(n,1,u_2)| \le r(n,1,r_0) \tag{29}$$

where $r(n) = r(n,1,r_0)$ *is the solution of the initial value problem*

$$\nabla^\alpha r(n+1) = g(n,r(n)), \quad n \in \mathbb{N}_1^+, \quad r(1) = r_0(=|u_1-u_2|). \tag{30}$$

Proof. Since $u(n,1,u_1)$ and $u(n,1,u_2)$ are solutions of (15), we have

$$\begin{aligned}
u(n,1,u_1) &= \binom{n+\alpha-2}{n-2}u_1 + \sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}f(j,u(j,1,u_1)), \\
u(n,1,u_2) &= \binom{n+\alpha-2}{n-1}u_2 + \sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}f(j,u(j,1,u_2)).
\end{aligned}$$

Then

$$\begin{aligned}
|u(n,1,u_1) - u(n,1,u_2)| \le &\binom{n+\alpha-2}{n-1}|u_1-u_2| \\
&+ \sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}|f(j,u(j,1,u_1)) - f(j,u(j,1,u_2))|.
\end{aligned}$$

Let $z(n) = |u(n,1,u_1) - u(n,1,u_2)|$. Then,

$$z(n) \le \binom{n+\alpha-2}{n-1} z_0 + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} g(j, z(j)). \tag{31}$$

Further, since $z_0 \le r_0$ and

$$r(n) = \binom{n+\alpha-2}{n-1} r_0 + \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1} g(j, r(j)), \tag{32}$$

the inequality (29) follows by using Theorem 1.4. □

Remark 2.4. If $r(n,1,0) = 0$ for all $n \in \mathbb{N}_1^+$ and $r(n,1,r_0) \to 0$ as $r_0 \to 0$, then from (29) it is clear that the solution $r(n,1,u_0)$ continuously depends on u_0.

Now we shall consider the following initial value problem.

$$\nabla^\alpha u(n+1) = f(n, u(n), p(n)), \quad n \in \mathbb{N}_1^+, \quad u(1) = u_0 \tag{33}$$

where $p(n) \in R$ is a parameter such that $|p(n) - p_0| \le \delta(> 0)$ and p_0 is a fixed scalar in $\mathbb{R}$. For a given $p(n)$ such that $|p(n) - p_0| \le \delta$ we shall assume that the solution $u(n, p(n)) = u(n, 1, u_0, p(n))$ of (33) exists on $\mathbb{N}_1^+$.

Theorem 2.5. *Let for all $n \in \mathbb{N}_0^+$, $u(n), p(n) \in \mathbb{R}$ such that $|p(n) - p_0| \le \delta$ the function $f(n, u(n), p(n))$ is defined, and the following inequalities hold*

$$|f(n, u(n), p(n)) - f(n, v(n), p(n))| \le \lambda(n)|u(n) - v(n)| \tag{34}$$

and

$$|f(n, u(n), p_1) - f(n, u(n), p_2)| \le \mu(n)|p_1 - p_2|, \tag{35}$$

where $\lambda(n)$ and $\mu(n)$ are nonnegative functions defined on $\mathbb{N}_0^+$. Then, for the solutions $u(n,1,u_1,p_1)$ and $u(n,1,u_2,p_2)$ of (33) the following inequality holds

$$|u(n) - v(n)| \le \binom{n+\alpha-2}{n-1} |u_0 - v_0| + |u_0 - v_0||p_1 - p_2| \sum_{j=1}^{n-1} \binom{n-j+\alpha-2}{n-j-1}$$
$$\Big[\binom{j+\alpha-1}{j} \lambda(j) + \mu(j)\Big] \prod_{k=j+1}^{n-1} \Big[1 + \binom{n-k+\alpha-2}{n-k-1} \lambda(k)\Big].$$

Proof. The proof is similar to the proof of Theorem 2.1. □

Theorem 2.6. *Let for all $n \in \mathbb{N}_0^+$, $u(n)$, $p(n) \in \mathbb{R}$ such that $|p(n) - p_0| \le \delta$ the function $f(n, u(n), p(n))$ is defined, and the partial derivatives $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial p}$ exist. Further, if $u(n, p(n)) = u(n, 1, u_0, p(n))$ is the solution of (33) on $\mathbb{N}_1^+$ then,*

$$\Phi(n, 1, u_0, p(n)) = \frac{\partial u(n, 1, u_0, p(n))}{\partial p} \tag{36}$$

exists, and is the solution of the initial value problem

$$\nabla^\alpha \Phi(n,1,u_0,p(n)) = G(n,1,u_0,p(n))\Phi(n,1,u_0,p(n)) + H(n,1,u_0,p(n)),$$
$$\Phi(1,1,u_0,p(n)) = 0 \tag{37}$$

where

$$G(n,1,u_0,p(n)) = \frac{\partial f(n,u(n,p),p)}{\partial u} \tag{38}$$

and

$$H(n,1,u_0,p(n)) = \frac{\partial f(n,u(n,p),p)}{\partial p}. \tag{39}$$

Proof. The proof is similar to the proof of Theorem 2.2. □

3. Method of nonlinear variation of parameters

The main purpose of this section is to develop the variation of parameters formula to represent the solution $v(n,1,u_0)$ of the perturbed problem (20) in terms of the solution $u(n,1,u_0)$ of the unperturbed problem (15).

Theorem 3.1. *Let for all $n \in \mathbb{N}_0^+$ and $u(n) \in \mathbb{R}$, the functions $f(n,u(n))$ and $g(n,u(n))$ be defined, and $\frac{\partial f}{\partial u}$ exists. If for each $u_0 \in \mathbb{R}$, the solution $u(n,1,u_0)$ of (15) exists on $\mathbb{N}_1^+$ and $\Phi(n,1,u_0) = \frac{\partial u(n,1,u_0)}{\partial u_0}$ is as defined in Theorem 2.2, then any solution $v(n) = v(n,1,u_0)$ of (20) satisfies the equation*

$$v(n,1,u_0) = u\Big(n,1,u_0 + \sum_{i=1}^{n-1}\Big[\Psi^{-1}(i+1,1,w(i),w(i+1)) \Big(\sum_{j=1}^{i}\binom{i-j+\alpha-1}{i-j} g(j,v(j))\Big)\Big]\Big) \tag{40}$$

where

$$\Psi(i,1,w(k),w(k+1)) = \int_0^1 \Phi(i,1,sw(k+1)+(1-s)w(k))ds, \tag{41}$$

$w(n)$ satisfies the implicit equation

$$w(n) = u_0 + \sum_{i=1}^{n-1}\Big[\Psi^{-1}(i+1,1,w(i),w(i+1))\Big(\sum_{j=1}^{i}\binom{i-j+\alpha-1}{i-j} g(j,v(j))\Big)\Big]. \tag{42}$$

Proof. The solution of (20) is given by

$$v(n+1,1,u_0) = \binom{n+\alpha-1}{n} v(1,1,u_0) + \sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}[f(j,v(j,1,u_0)) + g(j,v(j,1,u_0))]. \tag{43}$$

The method of variation of parameters requires determination of a function $w(n)$ so that $v(n,1,u_0) = u(n,1,w(n)),\quad w(1) = u_0$. Then, from (43), we have

$$(44)\quad u(n+1,1,w(n+1)) = \binom{n+\alpha-1}{n} u(1,1,u_0) + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} [f(j,u(j,1,w(j))) + g(j,u(j,1,w(j)))].$$

Since $u(n,1,u_0)$ is the solution of (15), we have

$$(45)\quad u(n+1,1,w(n)) = \binom{n+\alpha-1}{n} u(1,1,u_0) + \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} f(j,u(j,1,w(j))).$$

Using (44) and (45), we get

$$\sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} g(j,u(j,1,w(j))) = u(n+1,1,w(n+1)) - u(n+1,1,w(n)).$$

Using mean value theorem, we get

$$\int_0^1 \frac{\partial}{\partial u_0} u(n+1,1,sw(n+1)+(1-s)w(n))ds = \frac{u(n+1,1,w(n+1)) - u(n+1,1,w(n))}{w(n+1)-w(n)}$$

or $\Psi(n+1,1,w(n),w(n+1))\nabla w(n+1)$

$$= \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} g(j,u(j,1,w(j))).$$

Thus

$$w(n) = u_0 + \sum_{i=1}^{n-1} \Big[\Psi^{-1}(i+1,1,w(i),w(i+1)) \Big(\sum_{j=1}^{i} \binom{i-j+\alpha-1}{i-j} g(j,v(j)) \Big) \Big].$$

The proof is complete. □

Corollary 3.2. *Let the assumptions of Theorem 3.1 be satisfied. Then,*

$$(46)\quad v(n,1,u_0) = u(n,1,u_0) + \sum_{i=1}^{n-1} \Big(\Psi(n,1,w(n),u_0) \Psi^{-1}(i+1,1,w(i),w(i+1)) \Big[\sum_{j=1}^{i} \binom{i-j+\alpha-1}{i-j} g(j,v(j)) \Big] \Big).$$

Proof. Using mean value theorem, we get

$$\frac{u(n,1,w(n))-u(n,1,u_0)}{w(n)-u_0} = \int_0^1 \frac{\partial}{\partial u_0}[u(n,1,sw(n)+(1-s)u_0)]ds$$

$$\text{or } u(n,1,w(n)) = u(n,1,u_0)+[w(n)-u_0]\Psi(n,1,w(n),u_0). \tag{47}$$

Using (40) and (42) in (47), we get (46). □

Example 3.3. Solve $\nabla^\alpha v(n+1)=v(n)+n$, $n\in\mathbb{N}_1^+$, $v(1)=v_0$.

Solution: Let $u(n,1,u_0)$ be the solution of the unperturbed problem $\nabla^\alpha u(n+1)=u(n)$, $u(1)=u_0$. Then, from (17),

$$\begin{aligned} u(n,1,u_0) &= \binom{n+\alpha-2}{n-1}u_0+\sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1}u(j) \\ &= \binom{n+\alpha-2}{n-1}u_0+u_0\sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1} \\ &\quad \left[\binom{j+\alpha-2}{j-1}\right]\prod_{k=j+1}^{n-1}\left[1+\binom{n-k+\alpha-2}{n-k-1}\right]. \end{aligned} \tag{48}$$

Let $v(n,1,u_0)$ be the solution of the perturbed problem $\nabla^\alpha v(n+1)=v(n)+n$, $v(1)=u(1)=u_0$. Take $v(n,1,u_0)=u(n,1,w(n))$. Then, from (48),

$$\begin{aligned} v(n,1,u_0) = \binom{n+\alpha-2}{n-1}w(n)+w(n)\sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1} \\ \left[\binom{j+\alpha-2}{j-1}\right]\prod_{k=j+1}^{n-1}\left[1+\binom{n-k+\alpha-2}{n-k-1}\right]. \end{aligned} \tag{49}$$

Now

$$\begin{aligned} \Phi(n,1,u_0)=\frac{\partial}{\partial u_0}u(n,1,u_0) = \binom{n+\alpha-2}{n-1}+\sum_{j=1}^{n-1}\binom{n-j+\alpha-2}{n-j-1} \\ \left[\binom{j+\alpha-2}{j-1}\right]\prod_{k=j+1}^{n-1}\left[1+\binom{n-k+\alpha-2}{n-k-1}\right]. \end{aligned}$$

Then

$$\begin{aligned} \Phi(n+1,1,sw(k+1)+(1-s)w(k)) = \binom{n+\alpha-1}{n} \\ +\sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}\left[\binom{j+\alpha-2}{j-1}\right]\prod_{k=j+1}^{n}\left[1+\binom{n-k+\alpha-1}{n-k}\right]. \end{aligned}$$

Further

$$\Psi(n+1,1,w(n),w(n+1)) = \binom{n+\alpha-1}{n} + \sum_{j=1}^{n}\binom{n-j+\alpha-1}{n-j}\Big[\binom{j+\alpha-2}{j-1}\Big]\prod_{k=j+1}^{n}\Big[1+\binom{n-k+\alpha-1}{n-k}\Big].$$

Thus,

$$w(n) = u_0 + \sum_{i=1}^{n-1}\Big[\Psi^{-1}(i+1,1,w(i),w(i+1))\Big(\sum_{j=1}^{i}\binom{i-j+\alpha-1}{i-j}j\Big)\Big].$$

Hence

$$v(n,1,u_0) = u(n,1,u_0) + \sum_{i=1}^{n-1}\Big(\Psi(n,1,w(n),u_0) \\ \Psi^{-1}(i+1,1,w(i),w(i+1))\Big[\sum_{j=1}^{i}\binom{i-j+\alpha-1}{i-j}j\Big]\Big).$$

is the solution of the given nonlinear fractional order difference equation.

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