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UNIQUE FIXED POINT IN G-METRIC SPACE THROUGH GREATEST LOWER BOUND PROPERTIES

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Abstract. In this paper, we prove the celebrated Banach contraction mapping theorem and a result of Mustafa and Obiedat in a G-metric space using only elementary properties of greatest lower bound. This idea of using greatest lower bound properties in metric space was initiated by Joseph and Kwack in 1999. Also we introduce the notion of G-contractive fixed point and demonstrate that the unique fixed point will be a G-contractive fixed point for the underlying self-map in both the results. Our proof is highly distinct in repeatedly employing the rectangle inequality of the G-metric rather than using traditional iterative procedure.

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1. Introduction

Let A be a nonempty set of nonnegative real numbers which is bounded below. Then by the infimum property of $\mathbb{R}$ (Sec. 2.4, [1]), A will have a greatest lower bound, say a in $\mathbb{R}$. Also any number in A which exceeds a cannot be its lower bound.

The following is an easy consequence of properties of the infimum:

Lemma 1.1. *If A is a nonempty set of nonnegative real numbers with zero as its greatest lower bound, then there is a sequence $\langle r_n \rangle_{n=1}^{\infty}$ in A such that $\lim_{n\to\infty} r_n = 0$.*

Let M be a metric space with metric ρ. Using Lemma 1.1 and the repeated application of the triangle inequality of ρ, Joseph and Kwack [4] in 1999 proved the following theorem.

Theorem 1.2. *Let f be a self-map on M, and there exist constants $a_i \geq 0$, $i = 1, 2, ..., 5$ such that $0 \leq \sum_{i=1}^{5} a_i < 1$ and*

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$$(1.1) \qquad \rho(fx, fy) \le a_1\rho(x, y) + a_2\rho(x, fx) + a_3\rho(y, fy) + a_4\rho(x, fy) + a_5\rho(y, fx) \quad for\ all \quad x, y \in M.$$

If M is complete, then f will have a unique fixed point p.

We note that if $a_2 = \cdots = a_5 = 0$ in (1.1), f reduces to a contraction and Theorem 1.2, to the well-known Banach contraction mapping theorem.

In this paper, we present the proofs of G-contraction mapping theorem (See Theorem 2.2) and of results of Mustafa and Obiedat [5] and Mustafa et al. [6] only using the basic properties of greatest lower bound of a set of nonnegative real numbers. Interestingly, our technique focuses on repeatedly employing the axiom (A-5) (see below) instead of the routine iterative procedure.

Definition 1.3. Let X be a nonempty set and $d : X \times X \to \mathbb{R}$ satisfy the following axioms:

(A-1) $d(x, y, z) \ge 0$ for all $x, y, z \in X$ with $d(x, y, z) = 0$ if $x = y = z$,

(A-2) $d(x, x, y) > 0$ for all $x, y \in X$ with $x \ne y$,

(A-3) $d(x, x, y) \le d(x, y, z)$ for all $x, y, z \in X$ with $z \ne y$,

(A-4) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(z, x, y) = d(y, z, x) = d(z, y, x)$ for all $x, y, z \in X$

(A-5) $d(x, y, z) \le d(x, w, w) + d(w, y, z)$ for all $x, y, z, w \in X$

Then the function d is called a G-metric on X and the pair (X, d) a G-metric space, which was introduced by Mustafa and Sims [7] as a generalization of metric space.

Axiom (A-1) asserts that a G-metric d is nonnegative. Axiom (A-4) asserts that the value of $d(x, y, z)$ is independent of the order of x, y and z, and is usually known as the *symmetry* of d in them.

Example 1.4. Let X be a metric space with the metric $\rho(x, y)$.
For all $x, y, z \in X$, define

(a) $d_s(x, y, z) = \rho(x, y) + \rho(y, z) + \rho(z, x)$

(b) $d_m(x, y, z) = \max\{\rho(x, y), \rho(y, z), \rho(z, x)\}$.

Then d_s and d_m satisfy Axioms (A-1)-(A-5) and hence they are G-metrics on X.

Conversely, every G-metric d on X induces a metric ρ_G on it, given by

(c) $\rho_G(x, y) = d(x, y, y) + d(y, x, x)$ for all $x, y \in X$

(d) $\rho_G(x, y) = \max\{d(x, y, y), d(y, x, x)\}$ for all $x, y \in X$.

Geometrically, d_s represents the *perimeter* of a triangle with the vertices x, y and z in the plane. Further, if w is an interior point of the triangle, then (A-5) is the best possible. That is why Axiom (A-5) is referred to as the *rectangle inequality* of the G-metric d.

From this definition, it immediately follows that

(1.2-a) If x and y are points in X such that $d(x, x, y) = 0,$ then $x = y$

and

(1.2-b) $$d(x, y, y) \leq 2d(y, x, x) \text{ for all } x, y \in X.$$

Definition 1.5. Let (X, d) be a G-metric space. Then the G-metric d is called a *symmetric* [6] if

$$d(x, y, y) = d(x, x, y) \quad \text{for all} \quad x, y \in X.$$

The G-metric spaces in (a) and (b) of Example 1.4 are symmetric, while the following is nonsymmetric:

Example 1.6. Consider (X, d) with $X = \{x, y\}$ and

$$d(x, x, x) = d(y, y, y) = 0, d(x, x, y) = 1 \text{ and } d(x, y, y) = 2 \text{ for all } x, y \in X.$$

Then d is a G-metric, but not symmetric.

We require the following terminology and some topological concepts developed in [6] and [7]:

Lemma 1.7. *Consider a G-metric space (X, d) and the induced metric ρ_G given by (c) of Example 1.4. Then*

(i) $\rho_G(x, y) = 2d(x, y, y)$ *for all* $x, y \in X$, *provided* X *is symmetric,*

(ii) $\frac{3}{2}d(x, y, y) \leq \rho_G(x, y) \leq 3d(x, y, y)$ *for all* $x, y \in X$ *with* $x \neq y$*, if* X *is not symmetric. In general, these inequalities cannot be improved.*

Lemma 1.8. *Consider a symmetric G-metric space (X, d) the induced metric ρ_G given by (d) of Example 1.4. Then $\rho_G(x, y) = d(x, y, y)$ for all $x, y \in X$.*

Definition 1.9. A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said to be *G-convergent* with limit $p \in X$ if $\lim_{n,m \to \infty} d(p, x_n, x_m) = 0$, that is if for any $\epsilon > 0$ there is a positive integer N such that $n \geq N$ and $m \geq N \;\Rightarrow\; d(x_n, x_m, p) < \epsilon$, and we write $x_n \overset{G}{\to} p$.

An immediate consequence of Definition 1.9 is

Lemma 1.10. *In a G-metric space (X, d), the following statements are equivalent:*

(a) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is G-convergent with the limit* $p \in X$,

(b) $\lim_{n\to\infty} d(x_n, x_n, p) = 0$,

(c) $\lim_{n\to\infty} d(x_n, p, p) = 0$.

Also, it is known that d is jointly continuous in all the three variables, as a metric is continuous in two variables.

Definition 1.11. A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is *G-Cauchy* if

$$\lim_{n,m,l\to\infty} d(x_n, x_m, x_l) = 0$$

that is given $\epsilon > 0$, we can find a positive integer N such that $d(x_n, x_m, x_l) < \epsilon$ whenever $n \geq N$, $m \geq N$ and $l \geq N$.

It follows that a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is G-Cauchy if and only if for every $\epsilon > 0$, there is a positive integer N such that $d(x_n, x_m, x_m) < \epsilon$ whenever $n \geq N$ and $m \geq N$. Note that every G-convergent sequence is G-Cauchy (in a G-metric space).

Definition 1.12. A G-metric space X is said to be *G-complete* or simply complete if every G-Cauchy sequence in it is G-convergent with limit in it.

The G-metric space given in Example 1.6 is complete. Further, a G-metric space (X, d) is complete if and only if the induced metric space (X, ρ_G) is complete.

Definition 1.13. The self-map f on a G-metric space (X, d) is *G-continuous* at $x \in X$ if and only if for every sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with $x_n \overset{G}{\to} x$, we have $fx_n \overset{G}{\to} fx$.

2. Maim Results

In this paper, X denotes a G-metric space with G-metric d and f, a self-map on X.

First we have

Definition 2.1. The self-map f on X is a G-contraction if there is a constant α with the choice $0 \leq \alpha < 1$ such that

$$d(fx, fy, fz) \leq \alpha d(x, y, z) \quad \text{for all} \quad x, y, z \in X. \tag{2.1}$$

Now we have the following analogue of the celebrated Banach contraction mapping theorem for a G-metric space, which we shall call the *G-Contraction mapping theorem*:

Theorem 2.2. *Let f be a G-contraction with choice* (2.1). *Then f will have a unique fixed point p, provided X is G-complete.*

Proof. From (2.1), we get

$$d(fx,fy,fy) \le \alpha d(x,y,y) \quad \text{and} \quad d(fy,fx,fx) \le \alpha d(y,x,x),$$

which in view of Example 1.4-(d) gives

$$\rho_G(fx,fy) =\le \alpha \max\{d(x,y,y), d(y,x,x)\} = \alpha\rho_G(x,y) \quad \text{for all} \quad x,y \in X.$$

Thus the existence and uniqueness of the fixed point is ensured by the Banach contraction mapping theorem (BCT). □

Now we demonstrate in the next few lines that the existence of the fixed point can be effectively established using *only elementary properties* of a G-metric, without the application of BCT and usual iteration procedure.

Let $S = \{d(x,fx,fx) : x \in X\}$. Each S is a nonempty set of nonnegative numbers which is bounded below. Hence it has a greatest lower bound, say a.

Our claim is that $a = 0$. If possible, suppose that $a > 0$. Since $\alpha < 1$, we see that a/α being greater than a cannot be a lower bound of S. Thus $d(x,fx,fx) < \dfrac{a}{\alpha}$ or $\alpha d(x,fx,fx) < a$ for some $x \in X$, so that (2.1) gives $d(fx,f^2x,f^2x) \le \alpha d(x,fx,fx) < a$, which implies that a cannot be lower bound of S, as $d(fx,f^2x,f^2x) \in S$. This would contradict the choice of a. Therefore, $a = \inf\{d(x,fx,fx) : x \in X\} = 0$.

Then by Lemma 1.1, we choose points $x_1, x_2, ..., x_n, ...$ in X such that

$$d(x_n,fx_n,fx_n) \in S \text{ for } n = 1,2,... \text{ and } \lim_{n\to\infty} d(x_n,fx_n,fx_n) = 0. \tag{2.2}$$

Next we establish that

(e) $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy.

Repeatedly employing (A-5) and using (1.2-b) and (2.1), we get

$$\begin{aligned} d(x_n,x_m,x_m) &\le d(x_n,fx_n,fx_n) + d(fx_n,x_m,x_m) \\ &\le d(x_n,fx_n,fx_n) + d(fx_n,fx_m,fx_m) + 2d(fx_m,x_m,x_m) \\ &\le d(x_n,fx_n,fx_n) + \alpha d(x_n,x_m,x_m) + 2d(x_m,fx_m,fx_m) \end{aligned}$$

so that

$$(1-\alpha)d(x_n,x_m,x_m) \le d(x_n,fx_n,fx_n) + 2d(x_m,fx_m,fx_m)$$

for all $n \ge 1$ and all $m \ge 1$. Applying the limit as $n,m \to \infty$ and using (2.2), this gives $\lim_{n,m\to\infty} d(x_n,x_m,x_m) = 0$ proving (e).

Since X is G-complete, we can find a point $p \in X$ satisfying (b) of Lemma 1.10, that is

$$\lim_{n\to\infty} d(x_n,x_n,p) = 0. \tag{2.3}$$

Again from repeated application of (A-5); (2.1), and (1.2-b), we have

$$\begin{aligned} d(p,fp,fp) &\le d(p,fx_n,fx_n)+d(fx_n,fp,fp)\\ &\le d(p,x_n,x_n)+d(x_n,fx_n,fx_n)+\alpha d(x_n,p,p)\\ &\le [d(p,x_n,x_n)+d(x_n,fx_n,fx_n)]+2\alpha d(p,x_n,x_n)\\ &= (2\alpha+1)(d(p,x_n,x_n)+d(x_n,fx_n,fx_n). \end{aligned}$$

Applying the limit as $n \to \infty$ in this, and then using (2.2) and (2.3), we obtain that $d(p,fp,p) \le 0$ or $fp = p$, in view of (A-4) and (1.2-a).

That is, p is a fixed point of f.

Uniqueness: Let q be also a fixed point of f so that $fq = q$. Then from the condition (2.1), (A-4) and (1.2-a), we get $d(p,q,q) = d(fp,fq,fq) \le \alpha d(p,q,q)$ or $(1-\alpha)d(p,q,q) \le 0$ and hence $p = q$. Thus the fixed point of f is unique. We give an analogue of the notion of contractive fixed point [8] to a G-metric space:

Definition 2.3. A fixed point p of f on X is a G-contractive fixed point of it if the orbital sequence $x, fx, ..., f^n x, ...$ at each $x \in X$ G-converges to p.

We see that p is a *G-contractive fixed point* of f under the stated conditions of Theorem 2.2. In fact, for any $x \in X$ by repeatedly applying (2.1) n times,

$$d(f^n x,p,p) = d(f^n x, f^n p, f^n p) \le \alpha^n d(x,p,p). \tag{2.4}$$

But the rectangle inequality (A-5) and (2.1) give

$$d(x,p,p) = d(x,fp,fp) \le d(x,fx,fx)+d(fx,fp,fp) \le d(x,fx,fx)+\alpha d(x,p,p)$$

or $d(x,p,p) \le \frac{1}{1-\alpha} \cdot d(x,fx,fx)$. With this, (2.4) becomes

$$d(f^n x,p,p) \le \frac{\alpha^n}{1-\alpha} \cdot d(x,fx,fx) \quad \text{for all} \quad x \in X \quad \text{and all} \quad n \ge 1. \tag{2.5}$$

Since $\lim\limits_{n\to\infty} \alpha^n = 0$, from (2.5) it follows that $d(f^n x,p,p) \to 0$ as $n \to \infty$ for all $x \in X$. Thus in view of Lemma 1.3-(c), we get $fx_n \xrightarrow{G} p$ for each $x \in X$.

In other words, p is a G-contractive fixed point of f.

We now prove the following result due to Mustafa et al. [6], extending the same technique:

Theorem 2.4. *Suppose for all $x, y, z \in X$ that*

$$d(fx,fy,fz) \le \alpha d(x,fx,fx)+\beta d(y,fy,fy)+\gamma d(z,fz,fz)+\delta d(x,y,z) \tag{2.6}$$

where $\alpha+\beta+\gamma+\delta < 1$. If X is G-complete, then f will have a unique fixed point p and f is continuous at p.

Proof. From (2.6) with $y = z$, we have

$$d(fx,fy,fy) \le \alpha d(x,fx,fx)+(\beta+\gamma)d(y,fy,fy)+\delta d(x,y,y) \tag{2.7}$$

for all $x, y \in X$. As in the proof of Theorem 2.2, if $a > 0$, then (2.7) with $y = fx$ would give

$$d(fx, f^2x, f^2x) \le \alpha d(x, fx, fx) + (\beta+\gamma) d(fx, f^2x, f^2x) + \delta d(x, fx, fx)$$

or

$$d(fx, f^2x, f^2x) \le \frac{\alpha+\delta}{1-\beta-\gamma} \cdot d(x, fx, fx). \tag{2.8}$$

But $\dfrac{\alpha+\delta}{1-\beta-\gamma} < 1$, since $\alpha+\beta+\gamma+\delta < 1$. Then, from (2.8) we would get $d(fx, f^2x, f^2x) < a$ for some $x \in X$, which contradicts with the choice of a. Therefore, $a = 0$.

Hence, again by Lemma 1.1, we choose a sequence $\langle d(x_n, fx_n, fx_n)\rangle_{n=1}^{\infty}$ satisfying (2.2).

Now, repeatedly using (A-5), (2.7) and (1.2-b), we see that

$$\begin{aligned} d(x_n, x_m, x_m) &\le d(x_n, fx_n, fx_n) + d(fx_n, fx_m, fx_m) + d(fx_m, x_m, x_m) \\ &\le d(x_n, fx_n, fx_n) + \alpha d(x_n, fx_n, fx_n) + (\beta+\gamma) d(fx_m, x_m, x_m) \\ &\qquad + \delta d(x_n, x_m, x_m) + d(fx_m, x_m, x_m) \\ &\le (1+\alpha) d(x_n, fx_n, fx_n) + (\beta+\gamma+1) d(fx_m, x_m, x_m) \\ &\qquad\qquad + \delta d(x_n, x_m, x_m) \\ &\le (1+\alpha) d(x_n, fx_n, fx_n) + 2(\beta+\gamma+1) d(x_m, fx_m, fx_m) \\ &\qquad\qquad + \delta d(x_n, x_m, x_m) \end{aligned}$$

so that

$$d(x_n, x_m, x_m) \le \frac{1+\alpha}{1-\delta} \cdot d(x_n, fx_n, fx_n) + \frac{2(\beta+\gamma+1)}{1-\delta} \cdot d(x_m, fx_m, fx_m)$$

for all $n \ge 1$ and $m \ge 1$. Employing the limit as $m, n \to \infty$ in this and using (2.2), we get $\lim\limits_{n,m\to\infty} d(x_n, x_m, x_m) = 0$, proving (e).

Since X is G-complete, we can find a point $p \in X$ satisfying (2.3).

Again by repeated application of (A-5); from (A-4), (2.7) and (1.2-b), we have

$$\begin{aligned} d(p, fp, fp) &\le d(p, fx_n, fx_n) + d(fx_n, fp, fp) \\ &\le [d(p, x_n, x_n) + d(x_n, fx_n, fx_n)] + \alpha d(x_n, fx_n, fx_n) \\ &\qquad + (\beta+\gamma) d(p, fp, fp) + \delta d(x_n, p, p) \end{aligned}$$

or $(1-\beta-\gamma) d(p, fp, fp) \le d(x_n, x_n, p) + (1+\alpha) d(x_n, fx_n, fx_n) + \delta d(x_n, p, p)$. Proceeding the limit as $n \to \infty$ in this, and using (2.2), (2.3), Lemma 1.10, we obtain that $(1-\beta-\gamma) d(p, fp, fp) \le 0$ or $fp = p$, in view of (1.2-a). That is p is a fixed point of f. The uniqueness of the fixed point of f follows easily from (2.6).

To prove that f is G-continuous at p, consider $\langle y_n\rangle_{n=1}^{\infty} \subset X$ with $\lim\limits_{n\to\infty} y_n = p$. Then from (2.7), (A-5) and (1.2-b),

$$\begin{aligned} d(p, fy_n, fy_n) &= d(fp, fy_n, fy_n) \\ &\le \alpha d(p, fp, fp) + (\beta+\gamma) d(y_n, fy_n, fy_n) + \delta d(p, y_n, y_n) \\ &\le (\beta+\gamma)\,[d(y_n, p, p) + d(p, fy_n, fy_n)] + \delta d(p, y_n, y_n) \end{aligned}$$

or $d(p, fy_n, fy_n) \leq \frac{\beta+\gamma}{1-\beta-\gamma} \cdot d(y_n, p, p) + \frac{\delta}{1-\beta-\gamma} \cdot d(p, y_n, y_n)$.

Applying the limit as $n \to \infty$ in this and using Lemma 1.10, we find that $fy_n \xrightarrow{G} p = fp$. Thus f is G-continuous at p. □

Here also we see that p will be a G-contractive fixed point of f. Indeed, taking $y = z = p$ in (2.7) and using (A-5), we get

$$\begin{aligned} d(f^n x, p, p) &= d(f^n x, fp, fp) \\ &\leq \alpha d(f^{n-1}x, f^n x, f^n x) + (\beta+\gamma) d(p, fp, fp) + \delta d(f^{n-1}x, p, p) \\ &= \alpha d(f^{n-1}x, f^n x, f^n x) + \delta d(f^{n-1}x, p, p) \\ &\leq \alpha d(f^{n-1}x, f^n x, f^n x) + \delta \left[d(f^{n-1}x, f^n x, f^n x) + d(f^n x, p, p) \right] \end{aligned}$$

or

$$d(f^n x, p, p) \leq (\alpha+\delta) d(f^{n-1}x, f^n x, f^n x). \tag{2.9}$$

But again from (2.7) with $y = f^{n-1}x$, we have

$$\begin{aligned} d(f^{n-1}x, f^n x, f^n x) \leq \alpha d(f^{n-2}x, f^{n-1}x, f^{n-1}x) + (\beta+\gamma) d(f^{n-1}x, f^n x, f^n x) \\ + \delta d(f^{n-2}x, f^{n-1}x, f^{n-1}x) \end{aligned}$$

or $d(f^{n-1}x, f^n x, f^n x) \leq \frac{\alpha+\delta}{1-\beta-\gamma} \cdot d(f^{n-2}x, f^{n-1}x, f^{n-1}x)$.

Hence by the induction, it follows that

$$d(f^{n-1}x, f^n x, f^n x) \leq \left(\frac{\alpha+\delta}{1-\beta-\gamma} \right)^{n-1} d(x, fx, fx) \quad \text{for all} \quad n \geq 1.$$

Substituting this in (2.9), we get

$$d(f^n x, p, p) \leq (\alpha+\delta) \left(\frac{\alpha+\delta}{1-\beta-\gamma} \right)^{n-1} d(x, fx, fx) \text{ for all } x \in X \text{ and } n \geq 1,$$

which as $n \to \infty$ gives $f^n x \xrightarrow{G} p$ for each $x \in X$, since $\lim\limits_{n\to\infty} \left(\frac{\alpha+\delta}{1-\beta-\gamma} \right)^{n-1} = 0$.

Thus p is a G-contractive fixed point of f.

3. Discussion of the results

If X is symmetric, in view of Lemma 1.7-(i) and (2.7), from [5] we find that

$$\rho_G(fx, fy) \leq \frac{\alpha+\beta+\gamma}{2} \left[\rho_G(x, fx) + \rho_G(y, fy) \right] + \delta \rho_G(x, y).$$

Then the existence and uniqueness of the fixed point is ensured from the Reich theorem [9] in the metric space (X, ρ_G), since $\alpha+\beta+\gamma+\delta < 1$.

But if X is not symmetric, Lemma 1.7-(ii) and (2.7) would imply that

$$\rho_G(fx, fy) \leq \tfrac{2}{3}(\alpha+\beta+\gamma) \left[\rho_G(x, fx) + \rho_G(y, fy) \right] + \delta \rho_G(x, y),$$

which gives no information about f, since $\dfrac{2}{3}(\alpha+\beta+\gamma) + \dfrac{2}{3}(\alpha+\beta+\gamma) + \gamma$ may not be less than 1. This fact led the authors of [6] to implement the routine iteration procedure to prove the result.

However, if the induced metric in the above argument is replaced by that of Example 1.4-(d), iteration procedure can be avoided. Still, the unique fixed point is an immediate consequence of the Ćirić's result [2], for a complete metric space (X, ρ_G), as shown in a recent paper [3].

The significance of our proof technique is three-fold:

1. It asserts that a unique fixed point can be effectively obtained *using elementary properties* of a G-metric.
2. It does not utilize the traditional iteration procedure.
3. The results obtained are not consequences of the Banach, Ćirić and Reich contraction theorems.

Finally, writing $\alpha = \beta = \gamma = q$ and $\delta = 0$ in Theorem 2.4, we get $0 \leq q < \frac{1}{3}$ and hence the following result of [5]:

Corollary 3.1. *Suppose that there exists a constant q such that $0 \leq q < \frac{1}{3}$ and*

$$d(fx, fy, fz) \leq q\,[d(x, fx, fx) + d(y, fy, fy) + d(z, fz, fz)] \;\; for\ all\ x, y, z \in X.$$

If X is complete, then f will have a unique fixed point p and f is G-continuous at p.

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