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UNIFICATION OF λ-CLOSED SETS VIA GENERALIZED TOPOLOGIES

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Abstract. In this paper we introduce and study a new type of sets called $(\bigwedge, \mu\nu)$-closed sets by using the concept of generalized topology introduced by A. Császár.

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1. Introduction

For the last couple of years, different forms of open sets are being studied. Recently, a significant contribution to the theory of generalized open sets has been presented by A. Császár [10, 11, 12]. Especially, the author defined some basic operators on generalized topological spaces. It is observed that a large number of papers are devoted to the study of generalized open sets like open sets of a topological space, containing the class of open sets and possessing properties more or less similar to those of open sets.

We recall some notions defined in [10]. Let X be a non-empty set and let $expX$ denote the power set of X. We call a class $\mu \subseteq expX$ a generalized topology [10], (briefly, GT) if $\varnothing \in \mu$ and unions of elements of μ belong to μ. A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). The θ-closure, $cl_\theta(A)$ [23] (resp. δ-closure, $cl_\delta(A)$ [23]) of a subset A of a topological space (X, τ) is defined by $\{x \in X : clU \cap A \neq \varnothing$ for all $U \in \tau$ with $x \in U\}$ (resp. $\{x \in X : A \cap U \neq \varnothing$ for all regular open sets U containing $x\}$, where a subset A is said to be regular open if $A = int(cl(A))$). A is said to be δ-closed [23] (resp. θ-closed [23]) if $A = cl_\delta A$ (resp. $A = cl_\theta A$) and the complement of a δ-closed set (resp. θ-closed) set is known as a δ-open (resp. θ-open) set. A subset A of a topological space (X, τ) is said to be preopen [20] (resp. semi-open [17], α-open [21], b-open [1]) if $A \subseteq int(cl(A))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq int(cl(int(A)))$, $A \subseteq cl(int(A)) \cup int(cl(A))$). The complement of a semi-open set is called a semi-closed set. The semi-closure [18] of A, denoted by $scl(A)$, is the intersection of all semi-closed sets containing A. A point $x \in X$ is called a semi-θ-cluster point [18] of a set A if $sclU \cap A \neq \varnothing$ for each semi-open set U containing x. The set of all semi-θ-cluster points of A is denoted by $scl_\theta A$. If $A = scl_\theta A$, then A is known as semi-θ-closed and

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the complement of a semi-θ-closed set is called a semi-θ-open set [18]. We note that for any topological space (X, τ), the collection of all open (resp. preopen, semi-open, δ-open, α-open, b-open, θ-open, semi-θ-open) sets is denoted by τ (resp. $PO(X)$, $SO(X)$, $\delta O(X)$, $\alpha O(X)$, $BO(X)$ or $\gamma O(X)$, $\theta O(X)$, $S\theta O(X)$). Each of these collections is a generalized topology on X.

For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [10, 11]).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where the operator $\gamma : expX \to expX$ is said to be idempotent if $A \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if $A \subseteq B \subseteq X$ implies $\gamma(A) \subseteq \gamma(B)$. It is also well known from [11, 12] that if μ is a GT on X, $x \in X$ and $A \subseteq X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \varnothing$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

As the final prerequisites, we wish to recall a few definitions and results from [14].

Definition 1.1. [14] Let (X, μ) be a GTS and $A \subseteq X$. Then, the subset $\bigwedge_\mu(A)$ is defined as follows:

$$\bigwedge_\mu(A) = \begin{cases} \bigcap\{G : A \subseteq G, G \in \mu\}, & \text{if there exists } G \in \mu \text{ such that } A \subseteq G; \\ X & \text{otherwise.} \end{cases}$$

Proposition 1.2. *[14] Let A, B and $\{B_\alpha : \alpha \in \Omega\}$ be subsets of a GTS (X, μ). Then the following properties hold:*

(a) $B \subseteq \bigwedge_\mu(B)$;
(b) If $A \subseteq B$, *then* $\bigwedge_\mu(A) \subseteq \bigwedge_\mu(B)$;
(c) $\bigwedge_\mu(\bigwedge_\mu(B)) = \bigwedge_\mu(B)$;
(d) $\bigwedge_\mu[\bigcup_{\alpha\in\Omega} B_\alpha] = \bigcup_{\alpha\in\Omega} [\bigwedge_\mu(B_\alpha)]$;
(e) If $A \in \mu$, *then* $A = \bigwedge_\mu(A)$;
(f) $\bigwedge_\mu[\bigcap_{\alpha\in\Omega} B_\alpha] \subseteq \bigcap_{\alpha\in\Omega} [\bigwedge_\mu(B_\alpha)]$;

Definition 1.3. [14] In a GTS (X, μ), a subset B is called a $\bigwedge_\mu$-set if $B = \bigwedge_\mu(B)$.

Theorem 1.4. *[14] If (X, μ) is a GTS, then the intersection of $\bigwedge_\mu$-sets is a $\bigwedge_\mu$-set.*

2. $(\bigwedge, \mu\nu)$-closed sets and associated separation axioms

Definition 2.1. Let μ and ν be two GT's on X. A subset A of X is said to be $(\bigwedge, \mu\nu)$-closed if $A = U \cap F$, where U is a $\bigwedge_\mu$-set and F is a ν-closed set.

The family of all $(\bigwedge, \mu\nu)$-closed sets of (X, μ, ν) is denoted by $\bigwedge_{\mu\nu c}$.

Remark 2.2. In a topological space (X, τ), if $\mu = \nu = \tau$ (resp. $SO(X)$, $\alpha O(X)$, $\theta O(X)$, $\delta O(X)$, $S\theta O(X)$), then a $(\bigwedge, \mu\nu)$-closed set reduces to a λ-closed [2] (resp. semi-λ-closed [13], $(\bigwedge, \alpha)$-closed [6], $(\bigwedge, \theta)$-closed [5], $(\bigwedge, \delta)$-closed [16], $(\bigwedge, s\theta)$-closed [4]) set. On the other hand, if in a bi m-space (X, m_X, n_X), $\mu = m_X$ and $\nu = n_X$, then a $(\bigwedge, \mu\nu)$-closed set reduces to a $(\bigwedge, mn)$-closed [22] set.

Lemma 2.3. *Let μ and ν be two GT's on X, then the following properties are equivalent:*
(a) A is $(\bigwedge, \mu\nu)$-closed;
(b) $A = U \cap c_\nu(A)$, where U is a $\bigwedge_\mu$-set;
(c) $A = \bigwedge_\mu(A) \cap c_\nu(A)$.

Proof. **(a) ⇒ (b):** Let $A = U \cap F$, where U is a $\bigwedge_\mu$-set and F is a ν-closed set of X. Since $A \subseteq F$, we have $c_\nu(A) \subseteq F$. Thus $A \subseteq U \cap c_\nu(A) \subseteq U \cap F = A$.

(b) ⇒ (c): Let $A = U \cap c_\nu(A)$, where U is a $\bigwedge_\mu$-set. Since $A \subseteq U$, we have by Proposition 1.2, $\bigwedge_\mu(A) \subseteq \bigwedge_\mu(U) = U$ and hence, $A \subseteq \bigwedge_\mu(A) \cap c_\nu(A) \subseteq U \cap c_\nu(A) = A$. Thus, we obtain $A = \bigwedge_\mu(A) \cap c_\nu(A)$.

(c) ⇒ (a): We know that $c_\nu(A)$ is a ν-closed set and by Proposition 1.2(c), we have $\bigwedge_\mu(A)$ is a $\bigwedge_\mu$-set. Thus by (c), we have $A = \bigwedge_\mu(A) \cap c_\nu(A)$ and hence A is a $(\bigwedge, \mu\nu)$-closed set. □

It thus follows from Definition 2.1 that

Remark 2.4. Every $\bigwedge_\mu$-set is $(\bigwedge, \mu\nu)$-closed and every ν-closed set is $(\bigwedge, \mu\nu)$-closed.

Example 2.5. Let $X = \{a, b, c\}$, $\mu = \{\varnothing, \{a\}, \{a, b\}\}$ and $\nu = \{\varnothing, \{b\}, \{a, b\}\}$. Then, μ and ν are two GT's on X. It is easy to see that $\{a, c\}$ is a $(\bigwedge, \mu\nu)$-closed set but it is not a $\bigwedge_\mu$-set and $\{a, b\}$ is a $(\bigwedge, \mu\nu)$-closed set but it is not a ν-closed set.

Proposition 2.6. *Let μ and ν be two GT's on a set X. Then $\bigwedge_{\mu\nu c}$ is closed under arbitrary intersections.*

Proof. Suppose that $\{A_\alpha : \alpha \in I\}$ is a family of $(\bigwedge, \mu\nu)$-closed subsets of X. Then, for each $\alpha \in I$ there exist a $\bigwedge_\mu$-set U_α and a ν-closed F_α such that $A_\alpha = U_\alpha \cap F_\alpha$. Hence we have $\bigcap\limits_{\alpha\in I} A_\alpha = \bigcap\limits_{\alpha\in I} (U_\alpha \cap F_\alpha) = (\bigcap\limits_{\alpha\in I} U_\alpha) \cap (\bigcap\limits_{\alpha\in I} F_\alpha)$. We note that $\bigcap\limits_{\alpha\in I} U_\alpha$ is a $\bigwedge_\mu$-set (by Theorem 1.4) and $\bigcap\limits_{\alpha\in I} F_\alpha$ is ν-closed. Thus by Definition 2.1, it follows that $\bigcap\limits_{\alpha\in I} A_\alpha$ is a $(\bigwedge, \mu\nu)$-closed set. □

Example 2.7. Let $X = \{a, b, c\}$. Consider two GT's on X as $\mu = \{\varnothing, \{a\}, \{a, b\}\}$ and $\nu = \{\varnothing, \{a, b\}\}$. It is easy to see that $\{a\}$ and $\{c\}$ are two $(\bigwedge, \mu\nu)$-closed subsets of X but their union $\{a, c\}$ is not a $(\bigwedge, \mu\nu)$-closed set.

Definition 2.8. Let μ and ν be two GT's on X. Then a subset A of X is said to be generalized $\mu\nu$-closed (briefly, $\mu\nu g$-closed) if $c_\nu(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \mu$.

Observation 2.9. *Let μ and ν be two GT's on X and A, B be two subsets of X.*

(i) If A is ν-closed, then A is $\mu\nu g$-closed.
(ii) If A is $\mu\nu g$-closed and μ-open, then A is ν-closed.
(iii) If A is $\mu\nu g$-closed and $A \subseteq B \subseteq c_\nu(A)$, then B is $\mu\nu g$-closed.
(iv) A is $\mu\nu g$-closed if and only if $c_\nu(A) \subseteq \bigwedge_\mu(A)$.

Proof. The proofs of (i), (ii) and (iii) are straightforward, and we shall only prove (iv). Let A be a $\mu\nu g$-closed set and U be any μ-open set such that $A \subseteq U$. Then $c_\nu(A) \subseteq U$ and hence we obtain $c_\nu(A) \subseteq \bigwedge_\mu(A)$.

Conversely, suppose that $c_\nu(A) \subseteq \bigwedge_\mu(A)$ and $A \subseteq U \in \mu$. Then $c_\nu(A) \subseteq \bigwedge_\mu(A) \subseteq U$. This shows that A is $\mu\nu g$-closed. □

Example 2.10. Let $\mu = \{\varnothing, \{a\}, \{a,b\}, \{b,c\}, X\}$ and $\nu = \{\varnothing, \{a\}, \{a,c\}\}$ be two GT's on a set $X = \{a,b,c\}$. Then it is easy to see that $\{c\}$ is a $\mu\nu g$-closed set which is not a ν-closed set. Also, $\{b\}$ is a ν-closed set which is not a μ-open set.

Proposition 2.11. *Let μ and ν be two GT's on a set X. Then a subset A of X is ν-closed if and only if A is $\mu\nu g$-closed and $(\bigwedge, \mu\nu)$-closed.*

Proof. One part follows from Observation 2.9(i) and Remark 2.4. Conversely, let A be a $\mu\nu g$-closed as well as a $(\bigwedge, \mu\nu)$-closed set. Then by Observation 2.9(iv), $c_\nu(A) \subseteq \bigwedge_\mu(A)$. Thus by hypothesis and Lemma 2.3, $A = \bigwedge_\mu(A) \cap c_\nu(A) = c_\nu(A)$. So A is a ν-closed set. □

Definition 2.12. Let μ and ν be two GT's on a set X. Then (X, μ, ν) is said to be
(i) $\mu\nu$-T_0 if for any two distinct points $x, y \in X$, there exists a μ-open set U of X containing x but not y or a ν-open set V of X containing y but not x.
(ii) $\mu\nu$-$T_{1/2}$ if every singleton $\{x\}$ is either ν-open or μ-closed.

Theorem 2.13. *Let μ and ν be two GT's on a set X. Then (X, μ, ν) is $\mu\nu$-T_0 if and only if for each $x \in X$, the singleton $\{x\}$ is $(\bigwedge, \mu\nu)$-closed.*

Proof. Suppose that (X, μ, ν) be $\mu\nu$-T_0. For each $x \in X$, we have $\{x\} \subseteq \bigwedge_\mu(\{x\}) \cap c_\nu(\{x\})$. Let $y \neq x$. Then there exists a μ-open set U of X containing x but not y or a ν-open set V of X containing y but not x. In the first case, $y \notin \bigwedge_\mu(\{x\})$ and we have $y \notin \bigwedge_\mu(\{x\}) \cap c_\nu(\{x\})$. In the second case, $y \notin c_\nu(\{x\})$ and we have $y \notin \bigwedge_\mu(\{x\}) \cap c_\nu(\{x\})$. Thus $\bigwedge_\mu(\{x\}) \cap c_\nu(\{x\}) \subseteq \{x\}$. Hence we have $\bigwedge_\mu(\{x\}) \cap c_\nu(\{x\}) = \{x\}$. Hence by Lemma 2.3, $\{x\}$ is a $(\bigwedge, \mu\nu)$-closed set.

Conversely, suppose that (X, μ, ν) is not $\mu\nu$-T_0. Thus there exist distinct points $x, y \in X$ such that (i) $y \in U$ for every μ-open set U containing x and (ii) $x \in V$ for every ν-open set V containing y. Thus by (i) and (ii), $y \in \bigwedge_\mu(\{x\})$ and $y \in c_\nu(\{x\})$, respectively. Then by Lemma 2.3, $y \in \bigwedge_\mu(\{x\}) \cap c_\nu(\{x\}) = \{x\}$. This contradicts the fact that $x \neq y$. □

Theorem 2.14. *Let μ and ν be two GT's on a set X. Then the following statements are equivalent:*

(a) (X, μ, ν) *is* $\mu\nu$-$T_{1/2}$*;*
(b) Every $\mu\nu g$*-closed subset of* X *is* ν*-closed;*
(c) Every subset of X *is* $(\bigwedge, \mu\nu)$*-closed.*

Proof. **(a)** $\Rightarrow$ **(b):** Let (X, μ, ν) be $\mu\nu$-$T_{1/2}$. Suppose that there exists a $\mu\nu g$-closed set A of X which is not ν-closed. So, there exists $x \in c_\nu(A) \setminus A$. If $\{x\}$ is ν-open, then $x \in A$, which is a contradiction. In the case $\{x\}$ is μ-closed, we have $x \in X \setminus A$, and so $A \subseteq X \setminus \{x\} \in \mu$. So, by $\mu\nu g$-closedness of A, $c_\nu(A) \subseteq X \setminus \{x\}$, which is a contradiction.

(b) $\Rightarrow$ **(a):** Suppose that $\{x\}$ is not μ-closed. If X is not μ-open, then we have nothing to show. If $X \in \mu$, then the only μ-open set containing $X \setminus \{x\}$ is X. Thus $c_\nu(X \setminus \{x\}) \subseteq X$ and hence $X \setminus \{x\}$ is $\mu\nu g$-closed. Thus, by (b), $X \setminus \{x\}$ is ν-closed. So $\{x\}$ is ν-open. Therefore, (X, μ, ν) is $\mu\nu$-$T_{1/2}$.

(a) $\Rightarrow$ **(c):** Suppose that (X, μ, ν) is $\mu\nu$-$T_{1/2}$ and $A \subseteq X$. Then, for each $x \in X$, $\{x\}$ is ν-open or μ-closed. Let $B_\nu = \cap\{X \setminus \{x\} : x \in X \setminus A, \{x\}$ is ν-open$\}$ and $C_\mu = \cap\{X \setminus \{x\} : x \in X \setminus A, \{x\}$ is μ-closed$\}$. Then, B_ν is ν-closed, C_μ is a $\bigwedge_\mu$-set and $A = B_\nu \cap C_\mu$. Therefore, A is $(\bigwedge, \mu\nu)$-closed.

(c) $\Rightarrow$ **(a):** Suppose that A is a $\mu\nu g$-closed subset of X. Then, by the hypothesis, A is $(\bigwedge, \mu\nu)$-closed. Thus, by Proposition 2.11, A is ν-closed. Therefore, (X, μ, ν) is $\mu\nu$-$T_{1/2}$ (by (a) $\Leftrightarrow$ (b)). □

3. $g\bigwedge_{\mu\nu}$-sets

Definition 3.1. Let μ and ν be two GT's on a set X. Then a subset A of X is called a $g\bigwedge_{\mu\nu}$-set if $\bigwedge_\mu(A) \subseteq F$ whenever $A \subseteq F$ and F is a ν-closed set.

The family of all $g\bigwedge_{\mu\nu}$-sets is denoted by $g\bigwedge_{\mu\nu}$. The complement of a $g\bigwedge_{\mu\nu}$-set is called $g\bigwedge^*_{\mu\nu}$-set.

Remark 3.2. Let (X, τ) be a topological space. If $\mu = \nu = \tau$ (resp. $SO(X)$, $PO(X)$, $BO(X)$, $\delta O(X)$) then a $g\bigwedge_{\mu\nu}$-set is a generalized $\bigwedge$-set [19] (resp. generalized $\bigwedge_s$-set [3], generalized pre-$\bigwedge$-set [15], $g\bigwedge_b$-set [8], $g\bigwedge_\delta$-set [7]).

Proposition 3.3. *Let* μ *and* ν *be two GT's on a set* X *and* A *and* B *be two subsets of* X*, then the following properties hold:*
(a) If A *is a* $\bigwedge_\mu$*-set, then* A *is a* $g\bigwedge_{\mu\nu}$*-set.*
(b) If A *is a* $g\bigwedge_{\mu\nu}$*-set and* ν*-closed, then* A *is a* $\bigwedge_\mu$*-set.*
(c) If A *is a* $g\bigwedge_{\mu\nu}$*-set and* $A \subseteq B \subseteq \bigwedge_\mu(A)$*, then* B *is a* $g\bigwedge_{\mu\nu}$*-set.*

Proof. **(a)** Suppose that A is a $\bigwedge_\mu$-set and $A \subseteq F$, where F is a ν-closed set. Then $\bigwedge_\mu(A) = A \subseteq F$. Thus A is a $g\bigwedge_{\mu\nu}$-set.

(b) Let A be a $g\bigwedge_{\mu\nu}$-set and ν-closed. Then $\bigwedge_\mu(A) \subseteq A$. Thus, by Proposition 1.2(a), $\bigwedge_\mu(A) = A$ i.e., A is a $\bigwedge_\mu$-set.

(c) Let $B \subseteq F$, where F is a ν-closed set. Then, $A \subseteq F$ and A is a $g\bigwedge_{\mu\nu}$-set. Therefore, $\bigwedge_\mu(A) \subseteq F$. Now, by Proposition 1.2 we have, $\bigwedge_\mu(A) \subseteq$

$\bigwedge_\mu(B) \subseteq \bigwedge_\mu(\bigwedge_\mu(A)) = \bigwedge_\mu(A)$. Thus $\bigwedge_\mu(A) = \bigwedge_\mu(B)$ and hence $\bigwedge_\mu(B) \subseteq F$. Therefore, B is a $g\bigwedge_{\mu\nu}$-set. □

Example 3.4. Let $X = \{a, b.c\}$, $\mu = \{\varnothing, \{a, b\}\}$ and $\nu = \{\varnothing, \{c\}, \{a, c\}\}$. Then μ and ν are two GT's on X. It is easy to check that $\{a\}$ is a $g\bigwedge_{\mu\nu}$-set which is not a $\bigwedge_\mu$-set. We also note that $\{a, b\}$ and $\{b, c\}$ are two $g\bigwedge_{\mu\nu}$-sets but their intersection $\{b\}$ is not a $g\bigwedge_{\mu\nu}$-set.

Proposition 3.5. *Let μ and ν be two GT's on a set X. Then a subset A is a $g\bigwedge_{\mu\nu}$-set if and only if $\bigwedge_\mu(A) \cap U = \varnothing$ whenever $A \cap U = \varnothing$ and $U \in \nu$.*

Proof. Suppose that A is a $g\bigwedge_{\mu\nu}$-set. Let $A \cap U = \varnothing$ and $U \in \nu$. Then $A \subseteq X \setminus U$ and $X \setminus U$ is ν-closed. Therefore, $\bigwedge_\mu(A) \subseteq X \setminus U$ and hence $\bigwedge_\mu(A) \cap U = \varnothing$.

Conversely, let $A \subseteq F$ and F be ν-closed. Then $A \cap (X \setminus F) = \varnothing$ and $X \setminus F \in \nu$. So, by the hypothesis we have $\bigwedge_\mu(A) \cap (X \setminus F) = \varnothing$ and hence $\bigwedge_\mu(A) \subseteq F$. This shows that A is a $g\bigwedge_{\mu\nu}$-set. □

Proposition 3.6. *Let μ and ν be two GT's on a set X. Then a subset A of X is a $g\bigwedge_{\mu\nu}$-set if and only if $\bigwedge_\mu(A) \subseteq c_\nu(A)$.*

Proof. Suppose that A is a $g\bigwedge_{\mu\nu}$-set and $x \notin c_\nu(A)$. Then there exists a ν-open set U containing x such that $A \cap U = \varnothing$. Thus by Proposition 3.5, $\bigwedge_\mu(A) \cap U = \varnothing$ (as A is a $g\bigwedge_{\mu\nu}$-set). Hence $x \notin \bigwedge_\mu(A)$ and so we obtain $\bigwedge_\mu(A) \subseteq c_\nu(A)$.

Conversely, suppose that $\bigwedge_\mu(A) \subseteq c_\nu(A)$ and $A \subseteq F$, where F is ν-closed. Then $\bigwedge_\mu(A) \subseteq c_\nu(A) \subseteq F$ and thus A is a $g\bigwedge_{\mu\nu}$-set. □

Proposition 3.7. *Let μ and ν be two GT's on a set X. If $A_\alpha \in g\bigwedge_{\mu\nu}$ for each $\alpha \in I$, then $\bigcup_{\alpha\in I} A_\alpha \in g\bigwedge_{\mu\nu}$.*

Proof. Let $\bigcup_{\alpha\in I} A_\alpha \subseteq F$ and F be ν-closed. Then $A_\alpha \subseteq F$ and hence $\bigwedge_\mu(A_\alpha) \subseteq F$ for each $\alpha \in I$, since A_α is a $g\bigwedge_{\mu\nu}$-set. Thus by Proposition 1.2, we have $\bigwedge_\mu(\bigcup_{\alpha\in I} A_\alpha) = \bigcup_{\alpha\in I} \bigwedge_\mu(A_\alpha) \subseteq F$. This shows that $\bigcup_{\alpha\in I} A_\alpha \in g\bigwedge_{\mu\nu}$. □

Proposition 3.8. *Let μ and ν be two GT's on a set X and A be a $g\bigwedge_{\mu\nu}$-set of X. Then, for every ν-closed set F such that $(X \setminus \bigwedge_\mu(A)) \cup A \subseteq F$, $F = X$ holds.*

Proof. Let A be a $g\bigwedge_{\mu\nu}$-set and F a ν-closed set such that $(X \setminus \bigwedge_\mu(A)) \cup A \subseteq F$. Since $A \subseteq F$, $\bigwedge_\mu(A) \subseteq F$ and $X = (X \setminus \bigwedge_\mu(A)) \cup \bigwedge_\mu(A) \subseteq F$. Therefore, we have $X = F$. □

Proposition 3.9. *Let μ and ν be two GT's on a set X and A a $g\bigwedge_{\mu\nu}$-set of X. Then, $(X \setminus \bigwedge_\mu(A)) \cup A$ is ν-closed if and only if A is a $\bigwedge_\mu$-set.*

Proof. By Proposition 3.8, $(X \setminus \bigwedge_\mu(A)) \cup A = X$. Thus, $\bigwedge_\mu(A) \cap (X \setminus A) = \varnothing$ i.e., $\bigwedge_\mu(A) \subseteq A$. Thus by Proposition 1.2(a), $\bigwedge_\mu(A) = A$ i.e., A is a $\bigwedge_\mu$-set.

Conversely, if A is a $\bigwedge_\mu$-set, then $A = \bigwedge_\mu(A)$. So $(X \setminus \bigwedge_\mu(A)) \cup A = (X \setminus A) \cup A = X$ which is ν-closed. □

Proposition 3.10. *Let μ and ν be two GT's on a set X. Then, for each $x \in X$,*

(a) $\{x\}$ is either ν-open or $X \setminus \{x\}$ is a $g\bigwedge_{\mu\nu}$-set in X;

(b) $\{x\}$ is either a ν-open set or a $g\bigwedge^*_{\mu\nu}$-set in X.

Proof. **(a)** Suppose that $\{x\}$ is not ν-open. Then, the only ν-closed set F containing $X \setminus \{x\}$ is X. Thus, $\bigwedge_\mu(X \setminus \{x\}) \subseteq F = X$ and hence $X \setminus \{x\}$ is a $g\bigwedge_{\mu\nu}$-set.

(b) Follows from (a) and Definition 3.1. □

Theorem 3.11. *Let μ and ν be two GT's on a set X. Then (X, μ, ν) is $\mu\nu$-$T_{1/2}$ if and only if every $g\bigwedge_{\mu\nu}$-set is a $\bigwedge_\mu$-set.*

Proof. Let (X, μ, ν) be $\mu\nu$-$T_{1/2}$. Suppose that there exists a $g\bigwedge_{\mu\nu}$-set A in X which is not a $\bigwedge_\mu$-set. Then, there exists $x \in \bigwedge_\mu(A)$ such that $x \notin A$. Now since (X, μ, ν) is $\mu\nu$-$T_{1/2}$, $\{x\}$ is either ν-open or μ-closed. If $\{x\}$ is ν-open, then $A \subseteq X \setminus \{x\}$, where $X \setminus \{x\}$ is ν-closed. Since A is a $g\bigwedge_{\mu\nu}$-set, $\bigwedge_\mu(A) \subseteq X \setminus \{x\}$, and this is a contradiction. On the other hand, if $\{x\}$ is μ-closed then $A \subseteq X \setminus \{x\}$, where $X \setminus \{x\}$ is μ-open. Thus by Proposition 1.2, $\bigwedge_\mu(A) \subseteq \bigwedge_\mu(X \setminus \{x\}) = X \setminus \{x\}$. This is again a contradiction. Thus, every $g\bigwedge_{\mu\nu}$-set is a $\bigwedge_\mu$-set.

Conversely, assume that every $g\bigwedge_{\mu\nu}$-set is a $\bigwedge_\mu$-set. Suppose that (X, μ, ν) is not $\mu\nu$-$T_{1/2}$. Then by Theorem 2.14, there exists a $\mu\nu g$-closed set A which is not ν-closed. Since A is not ν-closed, there exists a point $x \in c_\nu(A)$ such that $x \notin A$. Thus, by Proposition 3.10, the singleton $\{x\}$ is either ν-open or $X \setminus \{x\}$ is a $g\bigwedge_{\mu\nu}$-set.

Case - 1: $\{x\}$ is ν-open: Then, since $x \in c_\nu(A)$, $x \in A$. This is a contradiction.

Case - 2: $X \setminus \{x\}$ is a $g\bigwedge_{\mu\nu}$-set: $\{x\}$ is either μ-closed or not μ-closed. If $\{x\}$ is not μ-closed, $X \setminus \{x\}$ is not μ-open and hence $\bigwedge_\mu(X \setminus \{x\}) = X$. Therefore, $X \setminus \{x\}$ is not a $\bigwedge_\mu$-set, which is a contradiction. If $\{x\}$ is μ-closed, then $A \subseteq X \setminus \{x\} \in \mu$ and A is $\mu\nu g$-closed. Hence, $c_\nu(A) \subseteq X \setminus \{x\}$ (by Definition 2.8). Thus, $x \notin c_\nu(A)$, which is a contradiction. □

References

[1] Andrijević, D., On b-open sets. Mat. Vesnik, 48 (1996), 59-64.

[2] Arenas, F. G., Dontchev, J., Ganster, M., On λ-sets and the dual of generalized continuity. Questions Answers Gen. Topology, 15 (1997), 3-13.

[3] Caldas, M., Dontchev, J., $G.\bigwedge_s$-sets and $G.\bigvee_s$-sets. Mem. Fac. Sci. Kochi Univ. Ser, Math., 21 (2000), 21-30.

[4] Caldas, M., Ganster, M., Georgiou, D. N., Jafari, S., Popa, V., On generalizations of closed sets. Kyungpook J. Math., 47 (2007), 155-164.

[5] Caldas, M., Georgiou, D. N., Jafari, S., Noiri, T., On $(\bigwedge, \theta)$-closed sets. Questions Answers Gen. Topology, 23 (1) (2005), 69-87.

[6] Caldas, M., Georgiou, D. N., Jafari, S., Study of $(\bigwedge, \alpha)$-closed sets and the related notions in topological spaces. Bull. Malays. Math. Sci. Soc. (2) 30 (1) (2007), 23-36.

[7] Caldas, M., Jafari, S., Generalized $\bigwedge_\delta$-sets and related topics. Georgian Math. J., 16 (2) (2009), 247-256.

[8] Caldas, M., Jafari, S., Noiri, T., On $\bigwedge_b$ sets and the associated topology $\tau^{\bigwedge_b}$. Acta Math. Hungar., 110 (2006), 337-345.

[9] Cammaroto, F., Noiri, T., On $\bigwedge_m$-sets and related topological spaces. Acta Math. Hungar., 109 (2005), 261-279.

[10] Császár, Á., Generalized topology, generalized continuity. Acta Math. Hungar., 96 (2002), 351-357.

[11] Császár, Á., Generalized open sets in generalized topologies. Acta Math. Hungar., 106 (2005), 53-66.

[12] Császár, Á., δ-and θ-modifications of generalized topologies. Acta Math. Hungar., 120 (2008), 275-279.

[13] Dontchev, J., Maki, H., On sg-closed sets and semi-λ-closed sets. Questions Answers Gen. Topology, 15 (1997), 259-266.

[14] Ekici, E., Roy, B., New generalized topologies on generalized topological spaces due to Császár. Acta Math. Hungar., 132 (2011), 117-124.

[15] Ganster, M., Jafari, S., Noiri, T. On pre-$\bigwedge$-sets and pre-$\bigvee$-sets. Acta Math. Hungar., 95 (2002), 337-343.

[16] Georgiou, D. N., Jafari, S., Noiri, T., Properties of $(\bigwedge, \delta)$-closed sets in topological spaces. Boll. Un. Mat. Ital. Sez. B Artic. Ric. Mat., (8) 7 (3)(2004), 745-756.

[17] Levine, N., Semi-open sets and semi-continuity in topological spaces. Amer. Math. Monthly, 70 (1963), 36-41.

[18] Di Maio, G., and Noiri, T., On s-closed spaces. Indian J. Pure Appl. Math., 18 (3) (1987), 225-233.

[19] Maki, H., Generalized $\bigwedge$-sets and the associated closure operator. The special issue in commemoration of Prof. Kazusada IKEDA's retirement, 1986, 139-146.

[20] Mashhour, A.S., Abd-El Monsef, M. E., El-Deeb, S. N., On precontinuous and weak precontinuous mappings. Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.

[21] Njåstad, O., On some classes of nearly open sets. Pacific J. Math., 15 (1965), 961-970.

[22] Sanabria, J., Rosas, E., Carpintero, C., The further unified theory for modifications of λ-closed sets and $g\bigwedge$-sets using minimal structures. Rend. Circ. Mat. Palermo, 58 (2009), 453-465.

[23] Veličko, N. V., H-closed topological spaces. Mat. Sb., 70 (1966), 98-112.

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