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ENDO-PRINCIPALLY PROJECTIVE MODULES

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Abstract. Let R be an arbitrary ring with identity and M a right R-module with $S = \mathrm{End}_R(M)$. In this paper, we introduce a class of modules that is a generalization of principally projective (or simply p.p.) rings and Baer modules. The module M is called *endo-principally projective* (or simply *endo-p.p.*) if for any $m \in M$, $l_S(m) = Se$ for some $e^2 = e \in S$. For an endo-p.p. module M, we prove that M is endo-rigid (resp., endo-reduced, endo-symmetric, endo-semicommutative) if and only if the endomorphism ring S is rigid (resp., reduced, symmetric, semicommutative), and we also prove that the module M is endo-rigid if and only if M is endo-reduced if and only if M is endo-symmetric if and only if M is endo-semicommutative if and only if M is abelian. Among others we show that if M is abelian, then every direct summand of an endo-p.p. module is also endo-p.p.

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1. Introduction

Throughout this paper R denotes an associative ring with identity, and modules will be unitary right R-modules. For a module M, $S = \mathrm{End}_R(M)$ denotes the ring of right R-module endomorphisms of M. Then M is a left S-module, a right R-module and an (S, R)-bimodule. In this work, for any of the rings T and R and any (T, R)-bimodule M, $r_R(.)$ and $l_M(.)$ denote the right annihilator of a subset of M in R and the left annihilator of a subset of R in M, respectively. Similarly, $l_T(.)$ and $r_M(.)$ will be the left annihilator of a subset of M in T and the right annihilator of a subset of T in M, respectively. A ring is *reduced* if it has no nonzero nilpotent elements. Recently, the reduced ring concept has been extended to modules by Lee and Zhou in [12], that is, a module M is called *reduced* if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. The module M is called *endo-semicommutative* if for any

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$f \in S$ and $m \in M$, $fm = 0$ implies $fSm = 0$, this class of modules is called S-semicommutative in [3]. *Baer rings* [10] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. A ring R is said to be *quasi-Baer* [7] if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. A ring R is called *right principally quasi-Baer* [5] if the right annihilator of a principal right ideal of R is generated by an idempotent. According to Rizvi and Roman [17], M is called a *Baer (resp. quasi-Baer) module* if for all R-submodules (resp. fully invariant R-submodules) N of M, $l_S(N) = Se$ with $e^2 = e \in S$. In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n.

2. Endo-Principally Projective Modules

Principally projective rings are introduced by Hattori [9] to study the torsion theory, that is, a ring R is called *left (right) p.p.* if every principal left (right) ideal is projective. The concept of left (right) p.p. rings has been comprehensively studied in the literature. In [12], Lee and Zhou introduced p.p. modules as follows: an R-module M is called *p.p.* if for any $m \in M$, $r_R(m) = eR$, where $e^2 = e \in R$. According to Baser and Harmanci [4], a module M is called *principally quasi-Baer* if for any $m \in M$, $r_R(mR) = eR$, where $e^2 = e \in R$. Motivated by these and the aforementioned definitions of Rizvi and Roman we give the following definition.

Definition 2.1. Let M be an R-module with $S = \mathrm{End}_R(M)$. The module M is called *endo-p.p.* if for any $m \in M$, $l_S(m) = Se$ for some $e^2 = e \in S$.

Note that a ring R is called *right (or left) p.p.* if every principal right (or left) ideal of R is a projective right (or left) R-module. Then, it is obvious that the module R is endo-p.p. if and only if the ring R is left p.p. It is clear that all Baer and quasi-Baer modules are endo-p.p.

Example 2.2. Let R be a Prüfer domain (i.e., a ring with an identity, no zero divisors and all finitely generated ideals are projective) and M the right R-module $R \oplus R$. By ([10], page 17) $S = \mathrm{End}_R(M)$ is isomorphic to the ring of 2×2 matrices over R, and it is a Baer ring. Hence M is Baer and so it is an endo-p.p. module.

Since $R \cong \mathrm{End}_R(R)$, the following example shows that endo-p.p. modules may not be quasi-Baer or Baer.

Example 2.3. ([6], Example 8.2) Consider the ring $S = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Let $T = \{(a_n)_{n=1}^{\infty} | a_n \text{ is eventually constant}\}$ and $I = \{(a_n)_{n=1}^{\infty} | a_n = 0 \text{ eventually}\}$. Then

$$R = \begin{bmatrix} T/I & T/I \\ 0 & T \end{bmatrix}$$

is a left p.p. ring which is neither right p.p. nor right principally quasi-Baer. It follows that R is an endo-p.p. module but not quasi-Baer or Baer.

Lemma 2.4. *If every cyclic submodule of M is a direct summand, then M is endo-p.p.*

Proof. Let $m \in M$. We prove $l_S(m) = Sf$ for some $f^2 = f \in S$. By hypothesis, $M = mR \oplus K$ for some submodule $K \leq M$. Let e denote the projection of M onto mR. It is easy routine to show that $l_S(m) = S(1-e)$. □

Note that the endomorphism ring of an endo-p.p. module may not be a right p.p. ring in general. For if M is an endo-p.p. module and $\varphi \in S$, then we have two cases. $Ker\varphi = 0$ or $Ker\varphi \neq 0$. If $Ker\varphi = 0$, then for any $f \in r_S(\varphi)$, $\varphi f = 0$ implies $f = 0$. Hence $r_S(\varphi) = 0$. Assume that $Ker\varphi \neq 0$. There exists a nonzero $m \in M$ such that $\varphi m = 0$. By hypothesis, $\varphi \in l_S(m) = Se$ for some $e^2 = e \in S$. In this case $\varphi = \varphi e$ and so $r_S(\varphi) \leq (1-e)S$. The following example shows that this inclusion is strict.

Example 2.5. Let Q be the ring and N the Q-module constructed by Osofsky in [13]. Since Q is commutative, we can just as well think of N as of a right Q-module. Let $S = \mathrm{End}_Q(N)$. By Lemma 2.4, N is an endo-p.p. module. Identify S with the ring $\begin{bmatrix} Q & 0 \\ Q/I & Q/I \end{bmatrix}$ in the obvious way, and consider $\varphi = \begin{bmatrix} 0 & 0 \\ 1+I & 0 \end{bmatrix} \in S$. Then $r_S(\varphi) = \begin{bmatrix} I & 0 \\ Q/I & Q/I \end{bmatrix}$. This is not a direct summand of S because I is not a direct summand of Q. Therefore, S is not a right p.p. ring.

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$. Abelian modules are introduced in the context of categories by Roos in [19] and studied by Goodearl and Boyle [8], Rizvi and Roman [18]. A module M is called *abelian* if for any $f \in S$, $e^2 = e \in S$, $m \in M$, we have $fem = efm$. Note that M is an abelian module if and only if S is an abelian ring. Recall that M is called *a duo module* [14] if every submodule N of M is fully invariant, i.e., $f(N) \leq N$ for all $f \in S$. Note that for a duo module M, if e is an idempotent and f is an element in S, then $(1-e)fem = 0 = ef(1-e)m$ for every $m \in M$. Thus every duo module is abelian.

Theorem 2.6. *Consider the following conditions for an R-module M.*
(1) *M is an endo-p.p. module.*
(2) *The left annihilator in S of every finitely generated R-submodule of M is generated (as a left ideal) by an idempotent.*
Then (2) ⇒ (1). *If M is duo, also* (1) ⇒ (2).

Proof. (2) ⇒ (1) Clear by definitions.
(1) ⇒ (2) Assume that M is a duo module and let N be a finitely generated R-submodule of M. By induction we may assume $N = m_1R + m_2R$. So $l_S(m_1R) = Se_1$ and $l_S(m_2R) = Se_2$ where $e_1^2 = e_1$, $e_2^2 = e_2 \in S$. Then $l_S(N) = (Se_1) \cap (Se_2)$. Clearly, $l_S(N) \subseteq Se_1e_2$. Let $ge_1e_2 \in Se_1e_2$. Since m_1R is fully invariant, $ge_1e_2N = ge_1e_2m_1R \leq ge_1m_1R = 0$. Hence $Se_1e_2 \subseteq l_S(N)$. Thus $l_S(N) = Se_1e_2$. Similarly, $l_S(N) = Se_2e_1$. And we have $Se_1e_2 = Se_2e_1$. So $e_1e_2 = fe_2e_1$ for some $f \in S$. Hence

$$e_1e_2 = e_1e_2e_1. \tag{1}$$

Similarly,

$$e_2e_1 = e_2e_1e_2 \tag{2}$$

Replacing (2) in (1) we obtain that e_1e_2 is an idempotent. This completes the proof. □

Proposition 2.7. *Let M be an abelian module and N a direct summand of M with S' =End$_R(N)$. If M is an endo-p.p. module, then N is also endo-p.p.*

Proof. Let N be a direct summand of M and $n \in N$. There exists $e^2 = e \in S$ with $l_S(n) = Se$. Since N is a direct summand of M and M is abelian, N is a fully invariant submodule of M. It follows that $eN \leq N$. Then the restriction $e' = e_{|N}$ belongs to S'. We claim that $l_{S'}(n) = S'e'$. Let $f \in l_{S'}(n)$. We extend f to $g = f \oplus 0 \in S$. Then $g \in l_S(n)$ and so $g = ge$. Hence $f = g_{|N} = (ge)_{|N} = fe' \in S'e'$. Thus $l_{S'}(n) \subseteq S'e'$. The reverse inclusion is clear. □

Let M be an R-module with $S = \mathrm{End}_R(M)$. The module M is called *endo-principally quasi-Baer* if for any $m \in M$, $l_S(Sm) = Se$ for some $e^2 = e \in S$, this class of modules is called principally quasi-Baer in [20]. Then the following lemma is obvious.

Lemma 2.8. *Consider the following conditions for an R-module M.*
(1) M is a Baer module.
(2) M is a quasi-Baer module.
(3) M is an endo-p.p. module.
(4) M is an endo-principally quasi-Baer module.
Then (1) ⇒ (2) ⇒ (4). If M is an endo-semicommutative module, then (2) ⇒ (1), (2) ⇒ (3) and (3) ⇔ (4).

3. Applications

If R is a ring, then some properties of R-modules do not characterize the ring R, namely there are reduced R-modules but R need not be reduced and there are abelian R-modules but R need not be an abelian ring. Because of that endo-reduced modules, endo-rigid modules, endo-symmetric modules, and endo-semicommutative modules are studied by the present authors in recent papers (see [2]). Our next endeavor is to investigate relationships between endo-reduced, endo-rigid, endo-symmetric, endo-semicommutative and abelian modules by using endo-p.p. modules.

Lemma 3.1. *Let M be an R-module. If M is an endo-semicommutative module, then S is a semicommutative ring. The converse holds if M is an endo-p.p. module.*

Proof. The first statement is from [2, Proposition 2.20]. Conversely, assume that M is an endo-p.p. module and S is a semicommutative ring. Let $fm = 0$ for $f \in S$ and $m \in M$. Since M is an endo-p.p. module, there exists $e^2 = e \in S$ such that $l_S(m) = Se$. Since $fm = 0$, $f \in l_S(m) = Se$ and then $fg \in Seg$ for all $g \in S$. By assumption, S is an abelian ring and so e is central in S. Then $eg = ge$ for all $g \in S$. Hence $fg \in Sge \subseteq Se = l_S(m)$. Thus $fgm = 0$ for all $g \in S$. This completes the proof. □

Lemma 3.2. *If a module M is endo-semicommutative, then M is abelian. The converse holds if M is an endo-p.p. module.*

Proof. One way is clear because S semicommutative implies S abelian and so M is abelian. Suppose that M is an abelian and endo-p.p. module. Let $f \in S$, $m \in M$ with $fm = 0$. Then $f \in l_S(m)$. Since M is an endo-p.p. module, there exists an idempotent e in S such that $l_S(m) = Se$ and so $Sem = 0$ and $fe = f$. By supposition, $eSm = 0$. Then $feSm = fSm = 0$. □

Recall that an R-module M is called *endo-reduced* if $fm = 0$ implies that $Imf \cap Sm = 0$ for each $f \in S$, $m \in M$, this class of modules is called reduced in [2]. Following the definition of a reduced module in [12] and [16], M is endo-reduced if and only if $f^2m = 0$ implies $fSm = 0$ for each $f \in S$, $m \in M$. Also, an R-module M is called *endo-rigid* [2] if for any $f \in S$ and $m \in M$, $f^2m = 0$ implies $fm = 0$. In this direction we have the following result.

Lemma 3.3. *If M is an endo-reduced module, then S is a reduced ring. The converse holds in case M is an endo-p.p. module.*

Proof. The first statement is from [2, Lemma 2.11 and Proposition 2.14]. Conversely, assume that M is an endo-p.p. module and S is a reduced ring. Then in particular S is an abelian ring. Let $fm = 0$ for $f \in S$ and $m \in M$, and $fm' = gm \in fM \cap Sm$. We may find an idempotent e in S such that $f \in l_S(m) = Se$. By assumption, e is central in S. So $f = fe = ef$. Multiplying $fm' = gm$ from the left by e, we have $fm' = egm = gem = 0$. Hence $fM \cap Sm = 0$. Thus M is endo-reduced. □

Lemma 3.4. *If a module M is endo-reduced, then it is endo-semicommutative. The converse is true if M is endo-p.p.*

Proof. Similar to the proof of Lemma 3.3. □

Lemma 3.5. *If M is an endo-rigid module, then S is a reduced ring. The converse holds if M is an endo-p.p. module.*

Proof. The first statement is from [2, Lemma 2.20]. Conversely, assume that M is an endo-p.p. module and S is a reduced ring. Let $f^2m = 0$ for $f \in S$ and $m \in M$. Since M is an endo-p.p. module, there exists $e^2 = e \in S$ such that $f \in l_S(fm) = Se$. Then $efm = 0$ and $f = fe$. By assumption, S is an abelian ring and so e is central in S. Then $fm = fem = efm = 0$. Hence M is an endo-rigid module. □

We now give a relation between endo-reduced modules and endo-rigid modules.

Lemma 3.6. *If M is an endo-reduced module, then M is an endo-rigid module. The converse holds if M is endo-p.p.*

Proof. The first statement is from [2, Lemma 2.14]. Conversely, let M be an endo-p.p. and endo-rigid module. Assume that $fm = 0$ for $f \in S$ and $m \in M$. Then there exists $e^2 = e \in S$ such that $f \in l_S(mR) = Se$. By Lemma 3.5, e is central in S and $fe = ef = f$ and $em = 0$. Let $fm' = gm \in fM \cap Sm$. Then $efm' = fm' = gem = 0$. Therefore M is endo-reduced. □

According to Lambek, a ring R is called *symmetric* [11] if whenever $a, b, c \in R$ satisfy $abc = 0$ implies $cab = 0$. A module M is called *symmetric* ([11] and [15]) if whenever $a, b \in R$, $m \in M$ satisfy $mab = 0$, we have $mba = 0$. Symmetric R-modules are also studied in [1] and [16]. In our case, we have the following.

Definition 3.7. Let M be an R-module with $S = \mathrm{End}_R(M)$. The module M is called *endo-symmetric* if for any $m \in M$ and f, $g \in S$, $fgm = 0$ implies $gfm = 0$.

Lemma 3.8. *If M is an endo-symmetric module, then S is a symmetric ring. The converse holds if M is an endo-p.p. module.*

Proof. Let $f, g, h \in S$ and assume $fgh = 0$. Then $fghm = 0$ for all $m \in M$. By hypothesis, $hfgm = 0$ for all $m \in M$. Hence $hfg = 0$. Conversely, assume that M is an endo-p.p. module and S is a symmetric ring. Let $fgm = 0$. There exists $e^2 = e \in S$ such that $f \in l_S(gm) = Se$. Then $f = fe$ and $egm = 0$. Similarly, there exists an idempotent $e_1 \in S$ such that $eg \in l_S(m) = Se_1$. Hence $eg = ege_1$ and $e_1m = 0$. By hypothesis, $Se_1m = 0$ implies $e_1Sm = 0$ and so $ege_1Sm = egSm = 0$. Thus $0 = egfm = gfem = gfm$. □

Lemma 3.9. *If M is endo-symmetric, then M is endo-semicommutative. The converse is true if M is an endo-p.p. module.*

Proof. Let $f \in S$ and $m \in M$ with $fm = 0$. Then for all $g \in S$, $gfm = 0$ implies $fgm = 0$. So $fSm = 0$. Conversely, let $f, g \in S$ and $m \in M$ with $fgm = 0$. Then $f \in l_S(gm) = Se$ for some $e^2 = e \in S$. So $f = fe$ and $egm = 0$. Since M is endo-semicommutative, $egSm = 0$. Therefore $gfm = gfem = gefm = egfm = 0$ because e is central. □

Lemma 3.10. *If M is an endo-reduced module, then M is endo-symmetric. The converse holds if M is an endo-p.p. module.*

Proof. The first statement is from [2, Lemma 2.18]. Conversely, let $f \in S$ and $m \in M$ with $f^2m = 0$. Then $f \in l_S(fm) = Se$ for some $e^2 = e \in S$. So $f = fe$ and $efm = 0$. By Lemma 3.9, M is endo-semicommutative, and so $efSm = 0$. Then $fgm = fegm = efgm = 0$ for any $g \in S$. Therefore $fSm = 0$. □

The next example shows that the reverse implication of the first statement in Lemma 3.10 is not true in general, i.e., there exists an endo-symmetric module which is neither endo-reduced nor endo-p.p. and nor endo-rigid.

Example 3.11. Consider a ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

and a right R-module

$$M = \left\{ \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $f \in S$ and $f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$. Multiplying the latter by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix}$. Similarly, let $g \in S$ and $g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ c' & d' \end{bmatrix}$. Then $g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c' \end{bmatrix}$.

For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $g \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix}$. Then it is easy to check that for any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$,

$$fg \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = f \begin{bmatrix} 0 & ac' \\ ac' & ad' + bc' \end{bmatrix} = \begin{bmatrix} 0 & ac'c \\ ac'c & ad'c + adc' + bc'c \end{bmatrix}$$

and,

$$gf \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = g \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix} = \begin{bmatrix} 0 & acc' \\ acc' & acd' + ac'd + bcc' \end{bmatrix}$$

Hence $fg = gf$ for all f, $g \in S$. Therefore S is commutative and so M is endo-symmetric. Define $f \in S$ by $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$, where $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$. Then $f \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $f^2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 0$. Hence M is neither endo-reduced nor endo-rigid. If $m = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $l_S(m) \neq 0$ since the endomorphism f defined preceding belongs to $l_S(m)$. M is indecomposable as a right R-module, therefore S does not have any idempotents other than zero and identity. Hence $l_S(m)$ can not be generated by an idempotent as a left ideal of S.

We now summarize the relations between endo-rigid, endo-reduced, endo-symmetric and endo-semicommutative modules and their endomorphism rings by using endo-p.p. modules.

Theorem 3.12. *If M is an endo-p.p. module, then we have the following.*
(1) *M is an endo-rigid module if and only if S is a reduced ring.*
(2) *M is an endo-reduced module if and only if S is a reduced ring.*
(3) *M is an endo-symmetric module if and only if S is a symmetric ring.*
(4) *M is an endo-semicommutative module if and only if S is a semicommutative ring.*

Proof. (1) Lemma 3.5, (2) Lemma 3.3, (3) Lemma 3.8, (4) Lemma 3.1. □

We wind up the paper with some observations concerning relationships between endo-reduced modules, endo-rigid modules, endo-symmetric modules, endo-semicommutative modules and abelian modules by using endo-p.p. modules.

Theorem 3.13. *If M is an endo-p.p. module, then the following conditions are equivalent.*
(1) *M is an endo-rigid module.*
(2) *M is an endo-reduced module.*
(3) *M is an endo-symmetric module.*
(4) *M is an endo-semicommutative module.*
(5) *M is an abelian module.*

Proof. (1) $\Leftrightarrow$ (2) Lemma 3.6. (2) $\Leftrightarrow$ (3) Lemma 3.10. (3) $\Leftrightarrow$ (4) Lemma 3.9. (4) $\Leftrightarrow$ (5) Lemma 3.2. □

We obtain the following well-known result as a direct consequence.

Corollary 3.14. *If R is a right p.p. ring, then the following conditions are equivalent.*
(1) *R is a reduced ring.*
(2) *R is a symmetric ring.*
(3) *R is a semicommutative ring.*
(4) *R is an abelian ring.*

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