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ON THE CRITICAL GROUP OF A FAMILY OF GRAPHS

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Abstract. The critical group is a subtle isomorphism invariant of the graph and closely connected with the graph Laplacian matrix. In this paper, the abstract structure of the critical group of a family of graphs $\mathcal{H}_n, n \geq 3$ is determined.

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1. Introduction

Let G be a finite multi-graph with n vertices. Let $A(G)$ and $D(G)$ be the adjacency and degree matrices of the graph G. Then, the Lapacian matrix $L(G)$ is defined as $L(G) = D(G) - A(G)$. The critical group of a graph G is closely related with the Lapacian matrix $L(G)$ as follows: thinking of $L(G)$ as a linear map $\mathbb{Z}^n \longrightarrow \mathbb{Z}^n$, its *cokernal* has the form $\text{coker}(G) = \frac{\mathbb{Z}^n}{L(G)\mathbb{Z}^n} \cong \mathbb{Z} \oplus K(G)$, where $K(G)$ is the *critical group* on G in the sense of isomorphism and the order of the critical group of a graph is equal to the number of spanning trees of the graph [3, 4, 10, 11, 14].

Let v_r be a vertex (called a root) of a graph G with n vertices. The critical group $K(G)$ of G is also the quotient group $\mathbb{Z}^n$ by the subgroup spanned by the n generators $\Delta_1, \ldots, \Delta_{r-1}, x_r, \Delta_{r+1}, \ldots, \Delta_n$, where $\Delta_i = d_i x_i - \sum\limits_{v_j \text{adjacent} v_i} a_{ij} x_j$ and $x_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$, whose unique nonzero entry 1 is in the position i, where $i = 1, 2, \ldots, n$. That is $K(G) = \frac{\mathbb{Z}^n}{\text{span}(\Delta_1, \ldots, \Delta_{r-1}, x_r, \Delta_{r+1}, \ldots, \Delta_n)}$. Notice that $K(G)$ is independent of the choice of v_r; for more details see [8].

The explicit determination of the structure of $K(G)$ in a given family of graphs is not always easy, and a series of paper whose goal is to explicitly determine the structure of the group $K(G)$ has appeared in the last ten year, see for example [1, 2, 6–9, 12, 13, 15–20].

We construct the family of graphs $\mathcal{H}_n$ by considering a cycle $C_{6n}:\ v_0, v_1, v_2, v_3, \ldots, v_{6n-1}, v_0$, where $n \geq 3$ and a new vertex v adjacent to n vertices $v_0, v_3, v_6, v_9, \ldots, v_{6n-2}$ of C_{6n}. This graph has order $6n + 1$ and size $7n$. The aim of this paper is to compute the structure of the critical group of this family of graphs $\mathcal{H}_n, n \geq 2$ by determine its Smith normal form.

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2. System of relations for the cokernel of the Laplacian of $\mathcal{H}_n$

In this section, we will first show that there are at most two generators for the critical group $K(\mathcal{H}_n)$ of the graph $\mathcal{H}_n$ and reduce the relation matrix to the special matrix B_n. Then, we will give some properties of the sequences concerning the entries of this matrix B_n.

Now, we work on the system of relations of the *cokernel* of the Laplacian of $\mathcal{H}_n$. Let $x_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{6n}$, whose unique nonzero 1 is in the position corresponding to the vertex v_i. Here we have chosen the vertex v as the root, such that $x_v = 0$. The relations of coker$L(\mathcal{H}_n)$ give rise to the following system of equations:

$$3x_{i-1} - x_i - x_{i-2} = 0; \qquad i \equiv 2 \pmod 6 \tag{2.1}$$

$$2x_{i-1} - x_i - x_{i-2} = 0; \qquad \text{otherwise} \tag{2.2}$$

Lemma 2.1. *There are two sequences (a_i) and (b_i) of integral numbers such that*

$$x_i = a_i x_2 - b_i x_1, \qquad 3 \le i \le 6n. \tag{2.3}$$

Moreover, the sequences have the following recurrence relations,

$$\begin{cases} a_i = 3a_{i-1} - a_{i-2}, & i \equiv 2 \ (mod\ 6) \\ a_i = 2a_{i-1} - a_{i-2}, & otherwise \\ b_i = 3b_{i-1} - b_{i-2}, & i \equiv 2 \ (mod\ 6) \\ b_i = 2b_{i-1} - b_{i-2}, & i \equiv 0, 3, 4, 5 \ (mod\ 6) \\ b_i = b_{i-1}, & i \equiv 1 \ (mod\ 6) \end{cases}$$

Proof. We know from the system of equations (2.1 & 2.2) that the group $K(\mathcal{H}_n)$ has at most 2 generators, i.e., each x_i can be expressed in terms of x_2 and x_1. So, there are at least $3n-2$ diagonal entries of Smith normal form of $L(\mathcal{H}_n)$ that are equal to 1, however, the remaining invariant factors of coker$(\mathcal{H}_n)$ hide inside the relations matrix induced by x_2 and x_1. Based on the structure of $\mathcal{H}_n$ and from equation 2.3, we have

$$B_n = \begin{pmatrix} a_{6n+1} & a_{6n} + 1 \\ b_{6n+1} + 1 & b_{6n} + 3 \end{pmatrix}. \tag{2.4}$$

From the above argument, one can reduce $L(\mathcal{H}_n)$ up to the equivalence $I_{6n-2} \oplus (B_n)$ by performing some row and column operations. Now, we only need to evaluate Smith normal form of the matrix B_n.

3. Analysis of the coefficients of the Smith normal form of B_n

In this section we will try to find the Smith normal form of B_n by calculating the diagonal entries. Let us define the following sequences of positive integers with the initial conditions, $\sigma_0 = 0,\ \sigma_1 = 1$

$$\begin{aligned} \sigma_m &= 8\sigma_{m-1} - \sigma_{m-2}, \\ \rho_m &= \sigma_m + \sigma_{m+1}, \\ r_m &= 241\sigma_{m-1} - 31\sigma_{m-2}, \\ s_m &= \sigma_{m+1} - \sigma_m, \\ t_m &= 6\sigma_m - \sigma_{m-1}. \end{aligned}$$

The following proposition is very easy to prove by induction.

Proposition 3.1.

- $2 \nmid r_m \quad \forall\ m,$
- $2 \nmid s_m \quad \forall\ m.$

Proposition 3.2. *The sequences r_m, s_m and t_m are relatively prime for each m i.e.,*

$$\gcd(r_m,\ s_m,\ t_m) = 1 \tag{3.1}$$

Proof. On contrary, suppose that there exists a prime p such that $p \mid r_m$, $p \mid s_m$ and $p \mid t_m$, then $p \mid \sigma_m$. Since $\sigma_m = 8\sigma_{m-1} - \sigma_{m-2} = s_m - t_m \Rightarrow p \mid \sigma_m$. Hence, we get $p \mid \sigma_{m-1}$ & $p \mid \sigma_{m-2}$. Again, we have, $\sigma_{m-1} = 8\sigma_{m-2} - \sigma_{m-3} \Rightarrow p \mid \sigma_{m-3}$ $\Rightarrow \cdots p \mid \sigma_{m-j} \cdots p \mid \sigma_1 = 1 \Rightarrow p = 1$, a contradiction, hence $(r_m,\ s_m,\ t_m) = 1$.

The Odd Case

Lemma 3.3. *If $n = 2m + 1$, then we have the following relation,*

$$\begin{aligned} a_{6n+1} &= 6s_m\rho_m \\ a_{6n} + 1 &= 6t_m\rho_m \\ b_{6n+1} + 1 &= 6t_m\rho_m \\ b_{6n} + 3 &= r_m\rho_m. \end{aligned}$$

Proof. It is easy to prove by induction.

Proposition 3.4.

$$\gcd(a_{6n+1},\ a_{6n} + 1,\ b_{6n+1} + 1,\ b_{6n} + 3) = \rho_m. \tag{3.2}$$

Proof. By Lemma 3.3 and then Proposition 3.2, we have the desired result.

Proposition 3.5. *If $n = 2m + 1$, then*

$$\det B_n = 3\rho_m^2$$

where B_n is defined in equation (2.4).

Proof.

$$\begin{aligned}
\det B_n &= 6\rho_m^2(s_m r_m - 6t_m^2) \\
&= 6\rho_m^2\Big[(\sigma_m - \sigma_{m-1})^2 - 3\sigma_m\sigma_{m-1}\Big] \\
&= 6\rho_m^2\Big[(\sigma_{m-1} - \sigma_{m-2})^2 - 2\sigma_{m-1}\sigma_{m-2}\Big] \\
&\vdots \\
&= 6\rho_m^2\Big[(\sigma_2 - \sigma_1)^2 - 2\sigma_2\sigma_1\Big] \\
&= 6\rho_m^2
\end{aligned}$$

Theorem 3.6. *If $n = 2m+1$, then the critical group of $\mathcal{H}_n$ is the direct product of two cyclic groups i.e.,*

$$K(\mathcal{H}_n) = \mathbb{Z}_{\rho_m} \oplus \mathbb{Z}_{6\rho_m}$$

Proof. Since the matrix B_n has Smith normal form as $\mathrm{diag}(s_{11}, s_{22})$ and s_{11} equals to the greatest common divisor of all the entries of B_n. So, by Proposition 3.4, we have

$$s_{11} = \rho_m. \tag{3.3}$$

Also $s_{11}s_{22} = \det B_n$ and then by Proposition 3.5, we have

$$s_{11}s_{22} = 6\rho_m^2. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$s_{22} = \rho_m, \tag{3.5}$$

which competes the proof.

The Even Case

If $n = 2m$, and consider the following sequence of positive integers with initial conditions, $\rho_0 = -1,\ \rho_1 = 1$,

$$\begin{aligned}
\rho_m &= 8\rho_{m-1} - \rho_{m-2}, \quad \rho_m = \sigma_m + \sigma_{m-1}, \\
\lambda_m &= \frac{1}{2}[241\rho_{m-1} - 31\rho_{m-2}], \\
\mu_m &= \frac{1}{2}[7\rho_m - \rho_{m-1}], \\
\nu_m &= 6\rho_m - \rho_{m-1}.
\end{aligned}$$

Lemma 3.7. *If $n = 2m$, then we have the following relation,*

$$\begin{aligned}
a_{6n+1} &= 12\mu_m\sigma_m \\
a_{6n} + 1 &= 6\nu_m\sigma_m \\
b_{6n+1} + 1 &= 6\nu_m\sigma_m \\
b_{6n} + 3 &= 2\lambda_m\sigma_m.
\end{aligned}$$

Proof. It is easy to prove by induction.

Proposition 3.8. *The sequences μ_m, ν_m and λ_m are relatively prime for each m, i.e.*

$$\gcd(\mu_m, \nu_m \lambda_m) = 1 \tag{3.6}$$

Proof. One can prove this proposition by similar arguments as in the proof of Proposition 3.2.

Proposition 3.9.

$$\gcd(a_{6n+1},\ a_{6n}+1,\ b_{6n+1}+1,\ b_{6n}+3) = 2\sigma_m \tag{3.7}$$

Proof. By Lemma 3.7 and then Proposition 3.2, we have the desired result.

Proposition 3.10. *If $n = 2m$, then*

$$\det B_n = 21\sigma_m^2,$$

where B_n is defined in equation 2.4.

Proof.

$$\begin{aligned}
\det B_n &= (a_{6n+1})(b_{6n}+3) - (a_{6n}+1)^2 \\
&= 6\sigma_m^2 \Big[(\rho_m - \rho_{m-1})^2 - 6\rho_m\rho_{m-1}\Big] \\
&= 6\sigma_m^2 \Big[(\rho_{m-1} - \rho_{m-2})^2 - 6\rho_{m-1}\rho_{m-2}\Big] \\
&\ \vdots \\
&= 6\sigma_m^2 \Big[(\rho_2 - \rho_1)^2 - 3\rho_2\rho_1\Big] \\
&= 60\sigma_m^2
\end{aligned}$$

Theorem 3.11. *If $n = 2m$, then the critical group of $\mathcal{H}_n$ is the direct product of two cyclic groups, i.e.*

$$K(\mathcal{H}_n) = \mathbb{Z}_{2\sigma_m} \oplus \mathbb{Z}_{30\sigma_m}$$

Proof. By Proposition 3.9 and Proposition 3.10, we have the desired result.

Proposition 3.12. *For each $m, n \geq 1$ we have*

$$\sigma_{m+n} = \sigma_{m+1}\sigma_n - \sigma_m\sigma_{n-1} \quad \textit{and} \quad \rho_{m+n} = \sigma_{m+1}\rho_n - \sigma_m\rho_{n-1}$$

Proof. Set $\mathcal{A} := \begin{pmatrix} 8 & -1 \\ 1 & 0 \end{pmatrix}$ we have that

$$\mathcal{A}^m := \begin{pmatrix} 8 & -1 \\ 1 & 0 \end{pmatrix}^m = \begin{pmatrix} \sigma_{m+1} & -\sigma_m \\ \sigma_m & -\sigma_{m-1} \end{pmatrix}$$

Since $\mathcal{A}^{m+n-1} = \mathcal{A}^m \mathcal{A}^{n-1}$, we have

$$\begin{pmatrix} \sigma_{m+n} & -\sigma_{m+n-1} \\ \sigma_{m+n-1} & -\sigma_{m+n-2} \end{pmatrix} = \begin{pmatrix} \sigma_{m+1} & -\sigma_m \\ \sigma_m & -\sigma_{m-1} \end{pmatrix} \begin{pmatrix} \sigma_n & -\sigma_{n-1} \\ \sigma_{n-1} & -\sigma_{n-2} \end{pmatrix}$$

Comparing the top left entry in the left-hand side with the corresponding in the right-hand side gives the first equality. For the second identity, we use the first one and the identity $\rho_m = \sigma_m + \sigma_{m-1}$

Theorem 3.13.

- *If a and b are both even and $a \mid b$, then $\sigma_a \mid \sigma_b$*
- *If a and b are both odd and $a \mid b$, then $\rho_a \mid \rho_b$*
- *If a is odd and b is even and $a \mid b$, then $\rho_a \mid \sigma_b$*
Moreover, we have that $\det(A_a)$ *divides* $\det(A_b)$.

Proof. For the first statement, we prove by induction on t that σ_a divides σ_{at}. This is true if $t = 0$. If σ_a divides σ_{at}, hence we have $\sigma_{a(t+1)} = \sigma_{at+1}\sigma_a - \sigma_{at}\sigma_{a-1}$. The inductive hypothesis implies that σ_a divides second term, hence it also divides $\sigma_{a(t+1)}$.

Now we prove that $2a+1$ divides $2b+1$, then ρ_a divides ρ_b. First notice that, by Proposition 3.12, we have $\sigma_{2m+1} = \sigma_{m+1}^2 - \sigma_m^2$. Let $2b+1 = (2a+1)(2t+1)$. We prove by induction on t that ρ_a divides $\rho_b = \rho_{2at+a+t}$. This is true if $t = 0$, if ρ_a divides $\rho_{2at+a+t}$, we have $\rho_{2a(t+1)+a+(t+1)} = \rho_{(2a+1)+(2at+a+t)} = \sigma_{2a+2}\rho_{2at+a+t} - \sigma_{2a+1}\rho_{2at+a+t-1}$. The first term is a multiple of ρ_a by induction hypothesis. Moreover, we have $\sigma_{2a+1} = \sigma_{a+1}^2 - \sigma_a^2 = (\sigma_{a+1} + \sigma_a)(\sigma_{a+1} - \sigma_a) = \rho_a(\sigma_{a+1} - \sigma_a)$. Hence the second term is a multiple of ρ_a.

Finally, we prove that if $2a + 1$ divides $2b$, then ρ_a divides σ_b.Let $2b = (2a+1)2t$. We have to prove that ρ_a divides $\sigma_{(2a+1)t}$. As we have already seen, ρ_a divides σ_{2a+1}. Moreover, we have that σ_{2a+1} is a divisor of $\sigma_{(2a+1)t} = \sigma_b$. By these facts and being $\det(A_a)$, we have the second statement.

By the statements verifying during the proof of Theorem 3.13, one can see that for a dividing b, each entry of the matrix (A_a), divides the corresponding on in the matrix $\det(A_b)$. This leads to the following theorem.

Theorem 3.14. *If $a \mid b$, then the critical group of $\mathcal{H}_a$ is isomorphic to a subgroup of the critical group of $\mathcal{H}_b$.*

4. The tree number

Let G be a graph, then the tree number $k(G)$ is equal to the number of spanning trees of the graph G. In this section, we will give the closed formula for the number of spanning trees for the graph $\mathcal{H}_n$, we refer [5] for the terminologies.

Proposition 4.1. *[5]*
Let G be a nearly regular graph of degree r and H be its subgraph obtained by removing the exceptional vertex, then

$$k(G) = P_H(r),$$

where $P_H(t)$ is the characteristic polynomial of the graph H.

Remark 4.2. The wheel graph W_n can be obtained from a cycle C_n by adding a new vertex connected by an edge to all vertices of C_n. Hence, W_n is nearly regular graph and by proposition 4.1, we get

$$k(W_n) = P_{C_n}(r).$$

The characteristic polynomial of a cycle C_n is given as

$$P_{C_n}(t) = 2T_n\left(\frac{t}{2}\right) - 2, \tag{4.1}$$

where

$$T_n(t) = \frac{n}{2}\sum_{m}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{n-m}\left(\begin{array}{c} n-m \\ m \end{array}\right)(2t)^{n-2m}$$

is the Chebyshev polynomial of the first kind. It is easy to see that it gives the same number of spanning trees of wheel graph given by N. Biggs in [2].

A very interesting application of the Proposition 4.1 is given as follows. The inner dual planner graph G^{**} is the subgraph of the usual dual G^* obtained by deleting the vertex corresponding to the infinite region of the original planer graph.

Let G be a plane graph in which any finite region is bounded by a cycle of fixed length r. Then, G^* is a nearly regular graph, so we have the following result.

Proposition 4.3. *[5]*
Let G is a plane graph in which any bounded region is a cycle of length r, then

$$k(G) = P^{**}_G(r),$$

where $P^{**}_G(t)$ is the characteristic polynomial of the graph G^{**}.

Theorem 4.4. *The tree number for the graph $\mathcal{H}_n$ is*

$$k(\mathcal{H}_n) = P_{C_n}(8) = 2T_n(4) - 2, \tag{4.2}$$

where $T_n(t)$ is the Chebyshev polynomial of the first kind.

Proof. Since $\mathcal{H}_n$ is a plane graph in which any bounded region is bounded by a cycle of length 8 and total number of bounded regions is n. Hence, in this case the inner dual will be a cycle of length n and its characteristic polynomial is defined in equation (4.1), and it follows the result.

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