Novi Sad J. Math.
Vol. 42, No. 2, 2012, 89-102

HANKEL DETERMINANT FOR p-VALENT STARLIKE AND CONVEX FUNCTIONS OF ORDER α

D. Vamshee Krishna[1] **and T. Ramreddy**[2]

Abstract. The objective of this paper is to obtain an upper bound to the second Hankel determinant $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for p-valent starlike and convex functions of order α, using Toeplitz determinants.

AMS Mathematics Subject Classification (2010): 30C45, 30C50

Key words and phrases: Analytic function, p-valent starlike and convex functions, upper bound, second Hankel functional, positive real function, Toeplitz determinants

1. Introduction

Let A_p (p is a fixed integer ≥ 1) denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in N = \{1, 2, 3, ...\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions.

In 1976, Noonan and Thomas [13] defined the q^{th} Hankel determinant of f for $q \geq 1$ and $n \geq 1$, which is stated by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

This determinant has been considered by several authors. For example, Noor [14] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with a bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [9]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in$ S. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$

[1]School of Sciences and Humanities, K L University, Green Fields, Vaddeswaram- 522 502, Guntur Dt., Andhra Pradesh, India, e-mail: vamsheekrishna1972@gmail.com
[2]Department of Mathematics, Kakatiya University, Warangal - 506009, Andhrapradesh, India, e-mail: reddytr2@yahoo.com

to the class of strongly starlike functions of order $\alpha(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as second Hankel determinant

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2|. \tag{1.3}$$

Janteng, Halim and Darus [8] have considered the functional$|a_2a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor [10]. In their work, they have shown that if $f \in$ RT then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. They [7] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and showed that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Mishra and Gochhayat [11] have obtained the sharp bound to the non-linear functional $|a_2a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \frac{\pi}{2})$, by making use of the fractional differential operator due to Owa and Srivastava [15]. They have shown that, if $f \in R_\lambda(\alpha, \rho)$ then

$$|a_2a_4 - a_3^2| \leq \left\{ \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2 cos^2\alpha}{9} \right\}.$$

Murugusundaramoorthy and Magesh [12] have obtained a sharp upper bound for the functional $|a_2a_4 - a_3^2|$ for the function $f \in R(\alpha)$, where

$$R(\alpha) = \left[f(z) \in A : Re\left\{ (1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \right\} > 0, \alpha > 0, \forall z \in E \right].$$

They have shown that if $f \in R(\alpha)$ then $|a_2a_4 - a_3^2| \leq \left\{ \frac{4}{(1+2\alpha)^2} \right\}$. Recently, Al-Refai and Darus [3] have obtained a sharp upper bound to the second Hankel determinant $|a_2a_4 - a_3^2|$ for the functions in the class denoted by $R_{\alpha,\beta}(\lambda, \rho)(0 \leq \alpha < 1, 0 \leq \beta < 1, -\frac{\pi}{2} < \lambda < \frac{\pi}{2}$and $0 \leq \rho \leq 1)$, defined as

$$R_{\alpha,\beta}(\lambda, \rho) = \left[f(z) \in A : Re\{e^{i\lambda}\frac{\Theta^{\alpha,\beta} f(z)}{z}\} > \rho\cos\lambda, \quad \forall z \in E \right],$$

where $\Theta^{\alpha,\beta}$ is the generalized Owa-Srivastava differential operator. They have shown that if $f \in R_{\alpha,\beta}(\lambda, \rho)$ then

$$|a_2a_4 - a_3^2| \leq \left\{ \frac{(1-\rho)^2(2-\alpha)^2(3-\alpha)^2(2-\beta)^2(3-\beta)^2 cos^2\lambda}{324} \right\}.$$

Very recently, Abubaker and Darus [1] have obtained a sharp upper bound to the non-linear functional $|a_2a_4 - a_3^2|$ for a new subclass of analytic functions denoted by $R_{\alpha,\mu}(\sigma, \rho)(0 \leq \mu \leq \alpha \leq 1, \rho, \sigma \in N_0)$, defined as

$$R_{\alpha,\mu}(\sigma, \rho) = \left[f(z) \in A : Re\left\{ (D^{\sigma,\rho}_{\alpha,\mu} f(z))' \right\} > 0, \text{forall z} \in E \right]$$

by making use of the linear differential operator $D^{\sigma;\rho}_{\alpha,\mu}$, defined by them. In their work they have shown that

$$|a_2a_4 - a_3^2| \leq \left\{ \frac{16}{9(1+\rho)^2 2(1+\rho)^2(1+2\alpha-2\mu+6\alpha\mu)^{2\sigma}} \right\}.$$

Motivated by the above mentioned results obtained by different authors in this direction, in this paper, we obtain an upper bound to the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function f belonging to p-valent starlike and convex functions, defined as follows.

Definition 1.1. A function $f(z) \in A_p$ is said to be p-valent starlike function $(\frac{f(z)}{z} \neq 0)$, if it satisfies the condition

$$Re\left\{\frac{zf'(z)}{pf(z)}\right\} > 0, \qquad \forall z \in E. \tag{1.4}$$

The set of all these functions is denoted by ST_p. It is observed that for $p = 1$, ST_p reduces to ST.

Definition 1.2. A function $f(z) \in A_p$ is said to be p-valent convex function, if it satisfies the condition

$$Re\left\{\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0, \qquad \forall z \in E. \tag{1.5}$$

The class of all these functions is denoted by CV_p. It is observed that for p=1, we obtain $CV_1 = CV$.

Definition 1.3. A function $f(z) \in A_p$ is said to be p-valent starlike function of order $\alpha(0 \leq \alpha < p)$ $(\frac{f(z)}{z} \neq 0)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad \forall z \in E. \tag{1.6}$$

The class of all these functions was introduced by Goodman [5] and denoted by $ST_p(\alpha)$. It is observed that for p $= 1$, $ST_p(\alpha)$ reduces to $ST(\alpha)$, class of starlike functions of order $\alpha(0 \leq \alpha < 1)$ and for p $= 1$ and $\alpha = 0$, we obtain $ST_1(0) = ST$.

Definition 1.4. A function $f(z) \in A_p$ is said to be p-valent convex function of order $\alpha(0 \leq \alpha < p)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad \forall z \in E. \tag{1.7}$$

The class of all these functions is denoted by $CV_p(\alpha)$. It is observed that for p $= 1$, we get $CV_p(\alpha) = CV(\alpha)$, class of convex functions of order $\alpha(0 \leq \alpha < 1)$ and for p $= 1$ and $\alpha = 0$, we obtain $CV_1(0) = CV$. From the relations (1.6) and (1.7), we observe that $f(z) \in CV_p(\alpha)$ if and only if $\frac{zf'(z)}{p} \in ST_p(\alpha)$. Further, we have $ST_p(\alpha) \subseteq ST_p(0)$, $CV_p(\alpha) \subseteq CV_p(0)$ and $CV_p(\alpha) \subset ST_p(\alpha) \subset A_p$, for $0 \leq \alpha < p$.

We first state some preliminary lemmas required for proving our results.

2. Preliminary Results

Let P denote the class of functions p analytic in E for which $\text{Re}\{p(z)\} > 0$,

$$p(z) = (1 + c_1 z + c_2 z^2 + c_3 z^3 + ...) = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \forall z \in E. \tag{2.1}$$

Lemma 2.1 ([16]). *If* $p \in \mathrm{P}$*, then* $|c_k| \leq 2$*, for each* $k \geq 1$*.*

Lemma 2.2 ([6]). *The power series for p given in (2.1) converges in the unit disc E to a function in* P *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$*, are all non-negative. They are strictly positive except for* $p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z)$*,* $\rho_k > 0$*,* t_k *real and* $t_k \neq t_j$*, for* $k \neq j$*; in this case* $D_n > 0$ *for* $n < (m-1)$ *and* $D_n \doteq 0$ *for* $n \geq m$*. This necessary and sufficient condition is due to Caratheodory and can be found in [6].*

We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4c_1^2] \geq 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \quad \text{for some x}, |x| \leq 1. \tag{2.2}$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2 \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\} \tag{2.4}$$

for some real value of z, with $|z| \leq 1$.

3. Main Results

Theorem 3.1. *If*

$$f(z) \in ST_p(\alpha) \ \left(0 \leq \alpha \leq \left(p - \frac{1}{2}\right)\right),$$

with $p \in N$, then

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq (p-\alpha)^2.$$

Proof. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be in the class $ST_p(\alpha)$, from Definition 1.3, there exists an analytic function $p \in P$ in the unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$\left\{\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)}\right\} = p(z) \tag{3.1}$$

$$\Rightarrow \{zf'(z) - \alpha f(z)\} = \{(p-\alpha)f(z)\}\, p(z).$$

Replacing $f(z)$, $f'(z)$ by their equivalent p-valent expressions and the equivalent expression for $p(z)$ in series in (3.1), we have

$$\left[z\left\{pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right\} - \alpha\left\{z^p + \sum_{n=p+1}^{\infty} a_n z^n\right\}\right]$$
$$= (p-\alpha) \times \left[\left\{z^p + \sum_{n=p+1}^{\infty} a_n z^n\right\} \times \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\right]$$

Upon simplification, we obtain

$$\begin{aligned} &[a_{p+1}z^{p+1} + 2a_{p+2}z^{p+2} + 3a_{p+3}z^{p+3} + ...] \\ &= (p-\alpha) \times [c_1 z^{p+1} + (c_2 + c_1 a_{p+1})\, z^{p+2} + (c_3 + c_2 a_{p+1} + c_1 a_{p+2})\, z^{p+3} + ...] \end{aligned} \tag{3.2}$$

Equating the coefficients of the like powers of z^{p+1}, z^{p+2} and z^{p+3} respectively on both sides of (3.2), we have

$$\begin{aligned} &[a_{p+1} = (p-\alpha)c_1; 2a_{p+2} = (p-\alpha)\{c_2 + c_1 a_{p+1}\}; \\ &\qquad 3a_{p+3} = (p-\alpha)\{c_3 + c_2 a_{p+1} + c_1 a_{p+2}\}] \end{aligned}$$

After simplifying, we get

$$\begin{aligned} &[a_{p+1} = (p-\alpha)c_1; a_{p+2} = \frac{(p-\alpha)}{2}\left\{c_2 + (p-\alpha)c_1^2\right\} \\ &\qquad a_{p+3} = \frac{(p-\alpha)}{6}\left\{2c_3 + 3(p-\alpha)c_1c_2 + (p-\alpha)^2 c_1^3\right\}] \end{aligned} \tag{3.3}$$

Considering the second Hankel functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function $f \in ST_p(\alpha)$ and substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from the relation (3.3), we have

$$|a_{p+1}a_{p+3}-a_{p+2}^2| = \left|(p-\alpha)c_1\times\frac{(p-\alpha)}{6}\left\{2c_3+3(p-\alpha)c_1c_2+(p-\alpha)^2c_1^3\right\} - \frac{(p-\alpha)^2}{4}\left\{c_2+(p-\alpha)c_1^2\right\}^2\right|$$

Upon simplification, we obtain

$$|a_{p+1}a_{p+3}-a_{p+2}^2| = \frac{(p-\alpha)^2}{12}\left|4c_1c_3-3c_2^2-(p-\alpha)^2c_1^4\right| \tag{3.4}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 in the right-hand side of (3.4), we have

$$\left|4c_1c_3-3c_2^2-(p-\alpha)^2c_1^4\right| = |4c_1\times\frac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\} - 3\times\frac{1}{4}\{c_1^2+x(4-c_1^2)\}^2-(p-\alpha)^2c_1^4|$$

After simplifying, we get

$$4\left|4c_1c_3-3c_2^2-(p-\alpha)^2c_1^4\right| = |\left\{1-4(p-\alpha)^2\right\}c_1^4+8c_1(4-c_1^2)z+ 2c_1^2(4-c_1^2)|x|-(c_1+2)(c_1+6)(4-c_1^2)z|x|^2| \tag{3.5}$$

Since $c_1\in[0,2]$, using the result $(c_1+a)(c_1+b)\geq(c_1-a)(c_1-b)$, where $a,b\geq 0$ in the relation (3.5), we get

$$4\left|4c_1c_3-3c_2^2-(p-\alpha)^2c_1^4\right| \leq |\left\{1-4(p-\alpha)^2\right\}c_1^4+8c_1(4-c_1^2)z+ 2c_1^2(4-c_1^2)|x|-(c_1-2)(c_1-6)(4-c_1^2)z|x|^2| \tag{3.6}$$

Choosing $c_1=c\in[0,2]$, applying Triangle inequality and replacing $|\ x\ |$ by μ in the right-hand side of (3.6), it reduces to

$$4\left|4c_1c_3-3c_2^2-(p-\alpha)^2c_1^4\right| \leq [\left\{4(p-\alpha)^2-1\right\}c^4+8c(4-c^2)+ 2c^2(4-c^2)\mu+(c-2)(c-6)(4-c^2)\mu^2] = F(c,\mu),\quad\text{for}\quad 0\leq\mu=|x|\leq 1 \tag{3.7}$$

where

$$F(c,\mu) = [\left\{4(p-\alpha)^2-1\right\}c^4+8c(4-c^2)+2c^2(4-c^2)\mu + (c-2)(c-6)(4-c^2)\mu^2] \tag{3.8}$$

We assume that the upper bound for (3.7) occurs at an interior point of the set $\{(\mu,c):\mu\in[0,1]\quad\text{and}\quad c\in[0,2]\}$.

Differentiating $F(c,\mu)$ in (3.8) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu}=[2c^2(4-c^2)+2(c-2)(c-6)(4-c^2)\mu] \tag{3.9}$$

For $0<\mu<1$ and for fixed c with $0<c<2$, from (3.9), we observe that $\frac{\partial F}{\partial \mu}>0$. Therefore, $F(c,\mu)$ is an increasing function of μ, which contradicts our assumption that the maximum value of it occurs at an interior point of the set $\{(\mu,c):\mu\in[0,1]\quad\text{and}\quad c\in[0,2]\}$. Also, for a fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1}F(c,\mu)=F(c,1)=G(c)(say). \tag{3.10}$$

Therefore, replacing μ by 1 in (3.8), upon simplification, we obtain

$$G(c)=4\left\{(p-\alpha)^2-1\right\}c^4+48 \tag{3.11}$$

$$G'(c)=16\left\{(p-\alpha)^2-1\right\}c^3 \tag{3.12}$$

$$G''(c)=48\left\{(p-\alpha)^2-1\right\}c^2 \tag{3.13}$$

For an optimum value of G(c), consider $G'(c)=0$. From (3.12), we get

$$16\left\{(p-\alpha)^2-1\right\}c^3=0.\Rightarrow\left\{(p-\alpha+1)(p-\alpha-1)c^3\right\}=0.$$

Since $\alpha<p\Rightarrow(p-\alpha+1)\neq0$. Therefore, we must have $(p-\alpha-1)c^3=0$. We now discuss the following cases.

Case 1. If $(p-\alpha)=1$ and for every $c\in[0,2]$, it is possible only when $p=1$ and $\alpha=0$, then we have $G'(c)=o$ and $G''(c)=0$. Therefore, in this case, we get $G(c)=48$, which is a constant. For these values i.e.,for $p=1$ and $\alpha=0$, from Definition 1.3, we obtain $ST_1(0)=ST$, for which the result can be found in [7].

Case 2. If $(p-\alpha)\neq1$ and $c=0$, then we get $G'(c)=o$ and $G''(c)=0$. In this case also, we obtain $G(c)=48$, which is a constant.

Therefore, From Cases 1 and 2, we conclude that the maximum value of $G(c)$ is 48, which occurs at $c=0$. From the expression (3.11), we get

$$G_{max}=G(0)=48. \tag{3.14}$$

From (3.7) and (3.14), upon simplification, we obtain

$$\left|4c_1c_3-3c_2^2-(p-\alpha)^2c_1^4\right|\le12 \tag{3.15}$$

From (3.4) and (3.15), after simplifying, we obtain

$$|a_{p+1}a_{p+3}-a_{p+2}^2|\le(p-\alpha)^2. \tag{3.16}$$

This completes the proof of the theorem. □

Remark.

1) For the choice of $p = 1$, from (3.16), we get

$$|a_2 a_4 - a_3^2| \leq (1-\alpha)^2 (0 \leq \alpha \leq \frac{1}{2}).$$

2) By choosing $p = 1$ and $\alpha = 0$, from (3.16), we obtain $|a_2 a_4 - a_3^2| \leq 1$. This inequality is sharp and it coincides with the result of Janteng, Halim and Darus [7].

Theorem 3.2. *If*

$$f(z) \in CV_p(\alpha)(0 \leq \alpha \leq \left(p - \frac{1}{2}\right)),$$

with $p \in N$, *then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2(p-\alpha)^2\left[6(p+1-\alpha)^2 + (p+1)(p+3)\left\{2\alpha(\alpha-2p)(p^2+4p+1) + (2p^4+8p^3+3p^2+4p+7)\right\}\right]}{(p+1)(p+2)^2(p+3)\left\{2\alpha(\alpha-2p)(p^2+4p+1)+(2p^4+8p^3+3p^2+4p+7)\right\}}.$$

Proof. Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be in the class $CV_p(\alpha)$, from Definition 1.4, there exists an analytic function $p \in P$ in the unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$\left\{\frac{\{f'(z) + zf''(z)\} - \alpha f'(z)}{(p-\alpha)f'(z)}\right\} = p(z) \tag{3.17}$$
$$\Rightarrow \{(1-\alpha)f'(z) + zf''(z)\} = (p-\alpha)\{f'(z)p(z)\}.$$

substituting the equivalent expressions for $f'(z)$, $f''(z)$ and $p(z)$ in series in the relation (3.17), we have

$$\left[(1-\alpha)\left\{pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right\} + z\left\{p(p-1)z^{p-2} + \sum_{n=p+1}^{\infty} n(n-1)a_n z^{n-2}\right\}\right]$$
$$= \left[(p-\alpha)\left\{pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right\} \times \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\right]$$

After simplifying, we get

$$[(p+1)a_{p+1}z^p + 2(p+2)a_{p+2}z^{p+1} + 3(p+3)a_{p+3}z^{p+2} + ...] \tag{3.18}$$
$$= (p-\alpha) \times [pc_1 z^p + \{pc_2 + (p+1)c_1 a_{p+1}\} z^{p+1} + \{pc_3 + (p+1)c_2 a_{p+1} + (p+2)c_1 a_{p+2}\} z^{p+2} + ...]$$

Equating the coefficients of like powers of z^p, z^{p+1} and z^{p+2} respectively on both sides of (3.18), upon simplification, we obtain

$$(3.19)\quad [a_{p+1} = \frac{p(p-\alpha)}{(p+1)}c_1; a_{p+2} = \frac{p(p-\alpha)}{2(p+2)}\left\{c_2 + (p-\alpha)c_1^2\right\};$$
$$a_{p+3} = \frac{p(p-\alpha)}{6(p+3)}\left\{2c_3 + 3(p-\alpha)c_1c_2 + (p-\alpha)^2c_1^3\right\}]$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from the relation (3.19) in the second Hankel functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function $f \in CV_p(\alpha)$, we have

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \left|\frac{p(p-\alpha)}{(p+1)}c_1 \times \frac{p(p-\alpha)}{6(p+3)}\left\{2c_3 + 3(p-\alpha)c_1c_2 + (p-\alpha)^2c_1^3\right\} - \frac{p^2(p-\alpha)^2}{4(p+2)^2}\left\{c_2 + (p-\alpha)c_1^2\right\}^2\right|$$

Upon simplification, we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2(p-\alpha)^2}{12(p+1)(p+2)^2(p+3)} \times \big|4(p+2)^2c_1c_3 + 6(p-\alpha)c_1^2c_2 - 3(p+1)(p+3)c_2^2 - (p^2+4p+1)(p-\alpha)^2c_1^4\big|$$

The above expression is equivalent to

$$(3.20)\quad |a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2(p-\alpha)^2}{12(p+1)(p+2)^2(p+3)} \times \big|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\big|.$$

where

$$(3.21)\quad \{d_1 = 4(p+2)^2; d_2 = 6(p-\alpha);$$
$$d_3 = -3(p+1)(p+3) = -3(p^2+4p+3); d_4 = -(p^2+4p+1)(p-\alpha)^2\}.$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 in the right-hand side of (3.20), we have

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|$$
$$= |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} + d_2c_1^2 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} + d_3 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 + d_4c_1^4|.$$

After simplifying, we get

$$(3.22)\quad 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| = |(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2d_1c_1(4-c_1^2)z + 2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x| - \left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\}(4-c_1^2)|x|^2z|.$$

Using the values of d_1, d_2, d_3 and d_4 from the relation (3.21), upon simplification, we obtain

$$(3.23)\quad \{(d_1+2d_2+d_3+4d_4) = \left\{-4(p^2+4p+1)(p-\alpha)^2+12(p-\alpha)+(p^2+4p+7)\right\}; \quad d_1=4(p+2)^2; (d_1+d_2+d_3)=(p^2+10p+7-6\alpha)\}.$$

$$(3.24)\quad \left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\} = \left\{(p^2+4p+7)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\}.$$

Consider

$$\left\{(p^2+4p+7)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\} = (p^2+4p+7)\times\left[c_1^2+\frac{8(p+2)^2}{(p^2+4p+7)}c_1+\frac{12(p+1)(p+3)}{(p^2+4p+7)}\right].$$

$$= (p^2+4p+7)\times \left[\left\{c_1+\frac{4(p+2)^2}{(p^2+4p+7)}\right\}^2-\frac{16(p+2)^4}{(p^2+4p+7)^2}+\frac{12(p+1)(p+3)}{(p^2+4p+7)}\right].$$

Upon simplification, the above expression can also be expressed as

$$\left\{(p^2+4p+7)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\}=(p^2+4p+7)\times \left[\left\{c_1+\frac{4(p+2)^2}{(p^2+4p+7)}\right\}^2-\left\{\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}^2\right].$$

$$(3.25)\quad \left\{(p^2+4p+1)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\} = (p^2+4p+7)\times \left[c_1+\left\{\frac{4(p+2)^2}{(p^2+4p+7)}+\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}\right] \times\left[c_1+\left\{\frac{4(p+2)^2}{(p^2+4p+7)}-\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}\right].$$

Since $c_1\in[0,2]$, using the result $(c_1+a)(c_1+b)\geq(c_1-a)(c_1-b)$, where $a,b\geq 0$ in the right-hand side of (3.25), upon simplification, we obtain

$$(3.26)\quad \left\{(p^2+4p+1)c_1^2+8(p+2)^2c_1+12(p+1)(p+3)\right\} \geq \left\{(p^2+4p+1)c_1^2-8(p+2)^2c_1+12(p+1)(p+3)\right\}.$$

From the relations (3.24) and (3.26), we obtain

$$(3.27)\quad -\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\}$$
$$-\leq\left\{(p^2+4p+1)c_1^2-8(p+2)^2c_1+12(p+1)(p+3)\right\}.$$

Substituting the calculated values from (3.23) and (3.27) in the right-hand side of the relation (3.22), we get

$$(3.28)\quad 4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|$$
$$\leq|\left\{-4(p^2+4p+1)(p-\alpha)^2+12(p-\alpha)+(p^2+4p+7)\right\}c_1^4$$
$$+8(p+2)^2c_1(4-c_1^2)z+2(p^2+10p+7-6\alpha)c_1^2(4-c_1^2)|x|$$
$$-\left\{(p^2+4p+1)c_1^2-8(p+2)^2c_1+12(p+1)(p+3)\right\}(4-c_1^2)|x|^2z|.$$

Choosing $c_1=c\in[0,2]$, applying Triangle inequality and replacing $|x|$ by μ in the right-hand side of (3.28), it reduces to

$$(3.29)\quad 4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|$$
$$\leq[\left\{-4(p^2+4p+1)(p-\alpha)^2+12(p-\alpha)+(p^2+4p+7)\right\}c^4$$
$$+8(p+2)^2c(4-c^2)+2(p^2+10p+7-6\alpha)c^2(4-c^2)\mu$$
$$+\left\{(p^2+4p+1)c^2-8(p+2)^2c+12(p+1)(p+3)\right\}(4-c^2)\mu^2]$$
$$=F(c,\mu),\quad \text{for}\quad 0\leq\mu=|x|\leq 1.$$

where

$$(3.30)\quad F(c,\mu)$$
$$=[\left\{-4(p^2+4p+1)(p-\alpha)^2+12(p-\alpha)+(p^2+4p+7)\right\}c^4$$
$$+8(p+2)^2c(4-c^2)+2(p^2+10p+7-6\alpha)c^2(4-c^2)\mu$$
$$+\left\{(p^2+4p+1)c^2-8(p+2)^2c+12(p+1)(p+3)\right\}(4-c^2)\mu^2]$$

We assume that the upper bound for (3.29) occurs at an interior point of the set $\{(\mu,c):\mu\in[0,1]\quad\text{and}\quad c\in[0,2]\}$. Differentiating $F(c,\mu)$ in (3.30) partially with respect to μ, we get

$$(3.31)\quad \frac{\partial F}{\partial\mu}=[2(p^2+10p+7-6\alpha)c^2(4-c^2)$$
$$+2\left\{(p^2+4p+1)c^2-8(p+2)^2c+12(p+1)(p+3)\right\}(4-c^2)\mu]$$

For $0<\mu<1$, for fixed c with $0<c<2$ and $(0\leq\alpha\leq\left(p-\frac{1}{2}\right))$, from (3.31), we observe that $\frac{\partial F}{\partial\mu}>0$. Therefore, $F(c,\mu)$ is an increasing function of μ, which contradicts our assumption that the maximum value of it occurs at an interior point of the set $\{(\mu,c):\mu\in[0,1]\quad\text{and}\quad c\in[0,2]\}$.

Further, for a fixed $c\in[0,2]$, we have

$$(3.32)\quad \max_{0\leq\mu\leq 1}F(c,\mu)=F(c,1)=G(c)(say).$$

From the relations (3.30) and (3.32), upon simplification, we obtain

$$G(c) = 2[-\left\{2\alpha(\alpha-2p)(p^2+4p+1)+ (2p^4+8p^3+3p^2+4p+7)\right\}c^4 + 24(p+1-\alpha)c^2 + 24(p+1)(p+3)]. \tag{3.33}$$

$$G'(c) = 2[-4\left\{2\alpha(\alpha-2p)(p^2+4p+1)+ (2p^4+8p^3+3p^2+4p+7)\right\}c^3 + 48(p+1-\alpha)c]. \tag{3.34}$$

$$G''(c) = 2[-12\left\{2\alpha(\alpha-2p)(p^2+4p+1)+ (2p^4+8p^3+3p^2+4p+7)\right\}c^2 + 48(p+1-\alpha)]. \tag{3.35}$$

The maximum or minimum value of $G(c)$ is obtained for the values of $G'(c) = 0$. From the expression(3.34), we get

$$-8c[\left\{2\alpha(\alpha-2p)(p^2+4p+1)+ (2p^4+8p^3+3p^2+4p+7)\right\}c^2 - 12(p+1-\alpha)] = 0. \tag{3.36}$$

We now discuss the following cases.

Case 1. If $c = 0$, then from (3.35), we obtain

$$G''(c) = 96(p+1-\alpha) > 0, \quad \text{because} \quad \alpha < p \Rightarrow (p-\alpha) > 0.$$

Therefore, by the second derivative test, $G(c)$ has a minimum value at $c = 0$, which is ruled out.

Case 2. If $c \neq 0$, then from (3.36), we obtain

$$c^2 = \left\{\frac{12(p+1-\alpha)}{2\alpha(\alpha-2p)(p^2+4p+1)+(2p^4+8p^3+3p^2+4p+7)}\right\} > 0, \quad \text{for}(0 \le \alpha \le \left(p-\frac{1}{2}\right)) \tag{3.37}$$

Using the value of c^2 given in (3.37) in (3.35), after simplifying, we get

$$G''(c) = -192(p+1-\alpha) > 0, \quad \text{because} \quad \alpha < p \Rightarrow (p-\alpha) > 0.$$

From the second derivative test, $G(c)$ has a maximum value at c, where c^2 is given by (3.37). From the expression (3.33), we have G-maximum value at c^2, after simplifying, it is given by

$$G_{max} = G(c) = 48 \times \left[\frac{6(p+1-\alpha)^2+(p+1)(p+3)\left\{2\alpha(\alpha-2p)(p^2+4p+1)+(2p^4+8p^3+3p^2+4p+7)\right\}}{\left\{2\alpha(\alpha-2p)(p^2+4p+1)+(2p^4+8p^3+3p^2+4p+7)\right\}}\right]. \tag{3.38}$$

Considering only the maximum value of G(c) at c, where c^2 is given by (3.37). From the expressions (3.29) and (3.38), upon simplification, we obtain

$$(3.39)\quad |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq 12 \times \left[\frac{6(p+1-\alpha)^2 + (p+1)(p+3)\left\{2\alpha(\alpha-2p)(p^2+4p+1) + (2p^4+8p^3+3p^2+4p+7)\right\}}{\{2\alpha(\alpha-2p)(p^2+4p+1) + (2p^4+8p^3+3p^2+4p+7)\}}\right].$$

From the expressions (3.20) and (3.39), after simplifying, we get

$$(3.40)\quad |a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2(p-\alpha)^2\left[6(p+1-\alpha)^2+(p+1)(p+3)\left\{2\alpha(\alpha-2p)(p^2+4p+1)+(2p^4+8p^3+3p^2+4p+7)\right\}\right]}{(p+1)(p+2)^2(p+3)\{2\alpha(\alpha-2p)(p^2+4p+1) + (2p^4+8p^3+3p^2+4p+7)\}}.$$

This completes the proof of the theorem. □

Remark.

1) For the choice of $p = 1$, from (3.40), we get

$$|a_2a_4 - a_3^2| \leq \left[\frac{(1-\alpha)^2(17\alpha^2 - 36\alpha + 36)}{144(\alpha^2 - 2\alpha + 2)}\right].$$

2) Choosing $p = 1$ and $\alpha = 0$, from (3.40), we obtain $|a_2a_4 - a_3^2| \leq \frac{1}{8}$. This inequality is sharp, and it coincides with the result of Janteng, Halim and Darus [7].

Acknowledgements

The authors are highly thankful to the referees for their valuable suggestions and comments that helped us in preparing this article.

References

[1] Abubaker, Afaf, Darus, M., Hankel Determinant for a class of analytic functions involving a generalized linear differential operator. Int. J. Pure Appl.Math. 69(4) (2011), 429-435.

[2] Ali, R. M., Coefficients of the inverse of strongly starlike functions. Bull. Malays. Math. Sci. Soc. (second series) 26(1) (2003), 63-71.

[3] Al-Refai, Oqlah., Darus, M., Second Hankel determinant for a class of analytic functions defined by a fractional operator. European J. Sci. Res. 28(2) (2009), 234-241.

[4] Ehrenborg, R., The Hankel determinant of exponential polynomials. Amer. Math. Monthly 107(6) (2000), 557-560.

[5] Goodman, A.W., Univalent functions Vol.I and Vol.II. Tampa, Florida: Mariner publishing Comp. Inc., 1983.

[6] Grenander, U., Szego, G., Toeplitz forms and their application. Berkeley and Los Angeles: Univ. of California Press, 1958.

[7] Janteng, A., Halim, S. A. Darus, M., Hankel Determinant for starlike and convex functions. Int. J. Math. Anal. 1(13) (2007), 619-625.

[8] Janteng, A., Halim, S. A. Darus, M., Coefficient inequality for a function whose derivative has a positive real part. J. Inequal. Pure Appl. Math 7(2) (2006), 1-5.

[9] Layman, J. W., The Hankel transform and some of its properties, J. Integer Seq. 4(1) (2001), 1-11.

[10] Mac Gregor, T.H., Functions whose derivative have a positive real part. Trans. Amer. Math. Soc. 104(3) (1962), 532-537.

[11] Mishra, A. K., Gochhayat, P., Second Hankel Determinant for a class of Analytic Functions Defined by Fractional Derivative. Int. J. Math. Math. Sci., vol. 2008, Article ID 153280 (2008), 1-10.

[12] Murugusundaramoorthy, Gangadharan., Magesh, N., Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant. Bull Math Anal. Appl. 1(3) (2009), 85-89.

[13] Noonan, J. W., Thomas, D. K., On the second Hankel determinant of a really mean p-Valent functions. Trans. Amer. Math. Soc. 223(2) (1976), 337-346.

[14] Noor, K. I., Hankel determinant problem for the class of functions with bounded boundary rotation. Rev. Roum. Math. Pures Et Appl. 28(8) (1983), 731-739.

[15] Owa, S. Srivastava, H. M., Univalent and starlike generalised hypergeometric functions. Canad. J. Math. 39(5) (1987), 1057-1077.

[16] Pommerenke, Ch., Univalent functions. Gottingen: Vandenhoeck and Ruprecht, 1975.

Received by the editors January 16, 2012