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SOME STRONGLY CONVERGENT DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

Kuldip Raj[1] **and Sunil K. Sharma**[2]

Abstract. In the present paper we introduce some strongly convergent difference sequence spaces defined by a sequence of modulus functions $F = (f_k)$. We also study some topological properties and inclusion relations between these spaces.

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1. Introduction and Preliminaries

Let $\Lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The generalized de la Valle-Poussin means is defined by $t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$, where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_n(x) \to L$ as $n \to \infty$ see [18]. If $\lambda_n = n$, then the (V, λ)-summability is reduced to ordinary $(C, 1)$-summability. A sequence $x = (x_k)$ is said to be strongly (V, λ)-summable to a number L if $t_n(|x - L|) \to 0$ as $n \to \infty$.
Let $A = (a_{nk})$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))_{n=1}^{\infty}$ if $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $n \in \mathbb{N}$. Spaces of strongly summable sequences were studied by Kuttner [17], Maddox [19] and others. The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox [20] as an extension of the definition of strongly Cesaro summable sequences. Connor [8] extended further this definition to a definition of strongly A-summability with respect to a modulus when A is a non-negative regular matrix.

Let w be the set of all sequences, real or complex numbers and l_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ normed by $||x|| = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers.

The notion of difference sequence spaces was introduced by Kızmaz [15], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [11] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$

[1]School of Mathematics, Shri Mata Vaishno Devi University, Katra - 182320, J&K, INDIA, e-mail: kuldipraj68@gmail.com
[2]School of Mathematics, Shri Mata Vaishno Devi University, Katra - 182320, J&K, INDIA, e-mail: sunilksharma42@yahoo.co.in

and let m, s be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_s^m) = \{x = (x_k) \in w : (\Delta_s^m x_k) \in Z\},$$

where $\Delta_s^m x = (\Delta_s^m x_k) = (\Delta_s^{m-1} x_k - \Delta_s^{m-1} x_{k+1})$ and $\Delta_s^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were studied by Et and Çolak [11]. Taking $m = s = 1$, we get the spaces which were introduced and studied by Kızmaz [15].

The difference space bv_p consisting of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ is studied in the case $1 \le p \le \infty$ by Başar and Altay [4] and in the case $0 < p < 1$ by Altay and Başar [2], respectively. Later, Altay [1] extended the space bv_p to the mth order difference space $\ell_p(\Delta^{(m)})$.

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \le f(x) + f(y)$ for all $x \ge 0$, $y \ge 0$,
3. f is increasing
4. f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus functions have been discussed in [3, 22, 24, 25, 27] and many others.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \ge 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \le p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda\ as\ n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0\ as\ n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0\ as\ n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [30, Theorem 10.4.2, P-183]).

Let $A = (a_{nk})$ be an infinite matrix of complex numbers, $u = (u_k)$ be a sequence of strictly positive real numbers, $p = (p_k)$ be a bounded sequence of positive real numbers such that $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$. $F = (f_k)$ also be a sequence of modulus functions. Now we define the following sequence spaces:

$$V_1^\lambda[A, \Delta_s^m, u, p, F] = \Big\{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k\in I_n} \frac{u_k\Big[f_k(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|)\Big]^{p_k}}{\lambda_n} = 0, \text{ for some } L\Big\},$$

$$V_0^\lambda[A, \Delta_s^m, u, p, F] = \Big\{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k\in I_n} \frac{u_k\Big[f_k(\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|)\Big]^{p_k}}{\lambda_n} = 0\Big\}$$

and

$$V_\infty^\lambda[A, \Delta_s^m, u, p, F] = \Big\{x = (x_k) \in w : \sup_n \sum_{k\in I_n} \frac{u_k\Big[f_k(\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|)\Big]^{p_k}}{\lambda_n} < \infty\Big\},$$

where $A_k(\Delta_s^m x_k) = \sum_{k=1}^\infty a_{nk} \Delta_s^m x_k$ for all $n \in \mathbb{N}$.

If $F(x) = x$, we get

$$V_1^\lambda[A, \Delta_s^m, u, p] = \Big\{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k\in I_n} \frac{u_k\Big[\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|\Big]^{p_k}}{\lambda_n} = 0, \text{ for some } L\Big\},$$

$$V_0^\lambda[A, \Delta_s^m, u, p] = \Big\{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k\in I_n} \frac{u_k\Big[\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|\Big]^{p_k}}{\lambda_n} = 0\Big\}$$

and

$$V_\infty^\lambda[A, \Delta_s^m, u, p] = \Big\{x = (x_k) \in w : \sup_n \sum_{k\in I_n} \frac{u_k\Big[\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|\Big]^{p_k}}{\lambda_n} < \infty\Big\}.$$

If $p = (p_k) = 1, \forall k \in \mathbb{N}$, we have

$$V_1^\lambda[A, \Delta_s^m, u, F] = \Big\{x = (x_k) \in w :$$

$$\lim_{n\to\infty} \sum_{k\in I_n} \frac{u_k\Big[f_k(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|)\Big]}{\lambda_n} = 0, \text{ for some } L\Big\},$$

$$V_0^\lambda[A, \Delta_s^m, u, F] = \Big\{x = (x_k) \in w :$$

$$\lim_{n\to\infty} \sum_{k\in I_n} \frac{u_k\Big[f_k(\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|)\Big]}{\lambda_n} = 0\Big\}$$

and

$$V_\infty^\lambda[A, \Delta_s^m, u, F] = \Big\{x = (x_k) \in w :$$

$$\sup_n \sum_{k\in I_n} \frac{u_k\Big[f_k(\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|)\Big]}{\lambda_n} < \infty\Big\}.$$

If we take $p = (p_k) = 1$ and $u = (u_k) = 1, \forall k \in \mathbb{N}$, we have

$$V_1^\lambda[A, \Delta_s^m, F] = \Big\{x = (x_k) \in w :$$

$$\lim_{n\to\infty} \sum_{k\in I_n} \frac{\Big[f_k(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|)\Big]}{\lambda_n} = 0, \text{ for some } L\Big\},$$

$$V_0^\lambda[A, \Delta_s^m, F] = \Big\{x = (x_k) \in w :$$

$$\lim_{n\to\infty} \sum_{k\in I_n} \frac{\Big[f_k(\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|)\Big]}{\lambda_n} = 0\Big\}$$

and

$$V_\infty^\lambda[A, \Delta_s^m, F] = \Big\{x = (x_k) \in w :$$

$$\sup_n \sum_{k\in I_n} \frac{\Big[f_k(\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|)\Big]}{\lambda_n} < \infty\Big\}.$$

If we take $F(x) = f(x), u = (u_k) = 1, s = 0, \|., \ldots, .\| = 1$, then the above spaces reduce to $V_1^\lambda[A, \Delta^m, p, f], V_0^\lambda[A, \Delta^m, p, f]$ and $V_\infty^\lambda[A, \Delta^m, p, f]$ which were studied by Ayhan Esi and Ayten Esi [10], and if we take $m = 0$ we get the spaces $V_1^\lambda[A, p, f], V_0^\lambda[A, p, f]$ and $V_\infty^\lambda[A, p, f]$ which were studied by Bilgin and Altun [5]. Throughout the paper Z will denote one of the notations $0, 1$ or ∞.

The following inequality will be used throughout the paper. If $0 < h = \inf p_k \le p_k \le \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also, $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In the present paper we introduce the sequence spaces defined by a sequence of modulus function $F = (f_k)$. We study some topological properties and prove some inclusion relations between these spaces.

2. Main Results

In this section we examine some topological properties of $V_Z^\lambda[A, \Delta_s^m, u, p, F]$ spaces and investigate some inclusion relations between these spaces.

Theorem 2.1. *Let $F = (f_k)$ be a sequence of modulus functions, $u = (u_k)$ be any sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $V_Z^\lambda[A, \Delta_s^m, u, p, F]$ is a linear space over the field $\mathbb{C}$ of complex numbers.*

Proof. Let $x = (x_k), y = (y_k) \in V_0^\lambda[A, \Delta_s^m, u, p, F]$ and $\alpha, \beta \in \mathbb{C}$. Then there exists integers M_α and N_β such that $|\alpha| \le M_\alpha$ and $|\beta| \le N_\beta$. By using inequality (1.1) and the properties of modulus function, we have

$$
\begin{aligned}
&\sum_{k \in I_n} \frac{u_k\Big[f_k\Big(\|\sum_{k=1}^{\infty} a_{nk}(\Delta_s^m(\alpha x_k + \beta y_k)), z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
\le\ & \sum_{k \in I_n} \frac{u_k\Big[f_k\Big(\|\sum_{k=1}^{\infty} \alpha a_{nk}\Delta_s^m x_k + \sum_{k=1}^{\infty} \beta a_{nk}\Delta_s^m y_k, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
\le\ & D \sum_{k \in I_n} \frac{u_k\Big[M_\alpha f_k\Big(\|\sum_{k=1}^{\infty} a_{nk}\Delta_s^m x_k, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
+\ & D \sum_{k \in I_n} \frac{u_k\Big[N_\beta f_k\Big(\|\sum_{k=1}^{\infty} a_{nk}\Delta_s^m y_k, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
\le\ & D M_\alpha^H \sum_{k \in I_n} \frac{u_k\Big[f_k\Big(\|\sum_{k=1}^{\infty} a_{nk}\Delta_s^m x_k, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
+\ & D N_\beta^H \sum_{k \in I_n} \frac{u_k\Big[f_k\Big(\|\sum_{k=1}^{\infty} a_{nk}\Delta_s^m y_k, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
\to\ & 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

This proves that $V_0^\lambda[A, \Delta_s^m, u, p, F]$ is a linear space. Similarly, we can prove that $V_1^\lambda[A, \Delta_s^m, u, p, F]$ and $V_\infty^\lambda[A, \Delta_s^m, u, p, F]$ are linear spaces. □

Theorem 2.2. *Let $F = (f_k)$ be a sequence of modulus functions. Then we have*

$$V_0^\lambda[A,\Delta_s^m,u,p,F] \subset V_1^\lambda[A,\Delta_s^m,u,p,F] \subset V_\infty^\lambda[A,\Delta_s^m,u,p,F].$$

Proof. The inclusion $V_0^\lambda[A,\Delta_s^m,u,p,F] \subset V_1^\lambda[A,\Delta_s^m,u,p,F]$ is obvious. Now, let $x = (x_k) \in V_1^\lambda[A,\Delta_s^m,u,p,F]$ such that $x = (x_k) \to L(V_1^\lambda[A,\Delta_s^m,u,p,F])$. By using inequality (1.1), we have

$$\begin{aligned}
&\sup_n \sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k), z_1,\ldots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
=\ & \sup_n \sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k) - L + L, z_1,\ldots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
\leq\ & D\sup_n \sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k) - L, z_1,\ldots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
+\ & D\sup_n \sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|L, z_1,\ldots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
\leq\ & D\sup_n \sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k) - L, z_1,\ldots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
+\ & D\max\{f_k(\|L, z_1,\ldots,z_{n-1}\|)^h, f_k(\|L, z_1,\ldots,z_{n-1}\|)^H\} \\
<\ & \infty.
\end{aligned}$$

Hence $x = (x_k) \in V_\infty^\lambda[A,\Delta_s^m,u,p,F]$. This proves that $V_1^\lambda[A,\Delta_s^m,u,p,F] \subset V_\infty^\lambda[A,\Delta_s^m,u,p,F]$. This completes the proof of the theorem. □

Theorem 2.3. *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k) \in l_\infty$. Then $V_0^\lambda[A,\Delta_s^m,u,p,F]$ is a paranormed space with the paranorm defined by*

$$g(x) = \sup_n \Big(\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k), z_1,\ldots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n}\Big)^{\frac{1}{M}},$$

where $M = \max(1, \sup_k p_k)$.

Proof. Clearly $g(-x) = g(x)$. It is trivial that $\Delta_s^m x_k = 0$ for $x = 0$. Hence we get $g(0) = 0$. Since $\frac{p_k}{M} \leq 1$ and $M \geq 1$, using Minkowski's inequality and definition of modulus function, for each x, we have

$$\begin{aligned}
&\Big(\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(||A_k(\Delta_s^m(x_k+y_k)), z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}}{\lambda_n}\Big)^{\frac{1}{M}} \\
&\le \Big(\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(||A_k(\Delta_s^m x_k), z_1,\ldots,z_{n-1}||\Big)+f_k\Big(||A_k(\Delta_s^m y_k), z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}}{\lambda_n}\Big)^{\frac{1}{M}} \\
&\le \Big(\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(||A_k(\Delta_s^m x_k), z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}}{\lambda_n}\Big)^{\frac{1}{M}} \\
&\quad +\Big(\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(||A_k(\Delta_s^m y_k), z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}}{\lambda_n}\Big)^{\frac{1}{M}}
\end{aligned}$$

Now it follows that g is subadditive. Finally, to check the continuity of multiplication, let us take any complex number α. By the definition of the modulus function F, we have

$$\begin{aligned}
g(\alpha x) &= \sup_n \Big(\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(||A_k(\Delta_s^m \alpha x_k), z_1,\ldots,z_{n-1}||\Big)\Big]^{p_k}}{\lambda_n}\Big)^{\frac{1}{M}} \\
&\le K^{\frac{H}{M}} g(x)
\end{aligned}$$

where $K = 1 + [|\alpha|]$ ($[|\alpha|]$ denotes the integer part of α). Since F is a sequence of modulus function, we have $x \to 0$ implies $g(\alpha x) \to 0$. Similarly, $x \to 0$ and $\alpha \to 0$ implies $g(\alpha x) \to 0$. Finally, we have fixed x and $\alpha \to 0$ implies $g(\alpha x) \to 0$. This completes the proof. □

Theorem 2.4. *Let $F = (f_k)$ be a sequence of modulus functions. Then*

$$V_Z^\lambda[A, \Delta_s^m, u, p] \subset V_Z^\lambda[A, \Delta_s^m, u, p, F].$$

Proof. Let $x = (x_k) \in V_1^\lambda[A, \Delta_s^m, u, p]$ and $\epsilon > 0$. We can choose $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for every $t \in [0, \infty)$ with $0 \le t \le \delta$. Then, we can write

$$\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k)-L,z_1,\dots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n}$$

$$\begin{aligned}
&= \sum_{k\in I_n,\ \|A_k(\Delta_s^m x_k)-L,z_1,\dots,z_{n-1}\|\le\delta} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k)-L,z_1,\dots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n}\\
&+ \sum_{k\in I_n,\ \|A_k(\Delta_s^m x_k)-L,z_1,\dots,z_{n-1}\|>\delta} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k)-L,z_1,\dots,z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n}\\
&\le \max\{f_k(\epsilon)^h, f_k(\epsilon)^H\}\\
&+ \max\{1,(2f_k(1)\delta^{-1})^H\} \sum_{k\in I_n,\ \|A_k(\Delta_s^m x_k)-L,z_1,\dots,z_{n-1}\|>\delta} \frac{u_k\Big(\|A_k(\Delta_s^m x_k)-L,z_1,\dots,z_{n-1}\|\Big)^{p_k}}{\lambda_n}.
\end{aligned}$$

Therefore, $x=(x_k)\in V_1^\lambda[A,\Delta_s^m,u,p,F]$. This completes the proof of the theorem. Similarly, we can prove the other cases. $\square$

Theorem 2.5. *Let $F=(f_k)$ be a sequence of modulus functions. If $\lim\limits_{t\to\infty}\dfrac{f_k(t)}{t}=s>0$, then $V_Z^\lambda[A,\Delta_s^m,u,p]=V_Z^\lambda[A,\Delta_s^m,u,p,F]$.*

Proof. The proof is easy, so we omit it. $\square$

Theorem 2.6. *t Let $0<p_k\le q_k$ for all $k\in\mathbb{N}$ and let $(\frac{q_k}{p_k})$ be bounded. Then*

$$V_Z^\lambda[A,\Delta_s^m,u,q,F]\subset V_Z^\lambda[A,\Delta_s^m,u,p,F].$$

Proof. Let $x=(x_k)\in V_Z^\lambda[A,\Delta_s^m,u,q,F]$. Let

$$t_k=u_k\Big[f_k\Big(||A_k(\Delta_s^m x_k)-L,z_1,\dots,z_{n-1}||\Big)\Big]^{q_k}$$

and $\lambda_k=(\frac{p_k}{q_k})$ for all $k\in\mathbb{N}$ so that $0<\lambda\le\lambda_k\le 1$. Define the sequences (u_k) and (v_k) as follows:

For $t_k\ge 1$, let $u_k=t_k$ and $v_k=0$ and for $t_k<1$, let $u_k=0$ and $v_k=t_k$.

Then clearly for all $k\in\mathbb{N}$, we have $t_k=u_k+v_k$, $t_k^{\lambda_k}=u_k^{\lambda_k}+v_k^{\lambda_k}$, $u_k^{\lambda_k}\le u_k\le t_k$ and $v_k^{\lambda_k}\le v_k^{\lambda}$. Therefore

$$\frac{1}{\lambda_n}\sum_{k\in I_n} t_k^{\lambda_k}\le\frac{1}{\lambda_n}\sum_{k\in I_n} t_k+[\frac{1}{\lambda_n}\sum_{k\in I_n} v_k]^\lambda.$$

Hence $x=(x_k)\in V_Z^\lambda[A,\Delta_s^m,u,p,F]$. Thus

$$V_Z^\lambda[A,\Delta_s^m,u,q,F]\subseteq V_Z^\lambda[A,\Delta_s^m,u,p,F].$$

This completes the proof of the theorem. $\square$

Corollary 2.7. *Let $F = (f_k)$ be a sequence of modulus functions. Then the following relation holds:*

(a) If $0 < \inf_k p_k \leq 1$ for all $k \in \mathbb{N}$, then $V_Z^\lambda[A, \Delta_s^m, u, F] \subset V_Z^\lambda[A, \Delta_s^m, u, p, F]$.

(b) If $1 \leq p_k \leq \sup_k p_k = H < \infty$ for all $k \in \mathbb{N}$, then

$$V_Z^\lambda[A, \Delta_s^m, u, p, F] \subset V_Z^\lambda[A, \Delta_s^m, u, F].$$

Proof. (a) It follows from Theorem 2.6 with $q_k = 1$ for all $k \in \mathbb{N}$.

(b) It follows from Theorem 2.6 with $p_k = 1$ for all $k \in \mathbb{N}$. □

Theorem 2.8. *Let $F = (f_k)$ be a sequence of modulus functions. If $0 < \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $V_Z^\lambda[A, \Delta_s^m, u, p, F] = V_Z^\lambda[A, \Delta_s^m, u, F]$.*

Proof. It is easy to prove so we omit it. □

Theorem 2.9. *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. Let $m \geq 1$ be a fixed integer, then $V_Z^\lambda[A, \Delta_s^{m-1}, u, p, F] \subset V_Z^\lambda[A, \Delta_s^m, u, p, F]$.*

Proof. The proof of the inclusion follows from the following inequality

$$\begin{aligned}
&\sum_{k \in I_n} \frac{u_k \Big[f_k\Big(\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
&\leq \quad D \sum_{k \in I_n} \frac{u_k \Big[f_k\Big(\|A_k(\Delta_s^{m-1} x_k), z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
&+ \quad D \sum_{k \in I_n} \frac{u_k \Big[f_k\Big(\|A_k(\Delta_s^m x_k), z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n}.
\end{aligned}$$

□

3. Statistical Convergence

The notion of statistical convergence of sequences was introduced by Fast [13], Buck [6], and Schoenberg [28] independently. It is also found in Zygmund [31]. Later on it was studied from sequence space point of view and linked with summability theory by Fridy [14], Connor [8], Salat [26], Maddox [21], Kolk [16], Rath and Tripathy [23], Tripathy [29], and many others. The notion depends on the density of subsets of the set N of natural numbers. A subset E of N is said to have density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$ exists, where χ_E is the characteristic function of E.

A complex number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\epsilon > 0$, $\lim_{n\to\infty} \frac{|K(\epsilon)|}{n} = 0$, where $|K(\epsilon)|$ denotes the number of elements in the set $K(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$.

A complex number sequence $x = (x_k)$ is said to be strongly generalized difference $S^{\lambda}(A, \Delta_s^m)$-statistically convergent to the number L if for every $\epsilon > 0$, $\lim_{n\to\infty} \frac{1}{\lambda_n}|KA(\Delta_s^m, \epsilon)| = 0$, where $|KA(\Delta_s^m, \epsilon)|$ denotes the number of elements in the set $KA(\Delta_s^m, \epsilon) = \{k \in I_n : |A_k(\Delta_s^m x_k) - L| \geq \epsilon\}$. The set of all strongly generalized difference statistically convergent sequences is denoted by $S^{\lambda}(A, \Delta_s^m)$. If $m = 0, \Delta = 0, S^{\lambda}(A, \Delta_s^m)$ reduces to $S^{\lambda}(A)$ which was defined and studied by Bilgin and Altun [5]. If A is identity matrix, and $\lambda_n = n, s = 0, S^{\lambda}(A, \Delta_s^m)$ reduces to $S^{\lambda}(\Delta^m)$, which was defined and studied by Et and Nuray [12]. If $m = 0, s = 0$, and $\lambda_n = n, S^{\lambda}(A, \Delta_s^m)$ reduces to S_A, which was defined and studied by Esi [9]. If $m = 0, s = 0, A$ is identity matrix and $\lambda_n = n$, strongly generalized difference $S^{\lambda}(A, \Delta_s^m)$-statistically convergent sequences reduces to ordinary statistical convergent sequences.

Theorem 3.1. *Let $F = (f_k)$ be a sequence of modulus functions. Then*

$$V_1^{\lambda}[A, \Delta_s^m, u, p, F] \subset S^{\lambda}(A, \Delta_s^m).$$

Proof. Let $x = (x_k) \in V_1^{\lambda}[A, \Delta_s^m, u, p, F]$. Then

$$\begin{aligned}
&\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
&\geq \sum_{k\in I_n,\ \|A_k(\Delta_s^m x_k)-L,z_1,\ldots,z_{n-1}\|>S} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n} \\
&\geq \sum_{k\in I_n,\ \|A_k(\Delta_s^m x_k)-L,z_1,\ldots,z_{n-1}\|>S} \frac{u_k\Big[f_k(\epsilon)\Big]^{p_k}}{\lambda_n} \\
&\geq \sum_{k\in I_n,\ \|A_k(\Delta_s^m x_k)-L,z_1,\ldots,z_{n-1}\|>S} \min\Big(f_k(\epsilon)^h, f_k(\epsilon)^H\Big) \\
&\geq \min\Big(f_k(\epsilon)^h, f_k(\epsilon)^H\Big)\frac{1}{\lambda_n}|KA(\Delta_s^m, \epsilon)|.
\end{aligned}$$

Hence $x = (x_k) \in S^{\lambda}(A, \Delta_s^m)$. □

Theorem 3.2. *Let $F = (f_k)$ be a bounded sequence of modulus functions. Then*

$$V_1^{\lambda}[A, \Delta_s^m, u, p, F] = S^{\lambda}(A, \Delta_s^m).$$

Proof. By Theorem 3.1, it is sufficient to show that

$$V_1^{\lambda}[A, \Delta_s^m, u, p, F] \supset S^{\lambda}(A, \Delta_s^m).$$

Let $x = (x_k) \in S^{\lambda}(A, \Delta_s^m)$. Since $F = (f_k)$ is bounded, so there exists an integer $K > 0$ such that $f_k(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|) \leq K$. Then for a given $\epsilon > 0$, we have

$$\sum_{k\in I_n} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n}$$

$$= \sum_{k\in I_n,\ \|A_k(\Delta_s^m x_k)-L,z_1,\ldots,z_{n-1}\|\le S} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n}$$

$$+ \sum_{k\in I_n,\ \|A_k(\Delta_s^m x_k)-L,z_1,\ldots,z_{n-1}\|> S} \frac{u_k\Big[f_k\Big(\|A_k(\Delta_s^m x_k) - L, z_1, \ldots, z_{n-1}\|\Big)\Big]^{p_k}}{\lambda_n}$$

$$\le \max\Big(f_k(\epsilon)^h, f_k(\epsilon)^H\Big) + K^H \frac{1}{\lambda_n}|KA(\Delta_s^m, \epsilon)|.$$

Taking the limit as $\epsilon \to 0$ and $n \to \infty$, it follows that

$$x = (x_k) \in V_1^\lambda[A, \Delta_s^m, u, p, F].$$

This completes the proof of the theorem. □

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