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EXTENSION OF RIDGELET TRANSFORM TO TEMPERED BOEHMIANS

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Abstract. We extend the ridgelet transform to the space of tempered Boehmians consistent with the ridgelet transform on the space of tempered distributions. We also prove that the extended ridgelet transform is continuous, linear, bijection and the extended adjoint ridgelet transform is also linear and continuous.

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1. Introduction

We denote by the set of all natural numbers, non-negative integers, real numbers and complex numbers respectively by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$. We also denote by $\mathscr{S}(\mathbb{R}^2)$ the Fréchet space of rapidly decreasing complex valued functions on $\mathbb{R}^2$ and $\mathscr{S}'(\mathbb{R}^2)$ by the space of all tempered distributions on $\mathbb{R}^2$ with weak* topology.

Let $\psi \in \mathscr{S}(\mathbb{R})$ be a real valued function satisfying the admissibility condition

$$\int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2/|\xi|^2 \, d\xi = 1.$$

For each $(a, b, \theta) \in \mathbb{Y} = \mathbb{R}^+ \times \mathbb{R} \times [0, 2\pi]$, the ridgelet is defined by

$$\psi_{a,b,\theta}(\mathbf{x}) = \psi_{a,b,\theta}(x_1, x_2) = \psi\left(\frac{x_1 \cos\theta + x_2 \sin\theta - b}{a}\right), \ \forall (x_1, x_2) \in \mathbb{R}^2.$$

The ridgelet transform [2, 25] of a square integrable function f on $\mathbb{R}^2$ is defined by

$$(1) \qquad (Rf)(a, b, \theta) = \int_{\mathbb{R}^2} f(\mathbf{x}) \psi_{a,b,\theta}(\mathbf{x}) \, d\mathbf{x}, \ \forall (a, b, \theta) \in \mathbb{Y}.$$

and adjoint ridgelet transform of a suitable function on $\mathbb{Y}$ is defined by

$$(2) \qquad (R^*F)(\mathbf{x}) = \int_0^{\infty} \int_{-\infty}^{\infty} \int_0^{2\pi} F(a, b, \theta) \psi_{a,b,\theta}(\mathbf{x}) \, \frac{da}{a^4} \, db \, d\theta$$

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and it is proved in [2] that the composition $(R^* \circ R)$ of R and R^* is the identity operator on $\mathscr{L}^2(\mathbb{R}^2)$.

Next, the ridgelet transform is consistently extended to the context of square integrable Boehmians [18] and studied. Though the space of square integrable Boehmians properly contains $\mathscr{L}^2(\mathbb{R}^2)$, it neither contains the tempered distributions nor contained in the tempered distributions. Later, in [24] the distributional ridgelet transform $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$ and the distributional adjoint ridgelet transform $\mathcal{R}^* : \mathscr{S}'(\mathbb{Y}) \to \mathscr{S}'(\mathbb{R}^2)$ are defined by

$$\langle \mathcal{R}u, F\rangle = \langle u, R^* F\rangle,\ \forall F \in \mathscr{S}(\mathbb{Y}),$$

$$\langle \mathcal{R}^*\Lambda, f\rangle = \langle \Lambda, Rf\rangle,\ \forall f \in \mathscr{S}(\mathbb{R}^2),$$

where $\mathscr{S}(\mathbb{Y})$ is the space consisting of all smooth functions on $\mathbb{Y}$, with

$$Q_{k,\alpha;l,\beta;m}(F) = \sup_{(a,b,\theta)\in\mathbb{Y}} |a^k b^l D_a^\alpha D_b^\beta D_\theta^m F(a,b,\theta)| < +\infty,\ \forall k,\alpha,l,\beta,m \in \mathbb{N}_0.$$

To extend the ridgelet transform further, we consider the space of tempered Boehmians which properly contains both the space of square integrable Boehmians and the space of tempered distributions.

2. Boehmian space

Motivated from the Boehme's regular operators [1], the concept of Boehmians is first introduced by J. Mikusiński and P. Mikusiński [5]. Later, many Boehmian spaces have been constructed to extend various integral transforms [3, 4, 7–10, 13–23, 26]

In this section, first we recall the construction of an abstract Boehmian space from [11] and tempered Boehmians [8] which is slightly modified in [9, 15] in two different ways. Next, we prove the auxiliary results required to construct the required Boehmian space which will be the range of the ridgelet transform on the tempered Boehmians.

To construct a Boehmian space, we need $G, (S, \odot), \bullet$ and Δ, where G is a sequential-convergence linear space [27, p. 6], $(S, \odot)$ is a commutative semigroup and $\bullet : G \times S \to G$ satisfying the following conditions.

Let $\alpha, \beta \in G$, $\zeta, \xi \in S$ and $c \in \mathbb{C}$ be arbitrary.

1. $(\alpha + \beta) \bullet \zeta = \alpha \bullet \zeta + \beta \bullet \zeta$;
2. $(c\alpha) \bullet \xi = c(\alpha \bullet \xi)$;
3. $\alpha \bullet (\zeta \odot \xi) = (\alpha \bullet \zeta) \bullet \xi$;
4. If $\alpha_n \to \alpha$ as $n \to \infty$ in G and $\xi \in S$ then $\alpha_n \bullet \xi \to \alpha \bullet \xi$ as $n \to \infty$,

and Δ is a collection of the sequences from S satisfying

(*a*) If $(\xi_n), (\zeta_n) \in \Delta$ then $(\xi_n \odot \zeta_n) \in \Delta$.

(*b*) If $\alpha \in G$ and $(\xi_n) \in \Delta$, then $\alpha \bullet \xi_n \to \alpha$ in G as $n \to \infty$.

Let $\mathscr{A}$ denote the collection of all pairs of sequences $((\alpha_n), (\xi_n))$, where $\alpha_n \in G$, $\forall n \in \mathbb{N}$ and $(\xi_n) \in \Delta$ satisfying the property

(3) $$\alpha_n \bullet \xi_m = \alpha_m \bullet \xi_n,\ \forall\ m, n \in \mathbb{N}.$$

Each element of $\mathscr{A}$ is called a quotient and is denoted by $\frac{\alpha_n}{\xi_n}$. Define a relation $\sim$ on $\mathscr{A}$ by

$$\frac{\alpha_n}{\xi_n} \sim \frac{\beta_n}{\zeta_n} \quad \text{if} \quad \alpha_n \bullet \zeta_m = \beta_m \bullet \xi_n, \ \forall \ \ m, n \in \mathbb{N}. \tag{4}$$

It is easy to verify that $\sim$ is an equivalence relation on $\mathscr{A}$ and hence it decomposes $\mathscr{A}$ into disjoint equivalence classes. Each equivalence class is called a Boehmian and is denoted by $\left[\frac{\alpha_n}{\xi_n}\right]$. The collection of all Boehmians is denoted by $\mathscr{B} = \mathscr{B}(G, (S, \odot), \bullet, \Delta)$. Every element α of G is identified uniquely as a member of $\mathscr{B}$ by $\left[\frac{\alpha \bullet \xi_n}{\xi_n}\right]$, where $(\xi_n) \in \Delta$ is arbitrary. In this case, we say that X represents α and we denote this by $X \in G$.

The set $\mathscr{B}$ becomes a vector space with addition and scalar multiplication defined as follow.

(i) $\left[\frac{\alpha_n}{\xi_n}\right] + \left[\frac{\beta_n}{\zeta_n}\right] = \left[\frac{\alpha_n \bullet \zeta_n + \beta_n \bullet \xi_n)}{\xi_n \odot \zeta_n}\right]$.

(ii) $c\left[\frac{\alpha_n}{\xi_n}\right] = \left[\frac{c\alpha_n}{\xi_n}\right]$.

The operation $\bullet$ can be extended to $\mathscr{B} \times S$ by the following definition.

Definition 1. *If* $X = \left[\frac{\alpha_n}{\xi_n}\right] \in \mathscr{B}$ *and* $\zeta \in S$, *then* $X \bullet \zeta = \left[\frac{\alpha_n \bullet \zeta}{\xi_n}\right]$.

Now we recall the notions of convergence on $\mathscr{B}$ from [6].

Definition 2 (δ-Convergence)**.** *We say that* $X_n \xrightarrow{\delta} X$ *as* $n \to \infty$ *in* $\mathscr{B}$ *if there exists* (ξ_n) *such that* $X_n \bullet \xi_k \in G$, $\forall n, k \in \mathbb{N}$, $X \bullet \xi_k \in G, \forall k \in \mathbb{N}$ *and for each* $k \in \mathbb{N}$,

$$X_n \bullet \xi_k \to X \bullet \xi_k \ \textit{as} \ n \to \infty \ \textit{in} \ G.$$

The following lemma gives an equivalent statement for δ-convergence.

Lemma 3. $X_n \xrightarrow{\delta} X$ *as* $n \to \infty$ *if and only if there exist* $\alpha_{n,k}, \alpha_k \in G$ *and* $(\xi_k) \in \Delta$ *such that* $X_n = \left[\frac{\alpha_{n,k}}{\xi_k}\right]$, $X = \left[\frac{\alpha_k}{\xi_k}\right]$ *and for every* $k \in \mathbb{N}$,

$$\alpha_{n,k} \to \alpha_k \ \textit{as} \ n \to \infty \ \textit{in} \ G.$$

Definition 4 (Δ-Convergence)**.** *We say that* $X_n \xrightarrow{\Delta} X$ *as* $n \to \infty$ *in* $\mathscr{B}$ *if there exists* (ξ_n) *such that* $(X_n - X) \bullet \xi_n \in G$, $\forall n \in \mathbb{N}$, *and*

$$(X_n - X) \bullet \xi_n \to 0 \ \textit{as} \ n \to \infty \ \textit{in} \ G.$$

The space of tempered Boehmians is introduced by P. Mikusiński [8] as $\mathscr{B}(\mathscr{I}, (\mathscr{D}, *), *, \Delta_0)$, where $\mathscr{I}$ is the space of all continuous functions on $\mathbb{R}^n$ with polynomial growth, $\mathscr{D}$ is the space of all smooth functions on $\mathbb{R}^n$ with compact supports, $*$ is the usual convolution between suitable real valued functions defined by

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})\, d\mathbf{y}, \ \forall \mathbf{x} \in \mathbb{R}^n, \tag{5}$$

and Δ_0 is the collection of all sequences (ϕ_k) satisfying the following conditions:

(1) $\int\limits_{\mathbb{R}^n} \phi_k(\mathbf{x})\, d\mathbf{x} = 1, \forall k \in \mathbb{N}$.

(2) $\int\limits_{\mathbb{R}^n} |\phi_k(\mathbf{x})|\, d\mathbf{x} \leq M, \forall k \in \mathbb{N}$, for some $M > 0$.

(3) If $s(\phi_k) = \sup\{\mathbf{x} \in \mathbb{R}^n : \phi_k(\mathbf{x}) \neq 0\}$, then $s(\phi_k) \to 0$ as $k \to \infty$.

In [15], the tempered Boehmians is slightly changed by replacing $\mathscr{I}$ by the space $\mathscr{S}'$ of tempered distributions. This change does not alter the set but it may increase the number of representatives of each Boehmian in the new setup. This changed version of tempered Boehmians is successfully used to extend Fourier transform [15], Radon transform [16] and Wavelet transform [21]. In this paper we also prefer to use the tempered Boehmians defined in [15] as $\mathscr{B}_1 = \mathscr{B}(\mathscr{S}'(\mathbb{R}^2), (\mathscr{D}(\mathbb{R}^2), *), *, \Delta_0)$, where $*$ is defined by

$$\text{for } \nu \in \mathscr{S}'(\mathbb{R}^2) \text{ and } \phi \in \mathscr{D}(\mathbb{R}^2),\ (\nu * \phi)(f) = \nu(f * \check{\phi}),\ \forall f \in \mathscr{S}(\mathbb{R}^2),$$

where $\check{\phi}(\mathbf{x}) = \phi(-\mathbf{x}),\ \forall \mathbf{x} \in \mathbb{R}^2$ and $f * \check{\phi}$ is the the usual convolution of f and $\check{\phi}$.

Remark 5. *Since the convolution on $\mathscr{D}'(\mathbb{R}^2) \times \mathscr{D}(\mathbb{R}^2)$ is consistent with the usual convolution defined in (5), it is customary to use the same notation for both convolutions.*

Definition 6 ([18])**.** *For $F \in \mathscr{S}(\mathbb{Y})$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$ define*

$$(F \star \phi)(a, b, \theta) = \int_{\mathbb{R}^2} F(a, b - \mathbf{x} \cdot \mathrm{e}^{i\theta}, \theta)\phi(\mathrm{x})\, d\mathbf{x},\ \forall (a, b, \theta) \in \mathbb{Y},$$

where $\mathbf{x} \cdot \mathrm{e}^{i\theta} = x_1 \cos\theta + x_2 \sin\theta$.

Theorem 7 ([18])**.** *If $f \in \mathscr{S}(\mathbb{R}^2)$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $R(f * \phi) = (Rf) \star \phi$.*

Definition 8. *For $\Lambda \in \mathscr{S}'(\mathbb{Y})$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$ define*

$$(\Lambda \otimes \phi)(F) = \Lambda(F \star \check{\phi}), \forall F \in \mathscr{S}(\mathbb{Y}).$$

To facilitate the understanding, we recall the multi-variate Faa di Bruno formula for the nth derivative of a composite function with a vector argument [12], which will be applied in the proof of the following lemma. If $h(t) = f[x_1(t), x_2(t)]$ and $n \in \mathbb{N}$, then

$$D_t^n h(t) = \sum_0 \sum_1 \cdots \sum_m \frac{m!}{\prod\limits_{r=1}^{m} (r!)^{k_r} \prod\limits_{r=1}^{m} q_{r,1} q_{r,2}} \frac{\partial^j f}{\partial x_1^{p_1} \partial x_2^{p_2}} \times \prod_{r=1}^{m} (x_1^{(r)})^{q_{r,1}} (x_2^{(r)})^{q_{r,2}},$$

where the respective sums are over all nonnegative integer solutions of the

Diophantine equations, as follows

$$\begin{aligned}
\sum_0 &\longrightarrow \quad k_1 + 2k_2 + \cdots + mk_m = m,\\
\sum_1 &\longrightarrow \quad q_{1,1} + q_{1,2} = k_1,\\
\sum_2 &\longrightarrow \quad q_{2,1} + q_{2,2} = k_2,\\
&\vdots\\
\sum_m &\longrightarrow \quad q_{m,1} + q_{m,2} = k_m,
\end{aligned}$$

$p_1 = q_{1,1} + q_{2,1} + \cdots + q_{m,1}$, $p_2 = q_{1,2} + q_{2,2} + \cdots + q_{m,2}$ and $k = p_1 + p_2 = k_1 + k_2 \cdots + k_m$.

Lemma 9. *Let $\phi \in \mathscr{D}(\mathbb{R}^2)$.*

(i) If $F \in \mathscr{S}(\mathbb{Y})$, then $F \star \phi \in \mathscr{S}(\mathbb{Y})$,

(ii) If $F_n \to 0$ as $n \to \infty$ in $\mathscr{S}(\mathbb{Y})$, then $F_n \star \phi \to 0$ as $n \to \infty$ in $\mathscr{S}(\mathbb{Y})$,

(iii) If $\Lambda \in \mathscr{S}'(\mathbb{Y})$, then $\Lambda \otimes \phi \in \mathscr{S}'(\mathbb{Y})$.

Proof. Let supp $\phi \subset K$ for some compact subset K of $\mathbb{R}^2$ and $M = \sup_{\mathbf{x}\in K} |\mathbf{x}|$. For $k, , \alpha, \beta, m \in \mathbb{N}_0$,

$$\begin{aligned}
&\left|a^k b^l D_a^\alpha D_b^\beta D_\theta^m ((F \star \phi)(a, b, \theta))\right|\\
&= \left|a^k b^l D_a^\alpha D_b^\beta D_\theta^m \int_K F(a, b - \mathbf{x}\cdot \mathrm{e}^{i\theta}, \theta)\phi(\mathbf{x})d\mathbf{x}\right|\\
&\le \int_K \left|a^k b^l D_a^\alpha D_b^\beta D_\theta^m F(a, b - \mathbf{x}\cdot \mathrm{e}^{i\theta}, \theta)\phi(\mathbf{x})\right| d\mathbf{x}\\
&\le \int_K \left|a^k b^l \sum_0 \cdots \sum_m \frac{m!}{\prod_{r=1}^m (r!)^{k_r} \prod_{r=1}^m q_{r,1}q_{r,2}} \frac{\partial^j G}{\partial g_1^{p_1} \partial g_2^{p_2}} \times \prod_{r=1}^m (g_1^{(r)})^{q_{r,1}} (g_2^{(r)})^{q_{r,2}}\right| |\phi(\mathbf{x})| d\mathbf{x}
\end{aligned}$$

where $G(g_1, g_2) = (D_a^\alpha D_b^\beta F)(a, g_1, g_2)$, $g_1(\theta) = b - \mathbf{x}\cdot \mathrm{e}^{i\theta}$, $g_2(\theta) = \theta, \forall \theta \in [0, 2\pi]$.

Since $g_1^{(r)}$ is a linear combination of x_1, x_2 with coefficients from $\{\pm\cos\theta, \pm\sin\theta\}$ and $g_2^{(r)} \in \{\theta, 1, 0\}$, we have

$$\left|(g_1^{(r)})^{q_{r,1}} (g_2^{(r)})^{q_{r,2}}\right| \le (2M)^{q_{r,1}} (2\pi)^{q_{r,2}}.$$

Thus the last term is dominated by

$$(2M)^{q_{r,1}}(2\pi)^{q_{r,2}}\int_K \sum_0\sum_1\cdots\sum_m \frac{m!}{\prod_{r=1}^m (r!)^{k_r}\prod_{r=1}^m q_{r,1}q_{r,2}} \left|a^k b^l (D_a^\alpha D_b^{\beta+p_1} D_\theta^{p_2} F)(a, b-\mathbf{x}\cdot \mathrm{e}^{i\theta},\theta)\right| |\phi(\mathbf{x})|\,d\mathbf{x}$$

Since for each $\mathbf{x}\in K$,

$$|b^l| \le (|b-\mathbf{x}\cdot\mathrm{e}^{i\theta}| + |\mathbf{x}\cdot\mathrm{e}^{i\theta}|)^l \le \begin{cases} 2^{l-1}(|b-\mathbf{x}\cdot\mathrm{e}^{i\theta}|^l + (2M)^l) & \text{for } l\ge 1\\ 1 & \text{for } l=0,\end{cases}$$

we have

$$\begin{aligned} Q_{k,\alpha;l,\beta;m}(F\star\phi) &\\ \le\ & (2M)^{q_{r,1}}(2\pi)^{q_{r,2}}\sum_0\sum_1\cdots\sum_m \frac{m!}{\prod_{r=1}^m (r!)^{k_r}\prod_{r=1}^m q_{r,1}q_{r,2}}\int_K|\phi(\mathbf{x})|\,d\mathbf{x}\\ & \left(2^{l-1}Q_{k,\alpha;l,\beta+p_1;p_2}(F) + ((2M)^l+1)Q_{k,\alpha;0,\beta+p_1;p_2}(F)\right) < +\infty \end{aligned} \tag{6}$$

Thus, $F\star\phi\in\mathscr{S}(\mathbb{Y})$.

We get the statement (ii) of this lemma as an immediate consequence of the estimate (6). To prove the statement (iii) of this lemma, let $F_1,F_2\in\mathscr{S}(\mathbb{Y})$ and $c_1,c_2\in\mathbb{C}$ be arbitrary. Then, we have

$$\begin{aligned}(\Lambda\otimes\phi)(c_1F_1+c_2F_2) &= \Lambda((c_1F_1+c_2F_2)\star\check{\phi})\\ &= c_1\Lambda(F_1\star\check{\phi}) + c_2\Lambda(F_2\star\check{\phi})\\ &= c_1(\Lambda\otimes\phi)(F_1) + c_2(\Lambda\otimes\phi)(F_2).\end{aligned}$$

Let $F_n\to 0$ as $n\to\infty$ in $\mathscr{S}(\mathbb{Y})$. From the statement (ii) of this lemma, we get $F_n\star\check{\phi}\to 0$ as $n\to\infty$ in $\mathscr{S}(\mathbb{Y})$. Since $\Lambda\in\mathscr{S}'(\mathbb{Y})$ and it is continuous on $\mathscr{S}(\mathbb{Y})$, it follows that $(\Lambda\otimes\phi)(F_n) = \Lambda(F_n\star\check{\phi})\to 0$ as $n\to\infty$. Therefore, $\Lambda\otimes\phi\in\mathscr{S}'(\mathbb{Y})$. □

Lemma 10. *If* $\Lambda_1,\Lambda_2\in\mathscr{S}'(\mathbb{Y})$, $\phi\in\mathscr{D}(\mathbb{R}^2)$ *and* $c\in\mathbb{C}$, *then*

(i) $(\Lambda_1+\Lambda_2)\otimes\phi = \Lambda_1\otimes\phi + \Lambda_2\otimes\phi$,

(ii) $(c\Lambda_1)\otimes\phi = c(\Lambda_1\otimes\phi)$.

Proof of this lemma is straightforward.

Lemma 11. *Let* $\phi_1,\phi_2\in\mathscr{D}(\mathbb{R}^2)$.

(i) If $F\in\mathscr{S}(\mathbb{Y})$, *then* $F\star(\phi_1*\phi_2) = (F\star\phi_1)\star\phi_2$,

(ii) If $\Lambda\in\mathscr{S}'(\mathbb{Y})$, *then* $\Lambda\otimes(\phi_1*\phi_2) = (\Lambda\otimes\phi_1)\otimes\phi_2$.

Proof. Let $a, b, \theta \in \mathbb{Y}$ be arbitrary. By applying Fubini's theorem, we get

$$
\begin{aligned}
(F \star (\phi_1 * \phi_2))(a,b,\theta) &= \int_{\mathbb{R}^2} F(a, b - \mathbf{x}\cdot \mathrm{e}^{i\theta}, \theta) \int_{\mathbb{R}^2} \phi_1(\mathbf{x}-\mathbf{y})\phi_2(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} F(a, b - \mathbf{x}\cdot \mathrm{e}^{i\theta}, \theta)\phi_1(\mathbf{x}-\mathbf{y})\, d\mathbf{x}\, \phi_2(\mathbf{y})\, d\mathbf{y} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} F(a, b - (\mathbf{z}+\mathbf{y})\cdot \mathrm{e}^{i\theta}, \theta)\phi_1(\mathbf{z})\, d\mathbf{z}\, \phi_2(\mathbf{y})\, d\mathbf{y} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} F(a, (b - \mathbf{z}\cdot \mathrm{e}^{i\theta}) - \mathbf{y}\cdot \mathrm{e}^{i\theta}, \theta)\phi_1(\mathbf{z})\, d\mathbf{z}\, \phi_2(\mathbf{y})\, d\mathbf{y} \\
&= \int_{\mathbb{R}^2} (F\star\phi_1)(a, b - \mathbf{z}\cdot \mathrm{e}^{i\theta}, \theta)\phi_2(\mathbf{y})\, d\mathbf{y} \\
&= ((F\star\phi_1)\star\phi_2)(a,b,\theta).
\end{aligned}
$$

Let $F \in \mathscr{S}(\mathbb{Y})$ be arbitrary.

$$
\begin{aligned}
(\Lambda \otimes (\phi_1 * \phi_2))(F) &= \Lambda(F \star (\phi_1 * \phi_2)\check{}) \\
&= \Lambda(F \star (\check{\phi}_1 * \check{\phi}_2)) \\
&= \Lambda(F \star (\check{\phi}_2 * \check{\phi}_1)) \text{ (since } * \text{ is commutative on } \mathscr{D}(\mathbb{R}^2)) \\
&= \Lambda((F \star \check{\phi}_2) \star \check{\phi}_1) \text{ (by using previous lemma)} \\
&= (\Lambda \otimes \phi_1)(F \star \check{\phi}_2) = ((\Lambda\otimes\phi_1)\otimes\phi_2)(F).
\end{aligned}
$$

□

Lemma 12. *Let If $F \in \mathscr{S}(\mathbb{Y})$, $\Lambda \in \mathscr{S}'(\mathbb{Y})$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$. Then,*

(i) $R^*(F \star \phi) = R^*F * \phi$,

(ii) $R^*(\Lambda \otimes \phi) = R^*\Lambda * \phi$.

Proof. For each $\mathbf{x} \in \mathbb{R}^2$, by using Fubini's theorem, we get

$$
\begin{aligned}
R^*(F\star\phi)(\mathbf{x}) &= \int_{\mathbb{Y}} (F\star\phi)(a,b,\theta)\psi_{a,b,\theta}(\mathbf{x})d\mu \\
&= \int_{\mathbb{Y}} \psi_{a,b,\theta}(\mathbf{x}) \int_{\mathbb{R}^2} F(a, b - \mathbf{y}\cdot \mathrm{e}^{i\theta}, \theta)\phi(\mathbf{y})d\mathbf{y}d\mu. \\
&= \int_{\mathbb{R}^2} \phi(\mathbf{y})d\mathbf{y} \int_{\mathbb{Y}} F(a, b - \mathbf{y}\cdot \mathrm{e}^{i\theta}, \theta)\psi_{a,b,\theta}(\mathbf{x})d\mu \\
&= \int_{\mathbb{R}^2} \phi(\mathbf{y})d\mathbf{y} \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} F(a, b - \mathbf{y}\cdot \mathrm{e}^{i\theta}, \theta)\psi\left(\frac{\mathbf{x}\cdot \mathrm{e}^{i\theta} - b}{a}\right)\frac{da}{a^4}db\frac{d\theta}{4\pi}
\end{aligned}
$$

$$= \int_{\mathbb{R}^2} \phi(\mathbf{y})d\mathbf{y} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} F(a,c,\theta)\psi\left(\frac{(\mathbf{x}-\mathbf{y})\cdot \mathrm{e}^{i\theta} - c}{a}\right) \frac{da}{a^4} dc \frac{d\theta}{4\pi}$$
$$= \int_{\mathbb{R}^2} (R^*F)(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})d\mathbf{y}$$
$$= (R^*F * \phi)(\mathbf{x})$$

For $f \in \mathscr{S}(\mathbb{R}^2)$,

$$\begin{aligned} R^*(\Lambda \otimes \phi)(f) &= (\Lambda \otimes \phi)(Rf) = \Lambda(Rf \star \check{\phi}) \\ &= \Lambda(R(f * \check{\phi})) = R^*\Lambda(f * \check{\phi}) \\ &= (R^*\Lambda * (f * \check{\phi})\check{})(0) = (R^*\Lambda * (\check{f} * \phi))(0) \\ &= (R^*\Lambda * (\phi * \check{f}))(0) = (R^*\Lambda * \phi) * \check{f})(0) \\ &= (R^*\Lambda * \phi)(f). \end{aligned}$$

Hence the lemma follows. □

Theorem 13 (Convolution Theorem)**.** *If $\nu \in \mathscr{S}'(\mathbb{R}^2)$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $\mathcal{R}(\nu * \phi) = \mathcal{R}\nu \otimes \phi$.*

Proof. Let $F \in \mathscr{S}(\mathbb{Y})$ be arbitrary.

$$\begin{aligned} \mathcal{R}(\nu * \phi)(F) &= (\nu * \phi)(R^*F) = ((\nu * \phi) * (R^*F)\check{})(0) \\ &= \nu * (\phi * (R^*F)\check{})(0) = \nu * ((R^*F)\check{} * \phi)(0) \\ &= \nu(R^*F * \check{\phi}) = \nu(R^*(F \star \check{\phi})) \\ &= \mathcal{R}\nu(F \star \check{\phi}) = (\mathcal{R}\nu \otimes \phi)(F). \end{aligned}$$

□

Then, we have the following corollary.

Corollary 14. *If $\Lambda \in \mathcal{R}(\mathscr{S}(\mathbb{R}^2))$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $\Lambda \otimes \phi \in \mathcal{R}(\mathscr{S}(\mathbb{R}^2))$.*

Lemma 15. *If $\Lambda_n \to \Lambda$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{Y})$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $\Lambda_n \otimes \phi \to \Lambda \otimes \phi$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{Y})$.*

Proof. Let $F \in \mathscr{S}(\mathbb{Y})$ be arbitrary. Then, by assumption, we have

$$\Lambda_n(F) \to \Lambda(F) \text{ as } n \to \infty \text{ in } \mathbb{C}.$$

Now, $(\Lambda \otimes \phi)(F) = \Lambda_n(F \star \check{\phi}) \to \Lambda(F \star \check{\phi}) = (\Lambda \otimes \phi)(F)$ as $n \to \infty$, since $F \star \check{\phi} \in \mathscr{S}(\mathbb{Y})$, by Lemma 9. □

In the following lemma, by "$\Lambda_n \to \Lambda$ as $n \to \infty$ in $\mathcal{R}(\mathscr{S}'(\mathbb{R}^2))$", we mean that $\Lambda_n \in \mathcal{R}(\mathscr{S}'(\mathbb{R}^2))$, $\forall n \in \mathbb{N}$, $\Lambda \in \mathcal{R}(\mathscr{S}'(\mathbb{R}^2))$ and $\Lambda_n \to \Lambda$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{Y})$.

Theorem 16. *If $\Lambda \in \mathcal{R}(\mathscr{S}'(\mathbb{R}^2))$ and $(\phi_n) \in \Delta_0$, then $\Lambda \otimes \phi_n \to \Lambda$ as $n \to \infty$ in $\mathcal{R}(\mathscr{S}'(\mathbb{R}^2))$.*

Proof. From $\Lambda \in \mathcal{R}(\mathscr{S}'(\mathbb{R}^2))$, there exists $\nu \in \mathscr{S}'(\mathbb{R}^2)$ such that $\Lambda = \mathcal{R}\nu$. It has been proved in [15, Lemma 3.5], that if $\nu \in \mathscr{S}'(\mathbb{R}^2)$ and $(\phi_n) \in \Delta_0$, then $\nu * \phi_n \to \nu$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{R}^2)$. Since $\Lambda \otimes \phi_n = \mathcal{R}\nu \otimes \phi_n = \mathcal{R}(\nu * \phi_n)$ and $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$ is continuous ([24, Theorem 4.5]), we have $\Lambda \otimes \phi_n \to \mathcal{R}\nu = \Lambda$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{Y})$. By using Corollary, we have $\Lambda \otimes \phi_n \in \mathcal{R}(\mathscr{S}'(\mathbb{R}^2))$, $\forall n \in \mathbb{N}$. Thus, we get $\Lambda \otimes \phi_n \to \Lambda$ as $n \to \infty$ in $\mathcal{R}(\mathscr{S}'(\mathbb{R}^2))$. □

Thus the Boehmian space $\mathscr{B}_2 = \mathscr{B}(\mathcal{R}(\mathscr{S}'(\mathbb{R}^2)), (\mathscr{D}, \star), \otimes, \Delta_0)$ has been constructed.

3. Extended ridgelet transform

Before defining the extended ridgelet transform, we consider the following.

If $\frac{\nu_n}{\phi_n}$ is a quotient in the context of $\mathscr{B}_1$, then we have $\nu_n \in \mathscr{S}'(\mathbb{R}^2)$, $\forall n \in \mathbb{N}$, $(\phi_n) \in \Delta_0$ and

$$\nu_n * \phi_m = \nu_m * \phi_n, \ \forall m, n \in \mathbb{N}.$$

By applying the ridgelet transform $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$ on both sides of the above equation and by invoking the Convolution Theorem (Theorem 13), we get

$$\mathcal{R}\nu_n \otimes \phi_m = \mathcal{R}\nu_m \otimes \phi_n, \ \forall m, n \in \mathbb{N}.$$

Therefore, $\frac{\mathcal{R}\nu_n}{\phi_n}$ is a quotient in the context of $\mathscr{B}_2$ and hence it represents a Boehmian in $\mathscr{B}_2$. Moreover, if $\frac{\nu_n}{\phi_n} \sim \frac{\mu_n}{\delta_n}$, then we have

$$\nu_n * \delta_m = \mu_m * \phi_n, \ \forall m, n \in \mathbb{N}.$$

Again, by using the same technique, we get

$$\mathcal{R}\nu_n \otimes \delta_m = \mathcal{R}\mu_m \otimes \phi_n, \ \forall m, n \in \mathbb{N}.$$

Thus $\frac{\mathcal{R}\nu_n}{\phi_n} \sim \frac{\mathcal{R}\mu_n}{\delta_n}$. With these observations, if we let $\mathscr{R}\left[\frac{\nu_n}{\phi_n}\right] = \left[\frac{\mathcal{R}\nu_n}{\phi_n}\right]$, then $\mathscr{R}$ is a well defined map from $\mathscr{B}_1$ into $\mathscr{B}_2$. We call $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ the extended ridgelet transform.

Lemma 17. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is consistent with the distributional ridgelet transform $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$.*

Proof. Let $\nu \in \mathscr{S}'(\mathbb{R}^2)$ be arbitrary. Then the tempered Boehmian representing ν is given by $\left[\frac{\nu * \phi_k}{\phi_k}\right]$ for any $(\phi_k) \in \Delta_0$. Therefore, by using Theorem 13, we get

$$\mathscr{R}\left[\frac{\nu * \phi_k}{\phi_k}\right] = \left[\frac{\mathcal{R}(\nu * \phi_k)}{\phi_k}\right] = \left[\frac{\mathcal{R}\nu \otimes \phi_k}{\phi_k}\right],$$

which is the Boehmian in $\mathscr{B}_2$ representing $\mathcal{R}\nu$. Hence $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is consistent with $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$. □

Theorem 18. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is a linear map.*

Proof. By using the linearity of $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$ and Theorem 13, the proof follows immediately. □

Theorem 19. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is injective.*

Proof. Let $\left[\frac{\nu_n}{\phi_n}\right], \left[\frac{\mu_n}{\delta_n}\right] \in \mathscr{B}_1$ be such that $\mathscr{R}\left[\frac{\nu_n}{\phi_n}\right] = \mathscr{R}\left[\frac{\mu_n}{\delta_n}\right]$. Then we have $\left[\frac{\mathcal{R}\nu_n}{\phi_n}\right] = \left[\frac{\mathcal{R}\mu_n}{\delta_n}\right]$. This implies that

$$\mathcal{R}\nu_n \otimes \delta_m = \mathcal{R}\mu_m \otimes \phi_n, \ \forall m, n \in \mathbb{N}.$$

Then, by using the convolution theorem, we get

$$\mathcal{R}(\nu_n * \delta_m) = \mathcal{R}(\mu_m * \phi_n), \ \forall m, n \in \mathbb{N}.$$

By applying the distributional adjoint ridgelet transform $\mathcal{R}^* : \mathscr{S}(\mathbb{Y}) \to \mathscr{S}(\mathbb{R}^2)$ on both sides, we get

$$\nu_n * \delta_m = \mu_m * \phi_n, \ \forall m, n \in \mathbb{N},$$

because $\mathcal{R}^* \circ \mathcal{R}$ is identity on $\mathscr{S}(\mathbb{R}^2)$. See [24].

Therefore, $\frac{\nu_n}{\phi_n}$ and $\frac{\mu_n}{\delta_n}$ represent the same Boehmian in $\mathscr{B}_1$. In other words, $\left[\frac{\nu_n}{\phi_n}\right] = \left[\frac{\mu_n}{\delta_n}\right]$. Hence, the theorem follows. □

Theorem 20. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is surjective.*

Proof. Let $\left[\frac{\Lambda_n}{\phi_n}\right] \in \mathscr{B}_2$ be arbitrary. Then $\Lambda_n \in \mathcal{R}(\mathscr{S}(\mathbb{R}^2))$, $\forall n \in \mathbb{N}$, $(\phi_n) \in \Delta_0$ and

$$\Lambda_n \otimes \phi_m = \Lambda_m \otimes \phi_n, \ \forall m, n \in \mathbb{N}.$$

Since $\Lambda_n = \mathcal{R}\nu_n$ for some $\nu_n \in \mathscr{S}'(\mathbb{R}^2)$, $\forall n \in \mathbb{N}$. Therefore, by using the convolution theorem, we get

$$\mathcal{R}(\nu_n * \phi_m) = \mathcal{R}(\nu_m * \phi_n), \ \forall m, n \in \mathbb{N}.$$

Again, by applying $\mathcal{R}^*$ on both sides and by using the fact that $\mathcal{R}^* \circ \mathcal{R}$ is identity on $\mathscr{S}(\mathbb{R}^2)$, we get

$$\nu_n * \phi_m = \nu_m * \phi_n, \ \forall m, n \in \mathbb{N}.$$

Therefore, $\frac{\nu_n}{\phi_n}$ is a quotient and hence $\left[\frac{\nu_n}{\phi_n}\right] \in \mathscr{B}_1$. Obviously, we have $\mathscr{R}\left[\frac{\nu_n}{\phi_n}\right] = \left[\frac{\Lambda_n}{\phi_n}\right]$. Hence, the theorem follows. □

Theorem 21. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is continuous with respect to δ-convergence.*

Proof. Let $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathscr{B}_1$. Then by Lemma 3, there exist $\nu_{n,k}, \nu_k \in \mathscr{S}'(\mathbb{R}^2)$, $n, k \in \mathbb{N}$ and $(\phi_k) \in \Delta_0$ such that

$$X_n = \left[\frac{\nu_{n,k}}{\phi_k}\right], X = \left[\frac{\nu_k}{\phi_k}\right] \text{ and } \nu_{n,k} \to \nu_k \text{ as } n \to \infty \text{ in } \mathscr{S}'(\mathbb{R}^2).$$

Since the distributional ridgelet transform $\mathcal{R} : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{Y})$ is continuous, we have

$$\mathcal{R}\nu_{n,k} \to \mathcal{R}\nu_k \text{ as } n \to \infty \text{ in } \mathscr{S}'(\mathbb{Y}) \text{ and hence in } \mathcal{R}(\mathscr{S}'(\mathbb{R}^2)).$$

Since

$$\mathscr{R}X_n = \left[\frac{\mathcal{R}\nu_{n,k}}{\phi_k}\right] \text{ and } \mathscr{R}X = \left[\frac{\mathcal{R}\nu_k}{\phi_k}\right],$$

again, by using Lemma 3, we get

$$\mathscr{R}X_n \xrightarrow{\delta} \mathscr{R}X \text{ as } n \to \infty \text{ in } \mathscr{B}_2.$$

Hence the theorem. □

Theorem 22. *If $X \in \mathscr{B}_2$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $\mathscr{R}(X * \phi) = \mathscr{R}X \otimes \phi$.*

Proof. Let $X = \left[\frac{\nu_k}{\phi_k}\right] \in \mathscr{B}_1$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$. Then by definition, $X * \phi = \left[\frac{\nu_k * \phi}{\phi_k}\right]$. Since $X * \phi \in \mathscr{B}_1$, we can apply $\mathscr{R}$ on $X * \phi$ and by using Theorem 13, we get

$$\mathscr{R}(X * \phi) = \left[\frac{\mathcal{R}(\nu_k * \phi)}{\phi_k}\right] = \left[\frac{\mathcal{R}\nu_k \otimes \phi}{\phi_k}\right] = \left[\frac{\mathcal{R}\nu_k}{\phi_k}\right] \otimes \phi = \mathscr{R}X \otimes \phi.$$

□

Theorem 23. *The extended ridgelet transform $\mathscr{R} : \mathscr{B}_1 \to \mathscr{B}_2$ is continuous with respect to Δ-convergence.*

Proof. If $X_n \xrightarrow{\Delta} X$ as $n \to \infty$ in $\mathscr{B}_1$, then there exist $\nu_n \in \mathscr{S}'(\mathbb{R}^2)$, $n \in \mathbb{N}$ and $(\phi_k) \in \Delta_0$ such that $(X_n - X) * \phi_n = \nu_n$, $\forall n \in \mathbb{N}$ and $\nu_n \to 0$ as $n \to \infty$ in $\mathscr{S}'(\mathbb{R}^2)$. Since

$$\begin{aligned}
(\mathscr{R}X_n - \mathscr{R}X) \otimes \phi_n &= \mathscr{R}(X_n - X) \otimes \phi_n \text{ (by Theorem 18)}\\
&= \mathscr{R}((X_n - X) * \phi_n) \text{ (by Theorem 22)}\\
&= \mathscr{R}\nu_n\\
&= \mathcal{R}\nu_n \text{ (by Theorem 17)}
\end{aligned}$$

we have $\mathscr{R}X_n \xrightarrow{\Delta} \mathscr{R}X$ as $n \to \infty$ in $\mathscr{B}_2$. □

Definition 24. *We define the extended adjoint ridgelet transform $\mathscr{R}^* : \mathscr{B}_2 \to \mathscr{B}_1$ by $\mathscr{R}^* \left[\frac{\Lambda_n}{\phi_n}\right] = \left[\frac{\mathcal{R}^*\Lambda_n}{\phi_n}\right]$, where $\mathcal{R}^* : \mathscr{S}'(\mathbb{Y}) \to \mathscr{S}'(\mathbb{R}^2)$. See [24].*

Lemma 25. *The map $\mathscr{R}^* : \mathscr{B}_2 \to \mathscr{B}_1$ is well defined.*

We call $\mathscr{R}^* : \mathscr{B}_2 \to \mathscr{B}_1$ the extended adjoint ridgelet transform.

Lemma 26. *The extended adjoint ridgelet transform $\mathscr{R}^* : \mathscr{B}_2 \to \mathscr{B}_1$ is consistent with the distributional adjoint ridgelet transform $\mathcal{R}^* : \mathscr{S}'(\mathbb{Y}) \to \mathscr{S}'(\mathbb{R}^2)$.*

Theorem 27. *The extended adjoint ridgelet transform $\mathscr{R}^* : \mathscr{B}_2 \to \mathscr{B}_1$ is linear.*

Theorem 28. *If $Y \in \mathscr{B}_2$ and $\phi \in \mathscr{D}(\mathbb{R}^2)$, then $\mathscr{R}^*(Y \otimes \phi) = \mathscr{R}^* Y * \phi$.*

Theorem 29. *The extended adjoint ridgelet transform $\mathscr{R}^* : \mathscr{B}_2 \to \mathscr{B}_1$ is continuous with respect to δ-convergence and Δ-convergence.*

Since the above theorems are analogous to the corresponding results of the extended ridgelet transform, we prefer to omit the details.

Theorem 30. *The composition $\mathscr{R}^* \circ \mathscr{R}$ of extended ridgelet transform and the extended adjoint ridgelet transform is identity on $\mathscr{B}_1$.*

Proof. Let $\left[\frac{\nu_n}{\phi_n}\right] \in \mathscr{B}_1$ be arbitrary. Then

$$(\mathscr{R}^* \circ \mathscr{R}) \left[\frac{\nu_n}{\phi_n}\right] = \mathscr{R}^* \left[\frac{\mathcal{R}\nu_n}{\phi_n}\right] = \left[\frac{\mathcal{R}^* \circ \mathcal{R}\nu_n}{\phi_n}\right] = \left[\frac{\nu_n}{\phi_n}\right],$$

since $\mathcal{R}^* \circ \mathcal{R}$ is identity on $\mathscr{S}'(\mathbb{R}^2)$. See [24]. □

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